LMI-based Method to Estimate the Domain of Attraction for Nonlinear Cascaded Systems with Delay

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Abstract: This paper deals with stability analysis for nonlinear systems with time delay. The proposed approach is based on the assumption that on a subset of the state space the system is represented by a continuous-time Takagi–Sugeno system with delay and cascaded structure. The first aim is to present linear matrix inequality conditions to assess non-local stability properties of the system. The second relevant contribution is to present linear matrix inequalities that allow to find an inner estimate of the domain of attraction for the system subject to constraints defining the subset under consideration. The proposed approach is based on common quadratic Lyapunov functions and the Razumikhin technique.

Key–Words: nonlinear delay system, Takagi–Sugeno system, system with constraints, LMI, domain of attraction

This paper aims at establishing a framework to estimate the domain of attraction (DA) of the equilibrium point (the origin) for nonlinear cascaded systems with time delay. It is assumed that there is a representation of the system by a continuous-time Takagi–Sugeno (TS) system with delay and cascade structure on a subset of the state space including the origin (see e.g. [1] and references therein). The method is beneficial in that stability conditions can be determined by solving some generalized eigenvalue minimization problem (GEVP) or a system of linear matrix inequalities (LMIs) which can be efficiently handled by means of convex optimization techniques [2]. But another important issue in stability analysis is how to estimate the DA. As the stability problems, such estimates can be obtained based on the Lyapunov function. Specifically, for a Lyapunov function which guarantees the local stability of the equilibrium, any sublevel set of the Lyapunov function is an inner estimate of the DA if the set belongs to the region where the function is positive definite and its derivative with respect to the system is negative definite. If we use a quadratic function for TS system, the above conditions are derived in terms of LMIs or GEVP. Moreover, if the LMIs are feasible, the resulted quadratic function seems to be a global Lyapunov function for the system. But asymptotic stability conditions in this case are valid only within the set where the convex sum property of TS systems holds. Therefore, when dealing with TS systems, what makes the problem more challenging is that additional constraints should be considered, because the system can be usually represented (exactly or approximately) in the TS form only on some subset of the state space including the origin ("modeling region"). Also, the system thus modeled may have physical constraints precisely reflected in the states belonging to some modeling region. So the problem arises to obtain the largest possible Lyapunov-based estimate of the DA of a nonlinear system with the asymptotically stable origin and subject to the given constraints. In other words, we need to find invariant subsets of the DA that fit into the modeling region.

In addition, the non-local stability of cascaded system requires a further research. The point is local asymptotic stability for each subsystem without interconnections implies the same for the whole cascade. But global asymptotic stability of the cascaded system does not necessarily follow from the same property of the subsystems. The known result is that global asymptotic stability of the subsystems implies the same for the whole system under the additional assumption that all solutions are uniformly bounded (for autonomous ordinary differential equations this assertion can be found e.g. in [3], for a non-autonomous nonlinear delay differential equations see e.g. [4]). The usual approach to prove local asymptotic stability for cascade is to find a Lyapunov function for each subsystem. But such a method does not give a common Lyapunov function with invariant sublevel sets.

So for the system with constraints the problem is to find a subset of the state space which both contains all the solutions starting in a neighborhood of the origin and is contained in the modeling region.

First, we obtain estimates for solutions of cascaded TS systems with delay by extending results from [4, 5]. Then, motivated by recent works (see, e.g., [7, 8]), GEVP is presented to find an inner estimate of the DA for some kinds of constraints which are turned into LMIs. Notice that both GEVPs and LMIs can be efficiently handled via available software, e.g. MatLab.

Taking into account time delay, we use the method of Lyapunov functions with additional restrictions, namely, the Razumikhin conditions [9]. We employ the sublevel set of a quadratic Lyapunov function as an inner estimate of the so called direct DA that does not depend on the delay [10].

Notations used throughout the paper is fairly standard. Time delay is denoted by r ($r > 0$), $R^+ = [0, +\infty)$, R^n denotes the n-dimensional space of vectors $x = (x_1, \ldots, x_n)^\top$ with the norm $|x| = \sqrt{\sum_{i=1}^n x_i^2}$, $C = C([-r, 0], R^n)$ is the Banach space with the supremum-norm $\|\cdot\|$. For a continuous function $x(t) \in C([\alpha - r, \alpha + \beta), R^n)$ ($\alpha \in R^+$, $\beta > 0$) an element $x_t \in C$ is defined for any $t \in [\alpha, \alpha + \beta)$ by $x_t(s) = x(t + s)$, $-r \le s \le 0$, $\dot{x}(t)$ stands for the right-hand derivative.

Then, $M < 0$ ($M \le 0$) means that M is a real symmetric and negative definite (semidefinite) matrix. The symbol $*$ within a matrix represents the symmetric of the matrix.

We will study systems dynamics of which are given by the following equation:

$$\dot{x}(t) = X(t, x_t), \quad X(t, 0) \equiv 0, \tag{1}$$

where $X: R^+ \times C \to R^n$, $n \ge 2$. The function $x(t)$ is said to be a solution of (1) if $x(t) \in C([\alpha - r, \alpha + \beta], R^n)$ for certain $\alpha \in R^+$, $\beta > 0$ and $x(t)$ is an absolutely continuous function on $[\alpha, \alpha+\beta]$ satisfying (1) almost everywhere on $[\alpha, \alpha + \beta]$. The solution of (1) satisfying the given initial condition $x_\alpha = \varphi$ is denoted by $x(t; \alpha, \varphi)$.

In the sequel, we assume that the system is considered on a set $D \subseteq R^n$ and the origin is an internal point of this set (the specification of such a set can be stipulated, e.g., by the physical meaning of state variables). Also, the right-hand side of (1) is assumed to ensure the existence, uniqueness, and continuability of solution for any initial function $\varphi \in C([-r, 0], D)$. In particular, if $\varphi = 0$ then $X(t, 0) \equiv 0$ implies $x(t; \alpha, 0) \equiv 0$ for any $\alpha \in R^+$.

Let $\xi(t)$ be a piecewise continuous vector function whose values at the current time t depend on t and on x_t and belong to the set $D_\xi \subset R^s$ whenever (t, x_t): $t \in R^+$, $x_t(s) \in D$ for all $s \in [-r, 0]$. For example, $\xi(t)$ can be a vector $(x^\top(t), x^\top(t - r))^\top$, then $D_\xi = D \times D$, $s = 2n$.

On the set D_ξ we define continuous functions μ^k, $k = 1, \ldots, p$ such that

$$\mu^k(\xi) \in [0, 1], \ \sum_{k=1}^p \mu^k(\xi) = 1 \text{ for all } \xi \in D_\xi. \tag{2}$$

Suppose that on the set D equation (1) is represented in the form of cascaded TS system (see e. g. [1] about ways of representation by TS systems):

$$\dot{z}(t) = \sum_{k=1}^p \mu_1^k(\xi(t))(A_1^k z(t) + A_{1\tau}^k z(t - \tau_1(t)) + g_k(z_t, y_t)), \tag{3}$$

$$\dot{y}(t) = \sum_{k=1}^p \mu_2^k(\xi(t))(A_2^k y(t) + A_{2\tau}^k y(t - \tau_2(t))). \tag{4}$$

Here $x^\top = (z^\top, y^\top)$, $z \in R^{n_1}$, $y \in R^{n_2}$, $n_1 + n_2 = n$, $x \in D \Leftrightarrow z \in D_z \subset R^{n_1}$, $y \in D_y \subset R^{n_2}$; $z_t \in C(z) := C([-r, 0], D_z)$ and $y_t \in C(y) := C([-r, 0], D_y)$ (norms in these spaces we endow by subscripts z and y). We also assume that g_k are continuous on $C(z) \times C(y)$, $g_k(z_t, 0) = 0$, $\tau_i(t) : R^+ \to [0, r]$ are piecewise continuous functions, μ_i^k meet (2), A_i^k, $A_{i\tau}^k$ are constant matrices of proper dimension ($k = 1, \ldots, p$, $i = 1, 2$).

The following statement can be proved using asymptotic stability theorem with Razumikhin conditions [11, 9] and follows from results of [5] and [5].

Theorem 1 *Assume that, for some positive numbers a_1, a_2, b_1, and b_2, and symmetric positive definite matrices $Q_1 \in R^{n_1 \times n_1}$, $Q_2 \in R^{n_2 \times n_2}$ the LMIs*

a) $\begin{pmatrix} -a_i Q_i & A_i^k Q_i \\ * & -Q_i \end{pmatrix} \le 0$, $\begin{pmatrix} -b_i Q_i & A_i^k Q_i \\ * & -Q_i \end{pmatrix} \le 0$,

b) $\begin{pmatrix} \frac{1}{\tau}\Phi_{ki} + (a_i + b_i) Q_i & A_{i\tau}^k Q_i \\ * & -\frac{1}{2} Q_i \end{pmatrix} < 0$,

hold for $k = 1, \ldots, p$, $i = 1, 2$ with $\Phi_{ki} = Q_i(A_i^k + A_{i\tau}^k)^\top + (A_i^k + A_{i\tau}^k) Q_i$. Then the zero solution of (3), (4) is uniformly asymptotically stable.

It should be noted that under the conditions of Theorem 1 the function $V_1(z) = z^\top P_1 z$ with $P_1 = Q_1^{-1}$ is the Lyapunov function for (3) with $y \equiv 0$, and the derivative of this function under the Razumikhin conditions meets the inequality $V' \le -c_1 V$; the function $V_2(y) = y^\top P_2 y$ with $P_2 = Q_2^{-1}$ is the Lyapunov function for (4), and the solutions of the system meet the inequality $|y(t; \alpha_0, \varphi_{20})| \le \sqrt{\frac{\lambda_{\max}(P_2)}{\lambda_{\min}(P_2)}} \|\varphi_{20}\|_y$.

Since LMIs in Theorem 1 depend on the parameters, using of LMI solver can be troublesome. A possible order of actions is as follows.

1. to find positive definite solutions Q_1, Q_2 of the following GEVPs:

$$\min_{Q_i} \alpha_i \text{ subject to}$$

$$\Phi_{ki} < \alpha_i Q_i,\ k = 1, \ldots, p\ (i = 1, 2);$$

2. to find the minimal positive a_i, b_i meeting ineqalities a) in Theorem 1 ($i = 1, 2$);

3. to find the maximal r such that LMIs b) in Theorem 1 are feasible.

So Theorem 1 gives the sufficient stability conditions whose solutions can be efficiently obtained by solving GEVP.

Now let us discuss the problem of constructing the attraction domain for system (3), (4). The results on this topic obtained through the Lyapunov direct method and other techniques are numerous (see e.g. [12] and references therein).

For system (3), (4) the problem has some special feature. Namely, outside of the set D the conditions of Theorem 1 do not ensure asymptotic stability of the zero solution. Moreover, the set D can have the sense of the domain of safe system operation. So, the problem is to construct a set $A \subset D$ such that the solutions beginning in the set should not leave the set D and must tend to equilibrium.

Within the framework of the direct Lyapunov method, the attraction domain is estimated by the set bounded by the level surfaces of the Lyapunov functions. Here we use quadratic Lyapunov functions. Thus, we want to find an estimate $A_0 \subset A$ that defined in the form $A_0 = B(P, c) = \{x \in D :\ x^\top P x \le c\}$ with some $c > 0$ and a positive definite matrix P.

Remember that global asymptotic stability of the cascaded system necessarily follow from the same property of the subsystems only under the additional assumption that all solutions of the whole system are uniformly bounded. Conditions guaranteeing this property are usually given in the form of some requirement for the interconnection term. In our case this term is discribed by functionals g_k.

Assume that for every $k = 1, \ldots, p$ the following estimate takes place: $|g_k(\varphi_1, \varphi_2)| \le C_g \|\varphi_2\|_y$ for all $t \in R^+$, $\varphi_1 \in C(z)$, and $\varphi_2 \in C(y)$. The simplest and natural (in the context of TS systems) example is the functionals $g_k(y_t) = B^k y(t) + B_\tau^k y(t - \tau_3(t))$.

Let $V_1(z) = z^\top P_1 z$, where $P_1 = Q_1^{-1}$, Q_1 is the matrix meeting conditions of Theorem 1. Then using the estimate for the derivative of the function with respect to (3) with $y \equiv 0$ and the estimate for g_k, we have that along a solution of (3), (4) we have $V' < 0$ (under the Razumikhin condition), if the initial point satisfies $\|\varphi_{20}\|_y \le \Delta$, the solution remains in D and $V > m = M\Delta^2$ with $M = \frac{(2\lambda_{\max}(P_1) C_g K)^2}{\lambda_{\min}(P_1)}$, $K = \sqrt{\frac{\lambda_{\max}(P_2)}{\lambda_{\min}(P_2)}}$.

Thus, the solutions of (3), (4) begining in the set $A_0(\Delta) = \{(z, y) :\ z^\top P_1 z \le M\Delta^2,\ |y| \le \Delta\}$ do not leave the set $A(\Delta) = \{(z, y) :\ z^\top P_1 z \le M\Delta^2,\ |y| \le K\Delta\}$ whenever $A(\Delta) \subset D$. The latter condition can be always satisfied for a sufficiently small Δ.

Assume now that the sets D_z and D_y are of the form

$$\begin{aligned} &D_z = \{z \in R^{n_1} : |z_i| \le d_i,\ i \in I \subset \{1, \ldots, n_1\}\}, \\ &D_y = \{y \in R^{n_2} : |y_j| \le c_j,\ j \in J \subset \{1, \ldots, n_2\}\}. \end{aligned} \quad (5)$$

Based on the assumption and definitions and following the concept of [13], we can translate the condition $A(\Delta) \subset D$ into LMIs. Namely, the following result is valid.

Theorem 2 *If there exist symmetric matrices $Q_1 > 0$, $Q_2 > 0$ such that the following system of LMIs has a solution:*

1. *LMIs in Theorem 1;*

2. $(\sum_{i \in I} \frac{1}{d_i^2} E_{ii}) Q_1 < E$*, where E is identity matrix, E_{ii} has 1 in the (i, i) position and zeros in all other positions,*

Then the zero solution of (3), (4) is uniformly asymptotically stable and for $P_1 = Q_1^{-1}$ we have $B(P_1, 1) \subset D_z$.

From Theorem (2) it follows that the inequality $\Delta \le \min\{\min_{j \in J} c_j / K, 1/\sqrt{M}\}$ implies $A(\Delta) \subset D$ and therefore the set $A_0(\Delta)$ is an inner estimate of the DA for system (3), (4) under constraints (5).

It is clear that the set $A_0(\Delta)$ can be generally enlarged as close as possible to that of $A_0(\Delta^*)$, where $\Delta^* = \max\{\Delta :\ A(\Delta) \subset D\}$. The number Δ^* can be numerically estimated by using the fact that the latter equality can be equivalently rewritten as $\Delta^* = \min\{\min_{j \in J} c_j / K, \sqrt{v_1 / M}\}$, where $v_1 = \min\{z^\top P_1 z :\ z \in \partial D_z\}$. Another way to find Δ^* is to solve the following problem: for given P_1 find the minimal $\beta \in (0, 1]$ such that $\sum_{i \in I} \frac{1}{d_i^2} E_{ii} < \beta P_1$. Then $B(P_1, 1/\beta) = B(\beta P_1, 1) \subset D_z$ and $\max\{c :\ B(P_1, c) \subset D_z\} = 1/\beta = v_1$. As the result, we obtain the set $A_0(\Delta^*)$ which is the maximal for the function $V_1(z) = z^\top P_1 z$ inner estimate of the DA.

Finally, notice that using the matrix Q_2 and the function $V_2(y) = y^\top P_2 y$ we can also construct an invariant inner estimate of the DA of the form $A_1(\delta) = \{(z, y) : \ z^\top P_1 z \leq M\delta^2, \ y^\top P_2 y \leq \lambda_{\min}(P_2)\delta^2\}$, where $\delta = \min\{\sqrt{v_1/M}, \sqrt{v_2/\lambda_{\min}(P_2)}\}$, $v_2 = \min\{y^\top P_2 y : y \in \partial D_y\}$.

In this paper the simplest case of common quadratic Lyapunov function for TS systems is considered. Similar ideas can be used for other techniques of stability analysis, such as fuzzy Lyapunov function or piecewise Lyapunov function approach (some recent results for ordinary differential and difference equations see e.g. [7, 8]). Also, the proposed approach can be naturally extended to more general conditions for functionals g_k.

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