with : $K_i = N_i E + L_i$.

If the following conditions are satisfied:

$$\begin{cases} P D_r = 0 \\ P = I+E C_a \\ N_i = P A_{ai} - K_i C_a \\ L_i = K_i - N_i E \\ G_{i1} = P B_{ai} \\ G_{i2} = P D_i \\ \sum_{i=1}^{M} \mu_i(\xi(t)) N_i e(t) \text{ stable} \end{cases} \quad (45)$$

The equation (43) is reduced to:

$$\dot{e}_a(t) = \sum_{i=1}^{M} \mu_i(\xi(t)) N_i e(t) \quad (46)$$

Thus, the constraints (45) allow to synthesise the multiple observer of a system with unknown inputs and outputs.

6.2 Global convergence of the multiple observer

In this section, we will develop the sufficient conditions of the asymptotic global convergence of the state reconstruction.

(46) is globally asymptotically stable if there exists a positive definite symmetric matrix X, such that [7]:

$$N_i^T X + X N_i < 0 \quad (47)$$

The design of the observer carries out to extract the following theorem:

6.2.1 Theorem:

The state estimation error converges towards zero, if all the pairs (A_{ai}, C_a) are observables and if the following conditions hold $\forall i \in \{1, ..., M\}$*:*

$$\begin{cases} N_i^T X + X N_i < 0 & (48a) \\ N_i = P A_{ai} - K_i C_a & (48b) \\ P = I + E C_a & (48c) \\ P D_r = 0 & (48d) \\ L_i = K_i - N_i E & (48e) \\ G_{i1} = P B_{ai} & (48f) \\ G_{i2} = P D_i & (48g) \end{cases} \quad (48)$$

where $X \in R^{n*n}$ *is a positive definite symmetric matrix.*

Using (48b), the expression (48a) can be written as:

$$(P A_{ai} - K_i C_a)^T X + X (P A_{ai} - K_i C_a) < 0, \quad \forall i \in \{1, ..., M\} \quad (49)$$

It is noted that the inequalities (49) are bilinear compared to variables X and Ki. To be reduced to the case of a linear problem, changes of variables are used.

6.3 Method of resolution

In order to solve the system (45), three steps are needed:

1. The matrix E is given, using the expression (48d), as:

$$E = - D_r (C_a D_r)^{(-)} \quad (50)$$

where $(C_a D_r)^{(-)}$ is the pseudo-inverse of $(C_a D_r)$ and the matrix P is deduced from (48c):

$$P = I - D_r (C_a D_r)^{(-)} C_a \quad (51)$$

2. By the variable change

$$W_i = X K_i \quad (52)$$

The inequalities (49) are written as:

$$(P A_{ai})^T X + X (P A_{ai}) - C_a^T W_i^T - W_i C_a < 0, \quad \forall i \in \{1, ..., M\} \quad (53)$$

The inequalities (53) are of LMI type and LMIMatlab Toolbox can be used for that resolution. The controller is

$$K_i = X^{-1} W_i \quad (54)$$

3. The other matrices defining the observer are deduced knowing E, P and K_i:

$$\begin{cases} N_i = P A_{ai} - K_i C_a & (55a) \\ L_i = K_i - N_i E & (55b) \\ G_{i1} = P B_{ai} & (55c) \end{cases} \quad (55)$$

7 Numerical example

Consider the Takagi-Sugeno multiple model with two local models, two states and two outputs.

$$\begin{cases} \dot{x}(t)= \sum_{i=1}^{2} \mu_i(\xi(t)) \Big[A_i\ x(t) + \\ \qquad + B_i\ u(t) + R\ v(t)\Big] \\ y(t)= C\ x(t) + D\ \bar{u}(t) \end{cases} \tag{56}$$

The numerical values of all these matrices are:

$$A_1 = \begin{bmatrix} -0{,}6 & -2 \\ 0{,}5 & -0{,}2 \end{bmatrix},\ A_2 = \begin{bmatrix} -0{,}7 & -0{,}3 \\ 2 & -0{,}3 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0{,}1 \\ 0{,}2 \end{bmatrix}, B_2 = \begin{bmatrix} 0{,}3 \\ 0{,}4 \end{bmatrix},\ R = \begin{bmatrix} 0{,}2 \\ 0{,}5 \end{bmatrix},$$

$$D = \begin{bmatrix} 0{,}15 \\ 0{,}35 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

The vector of decision is depending on the system input. The weighting functions μ_i are Gaussian and given in figure (1).

The new state z(t) satisfies the following equation:

$$\dot{z}(t) = \sum_{i=1}^{2} \mu_i(u(t)) \left[-\bar{A}\ z(t) + \bar{A}\ C\ x(t) + \bar{A}\ D\ \bar{u}(t)\right] \tag{57}$$

where $\bar{A} = 20 * I$.

The augmented system has the following expression:

$$\begin{cases} \dot{X}(t)= \sum_{i=1}^{2} \mu_i(u(t)) \Big[A_{ai}\ X(t) + \\ \qquad + B_{ai}\ u(t) + D\ \bar{u}(t)\Big] \\ Y(t) = C_a\ X(t) \end{cases} \tag{58}$$

with:

$$A_{a1} = \begin{bmatrix} -0{,}6 & -2 & 0 & 0 \\ 0{,}5 & -0{,}2 & 0 & 0 \\ 20 & 20 & -20 & 0 \\ 0 & 20 & 0 & -20 \end{bmatrix},\ B_{a1} = \begin{bmatrix} 0{,}1 \\ 0{,}2 \\ 0 \\ 0 \end{bmatrix},$$

$$A_{a2} = \begin{bmatrix} -0{,}7 & -0{,}3 & 0 & 0 \\ 2 & -0{,}3 & 0 & 0 \\ 20 & 20 & -20 & 0 \\ 0 & 20 & 0 & -20 \end{bmatrix},\ B_{a2} = \begin{bmatrix} 0{,}3 \\ 0{,}4 \\ 0 \\ 0 \end{bmatrix},$$

$$D_a = \begin{bmatrix} 0{,}2 & 0{,}5 & 0 & 0 \\ 0 & 0 & 3 & 7 \end{bmatrix}^T \text{ and } C_a = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

The multiple observer able to estimate the state of the multiple model (56) is as follows :

$$\begin{cases} \dot{X}(t)= \sum_{i=1}^{2} \mu_i(u(t)) \Big[N_i\ Z(t) + \\ \qquad + G_{i1}\ u(t) + L_i\ Y(t)\Big] \\ Y(t) = C_a\ X(t) - EY(t) \end{cases} \tag{59}$$

The computation of the matrices of the multiple observer (59) gives:

$$N_1 = \begin{bmatrix} -2{,}5 & 3 & 0 & 0 \\ -3{,}2 & -2{,}5 & 0 & 0 \\ 0 & 0{,}0002 & -20 & 0 \\ 0 & 0{,}0001 & 0 & -20 \end{bmatrix},\ G_{11} = \begin{bmatrix} -0{,}06 \\ -0{,}20 \\ 0 \\ 0 \end{bmatrix},$$

$$N_2 = \begin{bmatrix} -2{,}5 & 4{,}8 & 0 & 0 \\ -4{,}8 & -2{,}5 & 0 & 0 \\ 20 & 0{,}0002 & -20 & 0 \\ 0 & 0{,}0001 & 0 & -20 \end{bmatrix}, G_{21} = \begin{bmatrix} -0{,}02 \\ -0{,}40 \\ 0 \\ 0 \end{bmatrix},$$

$$L_1 = \begin{bmatrix} 1{,}50 & -2{,}14 \\ 2{,}70 & -7{,}56 \\ 20 & 0{,}0002 \\ 0 & 20{,}0001 \end{bmatrix}, L_2 = \begin{bmatrix} 0{,}20 & 2{,}54 \\ 2{,}80 & -8{,}84 \\ 20 & 0{,}0002 \\ 0 & 20{,}0001 \end{bmatrix}$$

$$\text{and } E = \begin{bmatrix} 0 & -0{,}8 \\ 0 & -2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Figure (5) represents the known input and Figure (6) visualizes the unknown input. The sensor fault affecting the system is given by Figure (7).

The simulation results are represented Figure (8). The proposed method provides good estimates of the system state. Indeed, the convergence of the state vector of the multiple observer towards those of the Takagi-Sugeno multiple model is quite good.

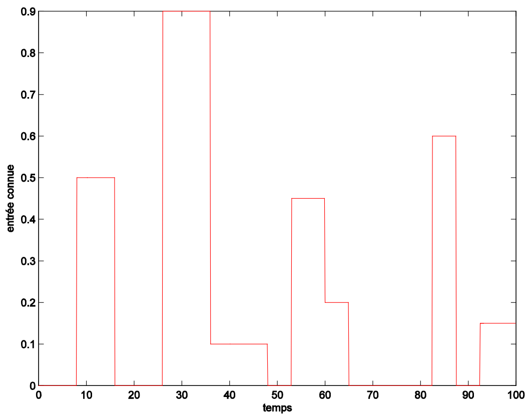

Figure 5: The known input u.

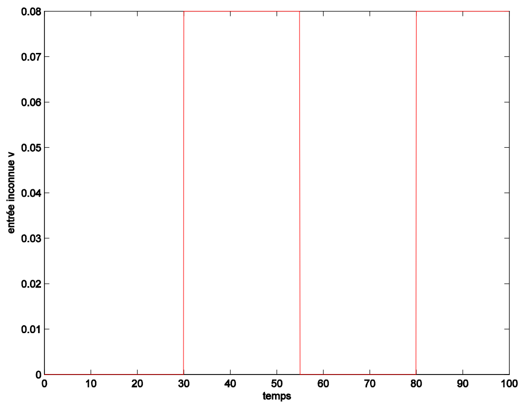

Figure 6: The unknown input.

8 Illustration example: Three column

The chosen system is a hydraulic process composed of three columns [1, 38]. The process is effected by an actuator fault v(t) and a sensor fault $\bar{u}(t)$. The three columns T_1, T_2 and T_3 have equal section A and connected to each others by identical section connectors S_n. The considered output is that of the column T_2; it ensures to empty the columns filled by pumps 1 and 2 with respective flows $Q_1(t)$ and $Q_2(t)$. Some combinaison of three levels are measured. The connexions between columns are assured by certain valves which are tuned manually to activate or not the corresponding pump. The three levels x_1, x_2 and x_3 satisfy $x_1 > x_2 > x_3$. The non linear model describing the process behaviour is given by [38]:

$$\begin{cases} A\,\dfrac{dx_1}{dt}(t) = -\alpha_1 S_n\left[2g\left(x_1(t)-x_3(t)\right)\right]^{1/2} \\ \qquad + Q_1(t) + Qf_1\,\bar{u}(t) \\ A\,\dfrac{dx_2}{dt}(t) = -\alpha_3 S_n\left[2g\left(x_3(t)-x_2(t)\right)\right]^{1/2} \\ -\alpha_2 S_n\left[2g\,x_2(t)\right]^{1/2} + Q_2(t) + Qf_2\,\bar{u}(t) \\ A\,\dfrac{dx_3}{dt}(t) = -\alpha_1 S_n\left[2g\left(x_1(t)-x_3(t)\right)\right]^{1/2} \\ -\alpha_3 S_n\left[2g\left(x_3(t)-x_2(t)\right)\right]^{1/2} + Qf_3\,\bar{u}(t) \end{cases} \tag{60}$$

where $\alpha_1 = \alpha_2 = 0.78$, $\alpha_3 = 0.75$, $g = 9.81\ ms^{-2}$, $S_n = 5\ 10^{-5}\ m^2$, $Qf_i = 10^{-4}$ and $A = 0.0154\ m^2$. $\bar{u}(t)$ is an unknown input. $Qf_i / f_i(t),\ i = 1, 2, 3$ are the additive massic flows in the columns.

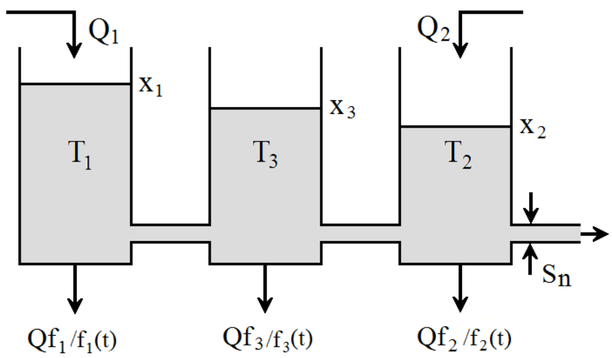

Figure 7. Three columns system.

The multi-model associated to the non linear system (60) is:

$$\dot{x}(t) = \sum_{i=1}^{4} \mu_i(\xi(t)) \left[A_i\, x(t) + B_i\, u(t) + R\, v(t) + d_i\right] \tag{61}$$

$$y(t) = C\,x(t) + D\,\bar{u}(t)$$

The matrices A_i, B_i and d_i are computed by linearizing model (60) around four set points chosen in system functioning zone. The numerical values of these matrices are:

$$A_1 = \begin{bmatrix} -0.0109 & 0 & 0.0109 \\ 0 & -0.0206 & 0.0106 \\ 0.0109 & 0.0106 & -0.0215 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -0.0110 & 0 & 0.0110 \\ 0 & -0.0205 & 0.0104 \\ 0.0110 & 0.0104 & -0.0215 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} -0.0084 & 0 & 0.0084 \\ 0 & -0.0206 & 0.0095 \\ 0.0084 & 0.0095 & -0.0180 \end{bmatrix},$$

$$A_4 = \begin{bmatrix} -0.0085 & 0 & 0.0085 \\ 0 & -0.0205 & 0.0095 \\ 0.0085 & 0.0095 & -0.0180 \end{bmatrix},$$

$$d_1 = 10^{-3} * \begin{bmatrix} -2.86 \\ -0.38 \\ 0.11 \end{bmatrix}, \ d_2 = 10^{-3} * \begin{bmatrix} -2.86 \\ -0.34 \\ 0.038 \end{bmatrix},$$

$$d_3 = 10^{-3} * \begin{bmatrix} -3.7 \\ -0.14 \\ 0.69 \end{bmatrix}, \ d_4 = 10^{-3} * \begin{bmatrix} -3.67 \\ -0.18 \\ 0.62 \end{bmatrix},$$

$$B_i = 1/A * \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

We choose $\bar{A}_1 = 30 * I$ where I is the 3-identity matrix.

The known input u(t) is defined by: $u(t) = \begin{bmatrix} u_1(t) & u_2(t) \end{bmatrix}^T$,

$$\begin{cases} u_1(t) = 0.5 \sin(0.15\,\pi t) \\ u_2(t) = 0.25 \sin(0.25\,\pi t) \end{cases}$$

The unknown input v(t) is defined by:

$$v(t) = \begin{cases} 0.01 * \sin(\pi t), & 0 < t \le 40s \\ 0, & t > 40s \end{cases}$$

The sensor fault $\bar{u}(t)$ is:

$$\bar{u}(t) = \begin{cases} 0, & t \le 10s \\ 0.1 * \sin(0.2\pi t), & 10s < t \le 45s \\ 0, & t > 45s \end{cases}$$

The multiple observer gains are:

$$L_1 = \begin{bmatrix} -19.5381 & 73.2584 & 51.0916 \\ 22.0748 & -93.8480 & -69.0445 \\ 31.7600 & -117.6084 & -88.2063 \\ 30.0183 & -0.0683 & -0.0509 \\ -0.0016 & 30.0058 & 0.0049 \\ 0.0004 & -0.0014 & 29.9990 \end{bmatrix},$$

$$L_2 = L_4 = \begin{bmatrix} -19.5180 & 73.1819 & 51.0341 \\ 22.0655 & -93.8111 & -69.0205 \\ 31.7663 & -117.6330 & 88.2247 \\ 30.0183 & -0.0683 & -0.0509 \\ -0.0016 & 30.0058 & 0.0049 \\ 0.0003 & -0.0014 & 29.9990 \end{bmatrix},$$

$$L_3 = \begin{bmatrix} -19.6094 & 73.5038 & 51.2952 \\ 22.1074 & -93.8275 & -68.9971 \\ 32.1424 & -119.0394 & -89.2798 \\ 30.0184 & -0.0684 & -0.0509 \\ -0.0016 & 30.0058 & 0.0049 \\ 0.0004 & -0.0014 & 29.9990 \end{bmatrix}.$$

In Figure 8 we plot the states and their estimations and Figure 9 draws the state estimation error. Figure 10 illustrates the sensor fault and figure 11 shows the evolution of state components (x_1 and x_2) for the Takagi-Sugeno multiple model described by relation (56) and the multiple observer given by relation (59). We note the perfect concordance between the state components and heir estimates by the multiple observer.

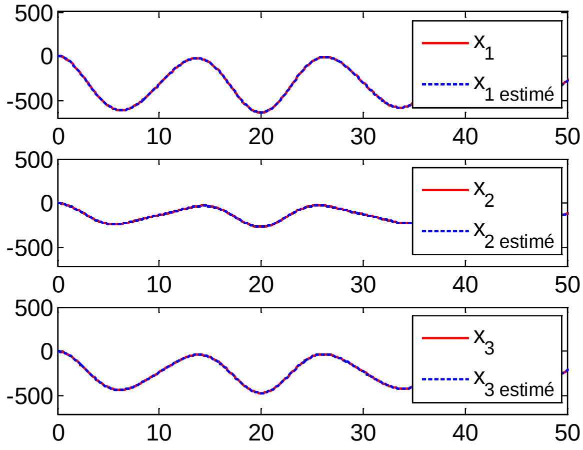

Figure 8: States and their estimations.

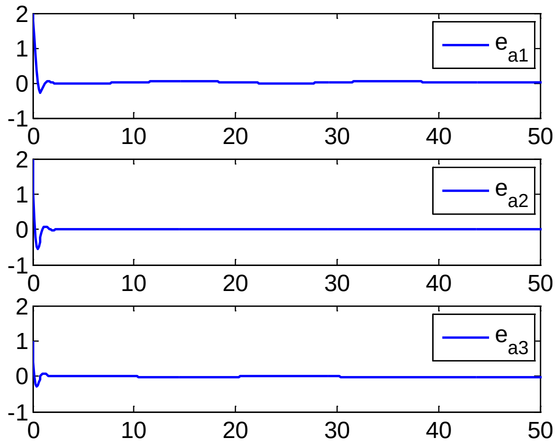

Figure 9: State estimation error.

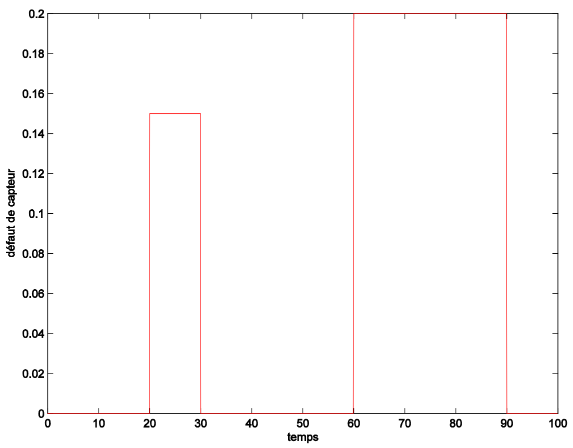

Figure 10: The sensor fault.

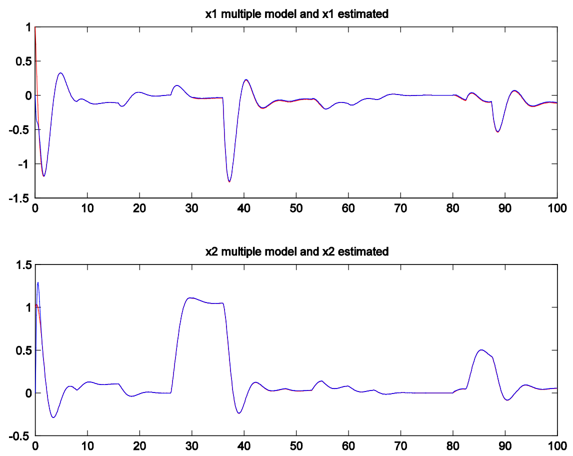

Figure 11: Multiple model (56) and multiple observer (59).

9 Conclusion

Using a multiple model representation, this paper has proposed new methods to design multiple observers for nonlinear systems submitted to unknown inputs and outputs. A mathematical transformation is used in order to formulate unknown outputs as unknown inputs. The proposed method is based on the principle of unknown input multiple observer which used the principle of the interpolation of local observers. The synthesis conditions of that observer are expressed in LMI terms. The simulation results show that we succeeds in making the estimation of state design the observer in spite of the existence of disturbances.

References:

[1] A. Akhenak, M. Chadli, J. Ragot, D. Maquin, *State estimation via Multiple observer with unknown input. Application to the three tank system*, 5th IFAC Symposium on Fault Detection Supervision and Safety for Technical Processes, Safe process, pp. 245-251, Washington, US, June 9-11, 2003.

[2] Akhenak A., M. Chadli, J. Ragot, D. Maquin, *Design of observers for Takagi-Sugeno fuzzy models for Fault Detection and Isolation*, 7th IFAC Symposium on Fault Detection, Supervision and Safety of Technical Processes, Safeprocess'2009, Barcelona, Spain, June 30th-July 3rd, 2009.

[3] A. Akhenak, M. Chadli, J. Ragot, D. Maquin, *Design of sliding mode unknown input observer for uncertain Takagi-Sugeno model*, 15th Mediterranean Conference on Control and

Automation, MED'07, Athens, Greece, June 27-29, 2007.
[4] Akhenak A., D. Maquin, J. Ragot, *Process input estimation with a multimodel. Application to communication*, Mediterranean Conference on Control and Automation, Limassol, Cyprus, June 27- 29, 2005.
[5] A. Akhenak, *Conception d'observateurs non linéaires par approche multimodèle: application au diagnostic*, Thèse de doctorat, Institut National Polytechnique de Lorraine, December 16, 2004.
[6] A. Akhenak, M. Chadli, J. Ragot, D. Maquin, *Estimation of state and unknown inputs of a nonlinear system represented by a multiple model*, 11th IFAC Symposium on Automation in Mining, Mineral and Metal processing, MMM 2004, Nancy, France, September 8-10, 2004.
[7] Ben Aïcha A., Guerfel M., Bouguila N., Ben Othman K., *New PCA-based methodology for sensor fault detection and localization*, 8th International Conference MOdeling and SIMulation (MOSIM'10), Hammamet, Tunisia, May 2010.
[8] Bouguila N., Khedher A., Ben Othman K., *Application of the upper bounding techniques for fault detection*, International Review of Automatic Control (I.RE.A.CO.), Vol. 5, n.2, pp. 292-297, March 2012.
[9] Boyd S., L. Elghaoui, E. Feron, V. Balakrishnan, *Linear matrix inequality in system and control theory*, Siam, 1994.
[10] Chadli M., D. Maquin, J. Ragot, *Multiple observers for discrete-time multiple models*, 5th IFAC Symposium on Fault Detection, Supervision and Safety for TechnicalProcesses, Safeprocess' 2003, Washington, D.C., USA, June 9-11, 2003.
[11] Darouach M., M. Zasadzinski, S.J. SXu, *Fullorder observers for linear systems with unknown inputs*, IEEE Transactions on Automatic Control, vol. 39, no. 3, pp. 606-609, 1994.
[12] Edwards C., *A comparison of sliding mode and unknown input observers for fault reconstruction*, IEEE Conference on Decision and Control, Atlantis, Paradise Island, Bahamas, pp. 5279-5284, December 14-17, 2004.
[13] Efendic H., Del Re L., *Automatic Iterative Fault Diagnosis Approach for Complex Systems*, WSEAS Transactions on Systems, Issue 2, Volume 5, February 2006.
[14] Fagarasan I., St. Lliescu S., Applications of Fault Detection Methods to Industrial Processes, WSEAS Transactions on Systems, Issue 6, Vol. 7, July 2008.
[15] Gao Z., Ho D.W.C. , *Proportional multiple-integral observer design for descriptor systems with measurement output disturbances*, IEE Proceedings - Control Theory and Applications, Volume 151, issue3, May 2004, p. 279 – 288.
[16] Guan Y., M. Saif, *A novel approach to the design of unknown input observe***rs**, IEEE Transactions on Automatic Control, vol. 36, no. 5, pp. 632- 635, 1991.
[17] Ichalal D., B. Marx, J. Ragot, D. Maquin, *Simultaneous state and unknown inputs estimation with PI and PMI observers for Takagi Sugeno model with unmeasurable premise variables*, 17th Mediterranean Conference on Control and Automation, MED'09, Thessaloniki, Greece, June 24-26, 2009.
[18] Jamel W., Bouguila N., Khedher A., Ben Othman K., *Observer design for nonlinear systems represented by Takagi-Sugeno models*, WSEAS Transactions on Systems, Issue 7, Vol. 9, pp. 804-813, July 2010.
[19] Jamel W., Khedher A., Bouguila N., Ben Othman K., *Design of Multiple Observers for a Class of Uncertain Takagi-Sugeno Multiple Models*, The 6th International Conference on Electrical Systems and Automatic Control, March 26-28, 2010, Hammamet, Tunisia.
[20] Jamel W., Bouguila N., Khedher A., Ben Othman K., *State estimation for uncertain Takagi-Sugeno multiple model*, 6th WSEAS International Conference on dynamical systems and control (CONTROL'10), pp. 74-79, Sousse, May 3 - 6, 2010.
[21]Jamel W., Khedher A., Bouguila N., Ben Othman K., *State Estimation via Observer with Unknown Inputs: Application to a Particular Class of Uncertain Takagi-Sugeno Systems*, Studies in Informatics and Control (S.I.C.), Vol.19, No.3, pp. 219-228, September 2010.
[22] Johansen T.A., A.B. Foss, *Constructing NARMAX using ARMAX models*, International Journal of Contol, vol. 58, no. 5, pp. 1125 1153, 1993.
[23] Johansen T.A., A.B. Foss, *Nonlinear local model representation for adaptive systems*, Singapore International Conference on Intelligent Control and Instrumentation, Singapore, February 17-21, 1992.

[24] Johansen T.A., R. Shorten, R. Murray-Smith, *Fuzzy identification of systems and its application to modelling and control*, IEEE Transactions on Systems Man and Cybernetics-part C, 15, pp. 116- 132, 1985.

[25] Khedher A., K. Benothman, D. Maquin, M. Benrejeb, *State and sensor faults estimation via a proportional integral observer*, 6th international multi-conference on Systems signals and devices, SSD'09, Djerba, Tunisia, March 23-26, 2009.

[26] Khedher A., Jamel W., Bouguila N., Ben Othman K., *Sensors faults detection and localization by generalized observers schemes*, The 6th International Conference on Electrical Systems and Automatic Control, March 26-28, 2010, Hammamet, Tunisia.

[27] Khedher A., K. Benothman, D. Maquin, M. Benrejeb, *Design of an Adaptive Tolerant: case of sensor Faults*, WSEAS Transactions on Systems, Issue 7, Vol. 9, pp. 804-813, July 2010.

[28] Luenberger D.G., *An introduction to observers*, IEEE Transactions on Automatic Control, vol. 16, no. 6, pp. 596-602, 1971.

[29] Murray-Smith R., T.A. Johansen, *Multiple model approaches to modelling and control*, Taylor and Francis, London, 1997.

[30] Orjuela R., B. Marx, J. Ragot, D. Maquin, *Observer design for nonlinear systems described by multiple models*, 7th IFAC Symposium on Fault Detection, Supervision and Safety of Technical Processes, Safeprocess'2009, Barcelona, Spain, June 30th-July 3rd, 2009.

[31] Orjuela R., *Contribution à l'estimation d'état et au diagnostic des systèmes représentés par des multimodèles*, Thèse de doctorat, Institut National Polytechnique de Lorraine, 2008.

[32] Park T.-G., Ryu J.-S., Lee K.-S, *Actuator fault estimation with disturbance decoupling*, IEE Proceedings - Control Theory and Applications, Volume 147, issue5, September 2000, p. 501 – 508.

[33] Sename O., *Unknown input robust observer for time delay system*, IEEE Conference on Decision and Control, 2, pp. 1629-1630, 1997.

[34] Sharma R., M. Aldeen, *Estimation of unknown disturbances in nonlinear systems*, Control 2004, University of Bath, UK, September 6-9, 2004.

[35] Tanaka K., T. Ikeda, H. Wang, *Fuzzy regulators and fuzzy observers: Relaxed stability conditions and LMI-based designs*, IEEE Transactions on Fuzzy Systems, 6(2), pp. 250-265, 1998.

[36] Wang S.H., E.J. Davison, P. Dorato, *Observing the states of systems with unmeasurable disturbances*, IEEE Transactions on Automatic Control, 20, pp. 716-717, 1995.

[37] ZHANG K., B. JIANG, V. COCQUEMPOT, *Adaptive Observer-based Fast Fault Estimation*, Internatinal Journal of Control, Automation, and Systems, vol. 6, no. 3, pp. 320-326, June 2008.

[38] Zolghadri A, Henry D, Monsion M., *Design of nonlinear odservers for fault diagnosis: a case study*, Control Engineering Practice, vol 4 (11), pp. 1535-1544, 1996.