

Mild solution for a stochastic partial differential equation with noise

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Abstract: This paper focuses on the study of the existence of a mild solution to time and space-fractional stochastic equation perturbed by multiplicative with noise. We use stochastic analysis techniques, fractional calculus, semi-group theory and we apply Sadovskii's fixed point theorem to prove our main result.

Key-Words: stochastic equation, mild solution, fractional derivative, Hausdorff measure of non-compactness, Sadovskii's fixed point

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1 Introduction

In this paper, we are interested in the existence of solutions for nonlinear fractional difference equations

$${}^c D_t^\alpha [u - h(u)] = \Delta u(t) + u \cdot \nabla u + g(u) W(t),$$

$$x \in D, t > 0,$$
(1)

subject to the initial condition

$$u(x, 0) = u_0(x), x \in D, t = 0,$$
(2)

and the Dirichlet boundary conditions

$$u(x, t) = 0, x \in \partial D,$$
(3)

where $D \subset \mathbb{R}^d$, $u(x, t)$ represents the velocity field of the fluid, the state $u(\cdot)$ takes values in a separable real Hilbert space H with inner product $\langle \cdot, \cdot \rangle$, the term $g(u) W(t) = g(u) \frac{d}{dt} W(t)$ describes a state dependent random noise, where $W(t)_{t \in [0, T]}$ is a F_t -adapted Wiener process defined in completed probability space (Ω, F, P) with expectation E and associate with the normal filtration $F_t = \sigma\{W(s) : 0 \leq s \leq t\}$. The operator Δ is the Laplacian. Here, ${}^c D_t^\alpha$ denotes the Caputo type derivative of order α ($0 < \alpha < 1$) for the function $u(x, t)$ with respect time t which is defined by

$$\begin{cases} {}^c D_t^\alpha u(t, x) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x, s)}{\partial s} \frac{ds}{(t-s)^\alpha}, & 0 < \alpha < 1, \\ \frac{\partial u(t, x)}{\partial t}, & \alpha = 1, \end{cases}$$
(4)

where $\Gamma(\cdot)$ stands for the Gamma function $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt$.

There has been a widespread interest during the last decade in constructing a stochastic integration theory with respect to fractional Brownian motion (FBM) and solving stochastic differential equations driven by FBM. In fact, stochastic perturbation factors, such as precipitation, absolute humidity, and temperature, have a significant impact on the infection force of all types of virus diseases to humans. Taking this into consideration enables us to present randomness into deterministic biological models to expose the environmental variability effect, whether it is environmental fluctuations in parameters or random noise in the differential systems [18]. Stochastic population dynamics perturbed by white noise (or Brownian motion) has been studied extensively by many authors [1, 12, 13]. It has been investigated in [14] that an environmental Brownian noise can suppress explosions in population dynamics. Yuan et al. [28] discussed the results of stochastic viral infection, immune response dynamics and analyzed the human immunodeficiency virus infection.

On the other hand, time-fractional differential equations are found to be quite effective in modelling anomalous diffusion processes as its can characterize the long memory processes [6, 19, 27, 22]. Hence, Burgers equation with time-fractional can be adapted to describe the memory effect of the wall friction through the boundary layer [8]. Furthermore, the analytical solutions of the time- and space-fractional Burgers equations have been investigated by variational iteration method and Adomian decomposition method [15]. Some relevant studies can be found in [4, 21, 25, 26, 29].

The existence of solution for partial neutral integro-differential equation with infinite delay in infinite dimensional spaces has been extensively studied by many authors, see for example [16]. Ezzinbi and al. [7] investigated the existence and regularity of solutions for some partial functional integro-differential equations in Banach spaces. Cui and Yan [5] investigated the existence of mild solutions for a class of fractional neutral stochastic integro-differential equations with infinite delay in Hilbert spaces by means of Sadovskii's fixed point theorem. In particular, the stability theory of stochastic differential equations has been popularly applied in variety fields of science and technology. Several authors have established the stability results of mild solutions for these equations by using various techniques, we refer the reader to [3, 10, 11, 17].

The main contribution of this paper is to establish the existence of mild solution for the problem (1)-(3). Using mainly the Sadovskii's fixed point theorem. The rest of the paper organised as follows, In Section 2, we will introduce some notations and preliminaries, which play a crucial role in our theorem analysis. In Section 3, the existence results on a mild solutions are derived.

2 Preliminaries

In this section, we give some notions and certain important preliminaries, which will be used in the subsequent discussions. Let $(\Omega, F, \{F_t\}_{t \geq 0}, P)$ be a filtered probability space with a normal filtration, where P is a probability measure on (Ω, F) and F is the Borel σ -algebra. Let $\{F_t\}_{t \geq 0}$ satisfying that F_0 contains all P -null sets. The operator A is the infinitesimal generator of a strongly continuous semigroup on a separable real Hilbert space H .

Denote the basic functional space $L^p(D)$, $1 \leq p < \infty$ and $H^s(D)$ by the usual Lebesgue and Sobolev space, respectively. We assume that A is the negative Laplacian $-\Delta$ in a bounded domain with zero Dirichlet boundary conditions in Hilbert space $H = L^2(D)$, which are given by

$$A = -\Delta, D(A) = H_0^1(D) \cap H^2(D),$$

since the operator A is self-adjoint, i.e., there exist the eigenvectors e_k corresponding to eigenvalues λ_k such that

$$Ae_k = \lambda_k e_k, e_k = \sqrt{2} \sin(k\pi), \lambda_k = \pi^2 k^2, k \in \mathbb{N}^+.$$

For any $\sigma > 0$, let H^σ be the domain of the frac-

tional power $A^{\frac{\sigma}{2}} = (-\Delta)^{\frac{\sigma}{2}}$, which can be defined by

$$\sigma > 0, A^{\frac{\sigma}{2}} e_k = \gamma_k^{\frac{\sigma}{2}} e_k, k = 1, 2, \dots$$

and

$$H^\sigma = D\left(A^{\frac{\sigma}{2}}\right) = \left\{v \in L^2(D), \right.$$

$$\left. \|v\|_{H^\sigma}^2 = \sum_{k=1}^{\infty} \gamma_k^{\frac{\sigma}{2}} v_k^2 < \infty \right\}$$

where $v_k = \langle v, e_k \rangle$ with the inner product $\langle \cdot, \cdot \rangle$ in $L^2(D)$, the norm $\|H^\sigma v\| = \left\|A^{\frac{\sigma}{2}} v\right\|$, the bilinear operator $B(u, v) = u \cdot \nabla v$ and $\mathcal{D}(B) = H_0^1(D)$ with the slight abuse of notation $B(u) = B(u, u)$. Then we can rewrite the equations (1)-(3) as follows in the abstract form

$$\begin{cases} {}^c D_t^\alpha [u(t) - h(u(t))] = Au(t) + B(u(t)) \\ \quad + g(u(t)) \frac{dW(t)}{dt}, t > 0, \\ u(0) = u_0, \end{cases} \quad (5)$$

where $\{W(t), t \geq 0\}$ is a Q -Wiener process with linear bounded covariance operator Q such that a trace class operator Q denote $Tr(Q) = \sum_{k=1}^{\infty} \lambda_k < \infty$, which satisfies that $Qe_k = \lambda_k e_k, k = 1, 2, \dots$, then the Wiener process is given by

$$W(t) = \sum_{k=1}^{\infty} \sqrt{\lambda_k} \beta_k(t) e_k,$$

where $\{\beta_k\}_{k=1}^{\infty}$ is a sequence of real-valued standard Brownian motions.

Let $L_0^2 = L^2\left(Q^{\frac{1}{2}}(H), H\right)$ be a Hilbert-Schmidt space of operators from $Q^{\frac{1}{2}}(H)$ to H with the norm

$$\|\phi\|_{L_0^2} = \left\| \phi Q^{\frac{1}{2}} \right\|_{H^\sigma} = \left(\sum_{n=1}^{\infty} \phi Q^{\frac{1}{2}} e_n \right)^{\frac{1}{2}},$$

i.e.,

$$L_0^2 = \left\{ \phi \in L(H) : \sum_{n=1}^{\infty} \left\| \lambda_n^{\frac{1}{2}} \phi Q^{\frac{1}{2}} e_n \right\|^2 < \infty \right\},$$

where $L(H)$ is the space of bounded linear operators from H to H .

For an arbitrary Banach space B , we denote

$$\|v\|_{L^p(\Omega, B)} = (E \|v\|_B^p)^{\frac{1}{p}}, \forall v \in L^p(\Omega, F, P, B),$$

for any $p \geq 2$.

We shall also need the following result with respect to the operator A_ν (see [22]).

Lemma 1. For any $\nu > 0$, an analytic semigroup $T(t) = e^{-tA_\nu}$, $t \geq 0$ is generated by the operator A_ν on L^p , there exists a constant C_ν dependent on ν such that

$$\|A_\nu T(t)\|_{L(L^p)} \leq C_\nu t^{-\nu}, \quad t > 0,$$

in which $L(B)$ denotes the Banach space of all bounded operators from K to itself.

Next we will introduce the following lemma to estimate the stochastic integrals, which contains the Burkholder-Davis-Gundy's inequality.

Lemma 2. [9] For any $0 \leq t_1 < t_2 \leq T$ and $p \geq 2$ and for any predictable stochastic process $v : [0, T] \times \Omega \rightarrow L^2_0$ which satisfies

$$E \left[\left(\int_0^T \|v(s)\|_{L^2_0}^2 ds \right)^{\frac{p}{2}} \right] < \infty,$$

then, we have

$$E \left[\left\| \int_{t_1}^{t_2} v(s) dW(s) \right\|^p \right] < C(p) E \left[\left(\int_{t_1}^{t_2} \|v(s)\|_{L^2_0}^2 ds \right)^{\frac{p}{2}} \right],$$

where $C(p) > 0$ is a constant depending only on p .

Inspired by the definition of the mild solution to the time-fractional differential equations (see [19]), we give the following definition of mild solution for our time-fractional stochastic equation.

Definition 3. An F_t -adapted stochastic process $(u(t), t \in [0, T])$ is called a mild solution to (5) if the following integral equation is satisfied

$$\begin{aligned} u(t) &= E_\alpha(t) u_0 + h(u(t)) \\ &+ \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(t-s) B(u(s)) ds \\ &+ \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(t-s) g(u(s)) dW(s), \end{aligned} \tag{6}$$

where the generalized Mittag-Leffler operators $E_\alpha(t)$ and $E_{\alpha,\alpha}(t)$ are defined, respectively, by

$$E_\alpha(t) = \int_0^\infty \zeta_\alpha(\theta) T(t^\alpha \theta) d\theta,$$

and

$$E_{\alpha,\alpha}(t) = \int_0^\infty \alpha \theta \zeta_\alpha(\theta) T(t^\alpha \theta) d\theta,$$

where $T(t) = e^{-tA}$, $t \geq 0$ is an analytic semi group generated by the operator $-A$ and the Mainardi's Wright-type function with $\alpha \in (0, 1)$ is given by

$$\zeta_\alpha(\theta) = \sum_{k=0}^\infty \frac{(-1)^k \theta^k}{k! \Gamma(1 - \alpha(1 + k))}.$$

Lemma 4. [5] For any $\alpha \in (0, 1)$ and $-1 < \nu < \infty$, it is not difficult to verify that

$$\zeta_\alpha(\theta) \geq 0 \text{ and } \int_0^\infty \theta^\nu \zeta_\alpha(\theta) d\theta = \frac{\Gamma(1 + \nu)}{\Gamma(1 + \alpha\nu)}, \tag{7}$$

for all $\theta \geq 0$.

The operators in (7) and $\{E_\alpha(t)\}_{t \geq 0}$ and $\{E_{\alpha,\alpha}(t)\}_{t \geq 0}$ have the following properties.

Lemma 5. For any $E_\alpha(t)$ and $E_{\alpha,\alpha}(t)$ are linear and bounded operators. Moreover, for $0 < \alpha < 1$ and $0 \leq \nu < 2$, there exists a constant $C > 0$ such that $E_\alpha(t)$ and $E_{\alpha,\alpha}(t)$ are defined, respectively, by

$$\|E_\alpha(t) \chi\|_{H^\nu} \leq Ct^{-\frac{\alpha\nu}{2}} \|\chi\|, \quad \|E_{\alpha,\alpha}(t) \chi\|_{H^\nu} \leq Ct^{-\frac{\alpha\nu}{2}} \|\chi\|. \tag{8}$$

Proof. For $T > 0$ and $0 \leq \nu < 2$, by means of Lemma 1 and Lemma 4, we have

$$\begin{aligned} \|E_\alpha(t) \chi\|_{H^\nu} &\leq \int_0^\infty \zeta_\alpha(\theta) \|A_\nu T(t^\alpha \theta) \chi\| d\theta \\ &\leq \int_0^\infty C_\nu t^{-\frac{\alpha\nu}{2}} \theta^{-\nu} \zeta_\alpha(\theta) \|\chi\| d\theta \\ &= \frac{C_\nu \Gamma(1 - \nu)}{\Gamma(1 - \alpha\nu)} t^{-\frac{\alpha\nu}{2}} \|\chi\|, \quad \chi \in L^2(D), \end{aligned}$$

and

$$\begin{aligned} \|E_{\alpha,\alpha}(t) \chi\|_{H^\nu} &\leq \int_0^\infty \alpha \theta \zeta_\alpha(\theta) \|A_\nu T(t^\alpha \theta) \chi\| d\theta \\ &\leq \int_0^\infty C_\nu \alpha t^{-\frac{\alpha\nu}{2}} \theta^{1-\nu} \zeta_\alpha(\theta) \|\chi\| d\theta \\ &= \frac{C_\nu \alpha \Gamma(2 - \nu)}{\Gamma(1 - \alpha\nu)} t^{-\frac{\alpha\nu}{2}} \|\chi\|, \quad \chi \in L^2(D), \end{aligned}$$

so, $E_\alpha(t)$ and $E_{\alpha,\alpha}(t)$ are linear and bounded operators. The proof is completed. \square

Lemma 6. For any $t > 0$, the operators $E_\alpha(t)$ and $E_{\alpha,\alpha}(t)$ are strongly continuous. Moreover, for $0 < \alpha < 1$ and $0 \leq \nu < 2$ and $0 \leq t_1 < t_2 \leq T$, there exists a constant $C > 0$ such that

$$\|(E_\alpha(t_2) - E_\alpha(t_1))\chi\|_{H^\nu} \leq C(t_2 - t_1)^{\frac{\alpha\nu}{2}} \|\chi\|, \quad (9)$$

and

$$\|(E_{\alpha,\alpha}(t_2) - E_{\alpha,\alpha}(t_1))\chi\|_{H^\nu} \leq C(t_2 - t_1)^{\frac{\alpha\nu}{2}} \|\chi\|. \quad (10)$$

Proof. For any $0 < T_0 \leq t_1 < t_2 \leq T$, it is easy to deduce that

$$\int_{t_1}^{t_2} \frac{dT(t^\alpha\theta)}{dt} dt = T(t_2^\alpha\theta) - T(t_1^\alpha\theta)$$

$$= \int_{t_1}^{t_2} \alpha t^{\alpha-1} \theta AT(t^\alpha\theta) dt,$$

and by (7) and Lemma 5, we have

$$\begin{aligned} & \|(E_\alpha(t_2) - E_\alpha(t_1))\chi\|_{H^\nu} \\ &= \|A_\nu(E_\alpha(t_2) - E_\alpha(t_1))\chi\| \\ &= \left\| \int_0^\infty \zeta_\alpha(\theta) A_\nu \left(T(t_2^\alpha\theta) - T(t_1^\alpha\theta) \right) \chi d\theta \right\| \\ &\leq \int_0^\infty \alpha \theta \zeta_\alpha(\theta) \int_{t_1}^{t_2} t^{\alpha-1} \|A_{2+\nu} T(t^\alpha\theta)\chi\|_{L^2} dt d\theta \\ &\leq \int_0^\infty C_\nu \alpha \theta^{-\frac{\nu}{2}} \zeta_\alpha(\theta) \left(\int_{t_1}^{t_2} t^{-\frac{\alpha\nu}{2}-1} dt \right) \|\chi\| d\theta \\ &= \frac{2C_\nu \Gamma(1 - \frac{\nu}{2})}{\nu \Gamma(1 - \frac{\alpha\nu}{2})} \left(t_1^{-\frac{\alpha\nu}{2}} - t_2^{-\frac{\alpha\nu}{2}} \right) \|\chi\| \\ &\leq \frac{2C_\nu \Gamma(1 - \frac{\nu}{2})}{\nu T_0^{\alpha\nu} \Gamma(1 - \frac{\alpha\nu}{2})} (t_2 - t_1)^{\frac{\alpha\nu}{2}} \|\chi\|, \chi \in L^2(D). \end{aligned}$$

Also

$$\begin{aligned} & \|(E_{\alpha,\alpha}(t_2) - E_{\alpha,\alpha}(t_1))\chi\|_{H^\nu} \\ &= \|A_\nu(E_{\alpha,\alpha}(t_2) - E_{\alpha,\alpha}(t_1))\chi\| \\ &= \left\| \int_0^\infty \alpha \theta \zeta_\alpha(\theta) A_\nu \left(T(t_2^\alpha\theta) - T(t_1^\alpha\theta) \right) \chi d\theta \right\| \\ &\leq \int_0^\infty \alpha^2 \theta^2 \zeta_\alpha(\theta) \int_{t_1}^{t_2} t^{\alpha-1} \|A_{2+\nu} T(t^\alpha\theta)\chi\|_{L^2} dt d\theta \\ &\leq \int_0^\infty C_\nu \alpha^2 \theta^{1-\frac{\nu}{2}} \zeta_\alpha(\theta) \left(\int_{t_1}^{t_2} t^{-\frac{\alpha\nu}{2}-1} dt \right) \|\chi\| d\theta \\ &= \frac{2\alpha C_\nu \Gamma(2 - \frac{\nu}{2})}{\nu \Gamma(1 + \alpha(1 - \frac{\nu}{2}))} \left(t_1^{-\frac{\alpha\nu}{2}} - t_2^{-\frac{\alpha\nu}{2}} \right) \|\chi\| \\ &\leq \frac{2C_\nu \Gamma(2 - \frac{\nu}{2})}{\nu T_0^{\alpha\nu} \Gamma(1 + \alpha(1 - \frac{\nu}{2}))} (t_2 - t_1)^{\frac{\alpha\nu}{2}} \|\chi\|, \chi \in L^2(D). \end{aligned}$$

It is obviously to see that the term

$$\|(E_\alpha(t_2) - E_\alpha(t_1))\chi\|_{H^\nu} \rightarrow 0,$$

and

$$\|(E_{\alpha,\alpha}(t_2) - E_{\alpha,\alpha}(t_1))\chi\|_{H^\nu} \rightarrow 0,$$

as $t_1 \rightarrow t_2$ which mean that the operators $E_\alpha(t)$ and $E_{\alpha,\alpha}(t)$ are strongly continuous. \square

3 Existence results

In this section, we present our main results on the existence of mild solutions of problem (5) and we define the following space

$$K = \left\{ u : u \in C([0, T], H^\nu), \sup_{t \in [0, T]} \|u\| < \infty \right\}.$$

To do this, we make the following hypotheses:

(H₁) A is the infinitesimal generator of a strongly continuous semigroup $\{T(t), t \geq 0\}$ on H . We will also suppose that the operator $E_\alpha(t), t > 0$ is compact.

(H₂) The function $g : \Omega \times H \rightarrow L_0^2$ satisfies the following global Lipschitz and growth conditions:

$$\|g(v)\|_{L_0^2} \leq C \|u\|, \|g(u) - g(v)\|_{L_0^2} \leq C \|u - v\|,$$

for any $u \in H, v \in H$.

(H₃) The initial value $u_0 : \Omega \rightarrow H^\nu$ is a F_0 -measurable random variable, it hold that

$$\|u_0\|_{L^p(\Omega, H^\nu)} < \infty, \text{ for any } 0 \leq \nu < \alpha < 2.$$

(H₄) The function $h : L_0^2 \rightarrow L_0^2$ is continuous and there exists $L_h > 0$ such that

$$E \|h(u_1(t)) - h(u_2(t))\|_{L_0^2}^p \leq L_h \|u_1(t) - u_2(t)\|_{L_0^2}^p, \\ t \in [0, T], u_1, u_2 \in L_0^2,$$

and

$$E \|h(u(t))\|_{L_0^2}^p \leq L_h E \|u(t)\|_{L_0^2}^p, t \in [0, T], u \in L_0^2.$$

(H₅) Let $C > 0$ be a real number, then the bounded bilinear operator $B : L^2(D) \rightarrow H^{-1}(D)$ satisfies the following properties

$$\|B(u)\|_{H^{-1}} \leq C \|u\|^2,$$

and

$$\|B(u) - B(v)\|_{H^{-1}} \leq C (\|u\| + \|v\|) \|u - v\|,$$

for any $u, v \in L^2(D)$.

Definition 7. (see [2]) Let C be a bounded set in metric space (X, d) , then Kuratowski measure of non-compactness, $\alpha(C)$ is defined as

$\inf \{ \epsilon : C \text{ covered by finitely many sets}$

$\text{such that the diameter of each set} \}.$

Definition 8. (see [2]). Let $\Phi : D(\Phi) \subset X \rightarrow X$ a bounded and continuous operator on Banach space X . Then Φ is called a condensing map if $\alpha(\Phi(Y)) < \alpha(Y)$ all bounded sets Y , where α denotes the Kuratowski measure of noncompactness.

Lemma 9 ([24], Example 11.7). The map $G_1 + G_2$ is a k -set contraction with $0 \leq k < 1$ and is thus condensing, if

- (i) $G_1, G_2 : D \subseteq X \rightarrow X$ are operators on the Banach space,
- (ii) G_1 is k -contractive, that is

$$\|G_1 u - G_1 v\| \leq k \|u - v\|,$$

for all $u, v \in D$ and fixed point $k \in [0, 1)$,

- (iii) G_2 is compact.

Our main results is based on the following Sadovskii fixed points theorem.

Theorem 10. [20](Sadovskii's fixed point theorem) Let Y be a convex, bounded, and closed subset of a Banach space X and let $\Phi : Y \rightarrow Y$ a condensing map. Then Φ has a fixed point.

In the proof of main result, we need the following Lemmas.

Lemma 11. Assume that conditions (H_1) and (H_2) hold. Let Φ_1 be the operator defined by for each $u \in K$

$$\Phi_1(u) = \int_0^t (t-s)^{\alpha-1} E_{\alpha, \alpha}(t-s) B(u(s)) ds. \tag{11}$$

Then Φ_1 is continuous and maps K into itself.

Proof. It is obvious that Φ_1 is continuous. Next we show that $\Phi_1(K) \subset K$. By (H_1) and (H_2) , from the equation (11) and by applying Holder inequality, we have

$$E \|(\Phi_1 u)(t)\|_{H^\nu}^p \\ = E \left\| \int_0^t (t-s)^{\alpha-1} A_1 E_{\alpha, \alpha}(t-s) A_{\nu-1} B(u(s)) ds \right\|_{H^\nu}^p \\ \leq C_\alpha^p \left(\int_0^t (t-s)^{\frac{p(\frac{\alpha-1}{2})}{p-1}} ds \right)^{p-1} \int_0^t E [\|A_{\nu-1} B(u(s))\|^p] ds \\ \leq C^p C_\alpha \left[\frac{2(p-1)}{p-2} \right]^{p-1} (T)^{\frac{p-2}{2}} \int_0^t E [\|u(t)\|_{H^\nu}^p] ds \\ = \gamma_1 \int_0^t E [\|u(s)\|_{H^\nu}^p] ds, \tag{12}$$

where $\gamma_1 = C^p C_\alpha \left[\frac{2(p-1)}{p-2} \right]^{p-1} (T)^{\frac{p-2}{2}}$. This complete the proof. \square

Lemma 12. Assume that conditions (H_1) and (H_2) hold. Let Φ_2 be the operator defined by for each $u \in K$

$$\Phi_2(u) = \int_0^t S_\alpha(t-s) f(u(s)) dW(s).$$

Then Φ_2 is continuous and maps K into itself.

Proof. Making use of the Holder inequality and Lemma 2, we obtain

$$\begin{aligned}
 & E \|(\Phi_2 u)(t)\|_{H^\nu}^p \\
 &= E \left\| \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(t-s) g(u(s)) dW(s) \right\|_{H^\nu}^p \\
 &\leq C(p) E \left[\left(\int_0^t \left\| (t-s)^{\alpha-1} E_{\alpha,\alpha}(t-s) \right\|^2 \|A_\nu g(u)\|_{L_0^2}^2 ds \right)^{\frac{p}{2}} \right] \\
 &\leq C(p) C_\alpha^p \left(\int_0^t (t-s)^{\frac{2p(\alpha-1)}{p-2}} \right)^{\frac{p-2}{2}} \int_0^t E \|A_\nu g(u)\|_{L_0^2}^p ds \\
 &\leq C(p) C_\alpha^p \left(\frac{p-2}{p(2\alpha-1)-2} \right)^{\frac{p-2}{2}} \int_0^t E \|A_\nu g(u)\|_{L_0^2}^p ds \\
 &= \gamma_2 \int_0^t E [\|u(s)\|_{H^\nu}^p] ds,
 \end{aligned} \tag{13}$$

where $\gamma_2 = C(p) C_\alpha^p C^p \left[\frac{p-2}{p(2\alpha-1)-2} \right]^{\frac{p-2}{2}}$.

That is $\Phi_2(B) \subset B$. □

Lemma 13. Assume that conditions (H_1) and (H_4) hold. Let Φ_3 be the operator defined by for each $u \in B$

$$(\Phi_3 u)(t) = E_\alpha(t) u_0 + h(u(t)).$$

Then Φ_3 is continuous and maps K into K .

Proof. The continuity in p -th moment of Φ_3 follows from (H_4) .

Next, we show that $\Phi_3(Y) \subset Y$. By (H_1) , (H_5) and from (13), we have

$$E \|(\Phi_3 u)(t)\|_{L_0^2}^p \leq E \|h(u(t))\|_{L_0^2}^p \leq L_h E \|u(t)\|_{L_0^2}^p.$$

So, we conclude $\Phi_3(K) \subset K$. □

Lemma 14. Assume that conditions (H_1) and (H_2) hold. Then

$$E [\|E_\alpha(t) u_0\|_{H^\nu}] \leq E [\|u_0\|_{H^\nu}].$$

Proof. By Lemma 5, we have

$$\begin{aligned}
 & E [\|E_\alpha(t) u_0\|_{H^\nu}] \\
 &\leq E \left[\int_0^\infty \zeta_\alpha(\theta) \left(\|A_\nu T(t^\alpha \theta) u_0\|^2 \right)^{\frac{1}{2}} d\theta \right] \\
 &\leq E \left[\int_0^\infty \zeta_\alpha(\theta) \left(\sum_{n=1}^\infty \langle A_\nu e^{-t^\alpha \theta A} u_0, e_n \rangle^2 \right)^{\frac{1}{2}} d\theta \right] \\
 &\leq E \left[\int_0^\infty \zeta_\alpha(\theta) \left(\sum_{n=1}^\infty \langle A_\nu u_0, e^{-t^\alpha \theta \lambda_n^{\frac{\nu}{2}}} e_n \rangle^2 \right)^{\frac{1}{2}} d\theta \right] \\
 &\leq E \left[\int_0^\infty \zeta_\alpha(\theta) \|u_0\|_{H^\nu} d\theta \right] = E [\|u_0\|_{H^\nu}].
 \end{aligned}$$

□

First, we define a map $F : K \rightarrow C([0, T], H^\nu)$ in the following manner: for any $u \in K$,

$$\begin{aligned}
 (Fu)(t) &= \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(t-s) B(u(s)) ds \\
 &\quad + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(t-s) f(u(s)) dW(s) \\
 &\quad + E_\alpha(t) u_0(s) + h(u(t)).
 \end{aligned}$$

Now, we set $F = F_1 + F_2$, where

$$(F_1 u)(t) = E_\alpha(t) u_0(s) + h(u(t)),$$

and

$$\begin{aligned}
 (F_2 u)(t) &= \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(t-s) B(u(s)) ds \\
 &\quad + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(t-s) f(u(s)) dW(s),
 \end{aligned}$$

for $t \in [0, T]$.

Lemma 15. Assume (H_2) , (H_4) , (H_5) hold and $0 < \nu < \alpha \leq 2$, $p \geq 2$, Then

$$E \|E_\alpha(t_2) - E_\alpha(t_1)\|_{H^\nu}^p \leq C_{\alpha,\nu}^p (t_2 - t_1)^{\frac{\alpha\nu}{2}} E \|u_0\|^p.$$

Proof. We set

$$I_1 = F_1(t_2) - F_1(t_1) = E_\alpha(t_2) u_0 - E_\alpha(t_1) u_0$$

For any $p \geq 2$, by virtue of Lemma 6, it follows that

$$E \left[\|I_1\|_{H^\nu}^p \right] = E \left[A \|E_\alpha(t_2) u_0 - E_\alpha(t_1) u_0\|^p \right] \\ \leq C_{\alpha,\nu}^p (t_2 - t_1)^{\frac{\alpha\nu}{2}} E \|u_0\|^p.$$

It is obviously to see that the term $\|(F_1(t_2) - F_1(t_1))\|_Y \rightarrow 0$ as $t_1 \rightarrow t_2$ which mean that the operators F_1 is strongly continuous. \square

Lemma 16. Assume $(H_2), (H_4), (H_5)$ hold and $0 < \nu < \alpha \leq 2, p \geq 2$, then the operator F_2 is uniformly bounded.

Proof. From Lemma 11, Lemma 12 and by means of extension of Gronwall's lemma, we have

$$\sup_{t \in [0, T]} E \left[\|F_2(u(t))\|_{H^\nu}^p \right] \leq \infty,$$

that is the operator F_2 is uniformly bounded. \square

Lemma 17. Assume $(H_2), (H_4), (H_5)$ hold and $0 < \nu < \alpha \leq 2, p \geq 2$. Then the operator F_2 is equicontinuous.

Proof. For any $0 \leq t_1 < t_2 \leq T$, from

$$(F_2 u)(t_2) - (F_2 u)(t_1) \\ = \int_0^{t_2} (t_2 - s)^{\alpha-1} E_{\alpha,\alpha}(t_2 - s) B(u(s)) ds \\ - \int_0^{t_1} (t_1 - s)^{\alpha-1} E_{\alpha,\alpha}(t_1 - s) B(u(s)) ds \\ + \int_0^{t_2} (t_2 - s)^{\alpha-1} E_{\alpha,\alpha}(t_2 - s) g(u) dW(s) \\ - \int_0^{t_1} (t_1 - s)^{\alpha-1} E_{\alpha,\alpha}(t_1 - s) g(u) dW(s) = I_2 + I_3, \tag{14}$$

where

$$I_2 = \int_0^{t_2} (t_2 - s)^{\alpha-1} E_{\alpha,\alpha}(t_2 - s) B(u(s)) ds \\ - \int_0^{t_1} (t_1 - s)^{\alpha-1} E_{\alpha,\alpha}(t_1 - s) B(u(s)) ds \\ + \int_0^{t_1} (t_1 - s)^{\alpha-1} [E_{\alpha,\alpha}(t_2 - s) \\ - E_{\alpha,\alpha}(t_1 - s)] B(u(s)) ds \\ + \int_0^{t_1} \left[(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} \right] \\ \times E_{\alpha,\alpha}(t_2 - s) B(u(s)) ds \\ + \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} E_{\alpha,\alpha}(t_2 - s) B(u(s)) ds \\ = I_{21} + I_{22} + I_{23}, \tag{15}$$

and

$$I_3 = \int_0^{t_2} (t_2 - s)^{\alpha-1} E_{\alpha,\alpha}(t_2 - s) f(u(s)) dW(s) \\ - \int_0^{t_1} (t_1 - s)^{\alpha-1} E_{\alpha,\alpha}(t_1 - s) f(u(s)) dW(s) \\ = \int_0^{t_1} (t_1 - s)^{\alpha-1} [E_{\alpha,\alpha}(t_2 - s) - E_{\alpha,\alpha}(t_1 - s)] f(u(s)) dW(s) \\ + \int_0^{t_1} \left[(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} \right] E_{\alpha,\alpha}(t_2 - s) f(u(s)) dW(s) \\ + \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} E_{\alpha,\alpha}(t_2 - s) f(u(s)) dW(s) \\ = I_{31} + I_{32} + I_{33}. \tag{16}$$

For the first term I_{21} in (15), applying the assumptions (H_5) and Lemma 10 and Holder inequality, we

have

$$\begin{aligned}
 E &= [\|I_{21}\|_{H^\nu}^p] \\
 &= E \left[\left\| \int_0^{t_1} (t_1 - s)^{\alpha-1} [E_{\alpha,\alpha}(t_2 - s) - E_{\alpha,\alpha}(t_1 - s)] B(u(s)) ds \right\|^p \right] \\
 &\leq C_{\alpha\nu}^p (t_2 - t_1)^{\frac{p\alpha(\nu+1)}{2}} \times \\
 &\quad \left(\int_0^{t_1} (t_1 - s)^{\frac{p(\alpha-1)}{p-1}} ds \right)^{p-1} \int_0^t E [\|A_{-1}B(u(s))\|_{H^1}^p] ds \\
 &\leq C^p C_{\alpha\nu}^p T^{p\alpha} \times \\
 &\quad \left(\frac{p-1}{p\alpha-1} \right)^{p-1} \left(\sup_{t \in [0, T]} E [\|u(s)\|_{H^1}^{2p}] \right) (t_2 - t_1)^{\frac{p\alpha(\nu+1)}{2}}. \tag{17}
 \end{aligned}$$

Using the assumptions (H_5) and Lemma 10 and Holder inequality, we have

$$\begin{aligned}
 E &[\|I_{22}\|_{H^\nu}^p] \\
 &= E \left[\left\| \int_0^{t_1} [(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}] \right. \right. \\
 &\quad \times [A_\nu E_{\alpha,\alpha}(t_2 - s)] B(u(s)) ds \left. \right\|^p] \\
 &\leq C_\alpha^p \left(\int_0^{t_1} \left\{ [(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1}] \right. \right. \\
 &\quad \times (t_2 - s)^{\frac{-\alpha(\nu+1)}{2}} \left. \left. \right\} ds \right)^{p-1} \int_0^t E [\|A_{-1}B(u(s))\|_{H^1}^p] ds \\
 &\leq C^p C_{\alpha\nu}^p T \left(\frac{p-1}{p(\alpha - \frac{\alpha(\nu+1)}{2})} \right)^{p-1} \\
 &\quad \times \left(\sup_{t \in [0, T]} E [\|u(s)\|_{H^1}^{2p}] \right) (t_2 - t_1)^{\frac{p\alpha(1-\nu)-2}{2}}, \tag{18}
 \end{aligned}$$

and

$$\begin{aligned}
 E &[\|I_{23}\|_{H^\nu}^p] \\
 &= E \left[\left\| \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} A_\nu E_{\alpha,\alpha}(t_2 - s) B(u(s)) ds \right\|^p \right] \\
 &\leq C_\alpha^p \left(\int_{t_1}^{t_2} (t_2 - s)^{\alpha-1 - \frac{\alpha(\nu+1)}{2}} ds \right)^{p-1} \\
 &\quad \times \int_{t_1}^{t_2} E [\|A_\nu B(u(s))\|_{H^1}^p] ds \\
 &\leq C^p C_\alpha^p \left(\frac{p-1}{p(\alpha - \frac{\alpha(\nu+1)}{2}) - 1} \right)^{p-1} \\
 &\quad \times \left(\sup_{t \in [0, T]} E [\|u(s)\|_{H^1}^{2p}] \right) (t_2 - t_1)^{\frac{p\alpha(1-\nu)}{2}}. \tag{19}
 \end{aligned}$$

Next, by following similar arguments as in the proof of (17)-(19) and using Lemma 2 there holds,

$$\begin{aligned}
 E &[\|I_{31}\|_{H^\nu}^p] \\
 &= E \left[\left\| \int_0^{t_1} (t_1 - s)^{\alpha-1} [E_{\alpha,\alpha}(t_2 - s) - E_{\alpha,\alpha}(t_1 - s)] f(u(s)) dW_s \right\|^p \right] \\
 &\leq C(p) E \left[\left(\int_0^{t_1} \left\| (t_1 - s)^{\alpha-1} A_\nu [E_{\alpha,\alpha}(t_2 - s) - E_{\alpha,\alpha}(t_1 - s)] \right\|^2 \|f(u(s))\|_{L_0^2}^2 ds \right)^{\frac{p}{2}} \right] \\
 &\leq C(p) C_{\alpha\nu}^p (t_2 - t_1)^{\frac{p\alpha\nu}{2}} \left(\int_0^{t_1} (t_1 - s)^{\frac{2p(\alpha-1)}{p-2}} ds \right)^{\frac{p-2}{2}} \\
 &\quad \times \int_0^{t_1} E \|f(u(s))\|_{L_0^2}^p ds \\
 &\leq C^p C_{\alpha\nu}^p T^{\frac{2p\alpha-p-1}{2}} \left(\frac{p-1}{2p\alpha-p-2} \right)^{p-1} \\
 &\quad \times \left(\sup_{t \in [0, T]} E [\|u(s)\|_{H^1}^p] \right) (t_2 - t_1)^{\frac{p\alpha\nu}{2}}, \tag{20}
 \end{aligned}$$

and

$$\begin{aligned}
 & E [\|I_{32}\|_{H^\nu}^p] \\
 &= E \left[\left\| \int_0^{t_1} \left[(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} \right] \right. \right. \\
 &\quad \times \left. \left. [A_\nu E_{\alpha,\alpha}(t_2 - s)] f(u(s)) qW_s \right\|_{L_0^2}^p \right] \\
 &\leq C(p) \times \\
 & E \left[\left(\int_0^{t_1} \left\| \left[(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} \right] [A_\nu E_{\alpha,\alpha}(t_2 - s)] \right\| \right. \right. \\
 &\quad \times \left. \left. \|f(u(s))\|_{L_0^2}^2 ds \right)^{\frac{p}{2}} \right] \\
 &\leq C(p) C_\alpha^p \times \\
 & \left(\int_0^{t_1} \left\{ \left[(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} \right] (t_2 - s)^{-\frac{\alpha\nu}{2}} \right\}^{\frac{2p}{p-2}} ds \right) \\
 &\quad \times \int_0^t E [\|f(u(s))\|_{L_0^2}^p] ds \\
 &\leq C(p) C^p C_\alpha^p T \left(\frac{2(p-2)}{2p\alpha(2-\nu) - 2(p+2)} \right)^{\frac{p-2}{2}} \\
 &\quad \times \left(\sup_{t \in [0, T]} E [\|u(t)\|^p] \right) (t_2 - t_1)^{\frac{2p\alpha(2-\nu) - 2(p+2)}{4}}, \tag{21}
 \end{aligned}$$

and

$$\begin{aligned}
 & E [\|I_{33}\|_{H^\nu}^p] \\
 &= E \left[\left\| \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} A_\nu E_{\alpha,\alpha}(t_2 - s) B(u(s)) ds \right\| \right]^p \\
 &\leq C(p) \times \\
 & E \left[\left(\int_0^{t_1} \left\| \left[(t_2 - s)^{\alpha-1} - (t_1 - s)^{\alpha-1} \right] [A_\nu E_{\alpha,\alpha}(t_2 - s)] \right\| \right. \right. \\
 &\quad \times \left. \left. \|f(u(s))\|_{L_0^2}^2 ds \right)^{\frac{p}{2}} \right] \\
 &\leq C(p) C_\alpha^p \left(\int_{t_1}^{t_2} (t_2 - s)^{\alpha-1-\frac{\alpha\nu}{2}} \right)^{\frac{p-2}{2}} \\
 &\quad \times \int_{t_1}^{t_2} E [\|f(u(s))\|_{L_0^2}^p] ds \\
 &\leq C(p) C^p C_\alpha^p \left(\frac{2(p-2)}{2p\alpha(2-\nu) - 2(p+2)} \right)^{\frac{p-2}{2}} \\
 &\quad \times \left(\sup_{t \in [0, T]} E [\|u(t)\|^p] \right) (t_2 - t_1)^{\frac{2p\alpha(2-\nu) - 2p}{4}} \tag{22}
 \end{aligned}$$

Taking expectation on the both side of (14) and in view of estimates (15) and (17)-(22), we conclude that

$$\|(F_2u)(t_2) - (F_2u)(t_1)\|_{L^p(\Omega, H^\nu)} \leq C(t_2 - t_1)^\gamma,$$

where $\gamma = \min \left\{ \frac{\alpha\nu}{2}, \frac{\alpha p(1-\nu)-2}{2p}, \frac{2p\alpha(2-\nu)-2(p+2)}{4p} \right\}$ when $0 < t_2 - t_1 < 1$.

Otherwise, if $t_2 - t_1 \geq 1$, then we set $\gamma = \max \left\{ \frac{\alpha(\nu+1)}{2}, \frac{\alpha(2-\nu-1)}{2}, \frac{2p\alpha(2-\nu)-2p}{4p} \right\}$. \square

Lemma 18. Assume the conditions (H_1) and (H_2) hold. Then F maps K into itself.

Proof. Let the nonlinear operator F defined by, for $\frac{t_2 - t_1}{2} \geq 0$,

$$\begin{aligned}
 (Fu)(t) &= E_\alpha(t) u_0 + h(u(t)) \\
 &\quad + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(t-s) B(u(s)) ds \\
 &\quad + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(t-s) g(u) dW(s).
 \end{aligned}$$

We prove that the operator F has a fixed point, which is a mild solution of the problem (1)-(3). We shall employ Theorem 10. For better readability, we break the proof into a sequence of steps.

Step 1. For $0 < \lambda < 1$, set $\{u \in K : u = \lambda Fu\}$ is bounded.

Let $u \in K$ be a possible solution of $u = \lambda Fu$ for some $0 < \lambda < 1$. Then by $(H_1) - (H_4)$ and applying the similar arguments in Lemma 13 and Lemma 16, we have

$$\begin{aligned}
 E \|u(t)\|_{H^\nu}^p &\leq 3^{p-1} \|E_\alpha(t) u_0\|_{H^\nu}^p \\
 &\quad + 3^{p-1} \|h(u(t))\|_{H^\nu}^p \\
 &\quad + 3^{p-1} E \|\Phi_1(u(t))\|_{H^\nu}^p \\
 &\quad + 3^{p-1} E \|\Phi_2(u(t))\|_{H^\nu}^p \\
 &\leq 3^{p-1} E [\|u_0\|_{H^\nu}] \\
 &\quad + 3^{p-1} (\gamma_1 + \gamma_2) \int_0^t E [\|u\|_{H^\nu}^p] ds.
 \end{aligned}$$

By means of the extension of Gronwall's lemma, it holds that

$$\sup_{t \in [0, T]} E \|u(t)\|_{H^\nu}^p < \infty.$$

This indicates that $u(\cdot)$ is bounded on $[0, T]$.

Step 2. $F : K \rightarrow C([0, T], H^\sigma)$ is continuous.

Let $\{u_n(t)\}_{n \geq 0}$ with $u_n \rightarrow u$ ($n \rightarrow \infty$) in K . Then there is a number $r > 0$ such that $E \|u_n(t)\|_{H^\nu}^2 \leq r$ for all n and a.e. $t \in [0, T]$, so $u_n \in B_r(0, Y) = \left\{ u \in K : \sup_{t \in [0, T]} \|u\|_{H^\sigma} \right\}$ and $u \in B_r(0, K)$. By the assumptions (H_2) and similar argument to obtain (12) and (13), we have

$$\begin{aligned} & E \|(Fu_n)(t) - (Fu)(t)\|_{H^\nu}^p \\ & \leq 3^{p-1} \|h(u_n(t)) - h(u(t))\|_{H^\nu}^p \\ & \quad + 3^{p-1} E \|\Phi_1(u_n(t) - u(t))\|_{H^\nu}^p \\ & \quad + 3^{p-1} E \|\Phi_2(u_n(t) - u(t))\|_{H^\nu}^p \\ & \leq 3^{p-1} \|h(u_n(t)) - h(u(t))\|_{H^\nu}^p \\ & \quad + 3^{p-1} (G\gamma_1 + K\gamma_2) \left(\int_0^t E \|u_n - u\|_{H^\nu}^p ds \right). \end{aligned}$$

Then, we have for all $t \in [0, T]$,

$$\|Fu_n - Fu\|_Y^p \rightarrow 0, \text{ as } n \rightarrow \infty.$$

Therefore F is continuous.

Step 3. We decompose F as $F = F_1 + F_2$ where F_1 and F_2 defined above.

(1) F_1 is a contraction on K . Let $u, v \in K$. It follows from Lemma 2 that

$$\begin{aligned} E \|F_1u - F_1v\|_{H^\nu}^p & \leq L_h E \|u(s) - v(s)\|_{H^\nu}^p \\ & \leq L_h \sup_{s \in [0, T]} E \|u(s) - v(s)\|_{H^\nu}^p ds \\ & \leq L_h \|u(s) - v(s)\|_Y^p. \end{aligned}$$

Taking supremum over t

$$\|F_1u - F_1v\|_Y^p \leq L_0 \|u(s) - v(s)\|_Y^p,$$

where $L_0 = L_h < 1$.

Hence F_1 is a contraction on K .

(2) F_2 is compact operator. Let $u, v \in K$. It follows from (H_2) , (H_5) and Lemma 2 that

$$\begin{aligned} & E \|F_2u - F_2v\|_{H^\nu}^2 \\ & \leq 2^{p-1} E \left\| \int_0^t (t-s)^{\alpha-1} E_{\alpha, \alpha}(t-s) A_\nu \right. \\ & \quad [g(u(s)) - g(v(s))] dW(s) \left. \right\|_{H^\nu}^2 \\ & \quad + 2^{p-1} E \left\| \int_0^t (t-s)^{\alpha-1} E_{\alpha, \alpha}(t-s) A_\nu \right. \\ & \quad \times [B(u(s)) - B(v(s))] ds \left. \right\|_{H^\nu}^p \\ & \leq (\gamma_1 + \gamma_2) E \left(\int_0^t \|u - v\|_{H^\nu}^2 ds \right), \end{aligned}$$

which implies

$$\sup_{t \in [0, T]} E \|F_2u - F_2v\|_{H^\nu}^2 = (\gamma_1 + \gamma_2) \sup_{t \in [0, T]} E \|u - v\|_{H^\nu}^2.$$

Since $0 < L = \gamma_1 + \gamma_2 < 1$, then F is contraction mapping on K .

From Lemma 16 and Lemma 17, the operator F_2 is relatively compact. together with Ascoli's theorem, we conclude that the operator F_2 is compact.

In view of Theorem 10, we conclude that F has at least one fixed point, which is a mild solution of the problem (1)-(2). \square

4 Conclusion

In this paper, we study a class of a stochastic partial differential equation with noise. By using stochastic analysis theory, fractional calculations, operator semi-group theory and applying Sadovskii's fixed point theorem, we obtain the existence of mild solutions for these equation. In the future, We develop a stochastic calculus for the fractional Brownian motion with Hurst parameter in different cases using the techniques of the Malliavin calculus.

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