Adaptive Decompositions of General Flows and Their Applications

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Abstract: Adaptive algorithms of spline-wavelet decomposition in a linear space over metrized fields are proposed. The algorithms provide a priori given estimate of the deviation of the main flow from the initial one. Comparative estimates of data of the main flow under different characteristics of the irregularity of the initial flow are done. The limiting characteristics of data, when the initial flow is generated by abstract differentiable functions, are discussed. The constructions of adaptive grid and pseudo-equidistant grid and relative quantity of their knots are considered, flows of elements of linear normed spaces and formulas of decomposition and reconstruction are discussed. Wavelet decomposition of the flows is obtained with using of spline-wavelet decomposition. Sufficient condition of the construction is obtained. Applications to different spaces of matrix of fixed order and to spaces of infinite-dimension vectors with numerical elements (rational, real, complex and p-adic elements) are considered.

Key–Words: signal processing, matrix flows, adaptive spline-wavelets, general flows, p-adic flows

1 Introduction

Many studies have been devoted to the investigation of numerical flows (signals). There is the theory of filtration, the theory of classical wavelets, the theory of spline-wavelets (see, for example, monographs [1] – [3] and bibliography there). There exist many implementations of wavelets in different practical investigations (for instance, see [4] – [6], [27] – [29]).

For classical wavelet decomposition (see [2] - [18]) the translation invariance of the spaces, the multiple-scale analysis, and Fourier transformer are required; that creates great difficulties for the construction of adaptive algorithms for processing numerical flows. Adaptive spline-wavelet expansions use approximate relations for constructing nested spline spaces on non-uniform grids (see [24] – [26]).

In papers [24] - [25], algorithms of adaptive spline-wavelet decomposition for numerical flows are proposed. The construction of spline-wavelet decompositions of flow of a more general nature than real numerical flow (i.e. flow of elements of linear normed space, flow of matrices or flow of p-adic numbers), encounters difficulties in implementing relevant generalizations of splines. We overcome these difficulties by a special construction: according to properties of spline-wavelet decomposition (see [24]) the construction of the main flow reduces to the trace operation over initial flow on the enlargement of the initial grid. Thus, for obtaining the adaptive main flow of spline-wavelet decomposition it is sufficient to construct adaptive approximation of the initial flow.

Aim of this paper to propose algorithms for the construction of the main flow in adaptive spline-wavelet decomposition for flows of the elements of a linear normed space. Under condition of the same approximation we consider the ratio of the volume of the main flow mentioned above to the volume of the main flow obtained with a pseudo-equidistance grid. The limit characteristics are discussed in the case of the flow generated by differentiable function.

In the paper we consider construction of adaptive grid and pseudo-equidistant grid, relative quantity of knots, flows of elements of linear normed spaces, approximations of these flows connected with different grids, embedded spaces, calibration relations and formulas of decomposition and reconstruction. Wavelet decomposition of the flows is obtained with using of spline-wavelet decomposition. Sufficient condition of the construction is linear independence over certain space (note the condition are right if the normed space is $\mathbb{R}^1$). Applications are discussed at the end of the paper: obtained results are applied to different spaces of matrix of fixed order with numerical elements (rational, real, complex and p-adic elements). The results are also applied to spaces of infinite-dimension vectors.

2 Auxiliary assertions

Here we introduce some notation used in the following.

2.1 Construction of adaptive grid

Let (α,β) be an interval of real axis $\mathbb{R}^1$, let Ξ be a grid with rational knots $\xi_i \in (\alpha,\beta)$, $i \in \mathbf{Z}$,

$$\Xi: \quad \ldots < \xi_{-2} < \xi_{-1} < \xi_0 < \xi_1 < \xi_2 \ldots, \quad (1)$$

$$\lim_{i\to-\infty} \xi_i = \alpha, \quad \lim_{i\to+\infty} \xi_i = \beta.$$

If $d \in \Xi$ then there exists $i \in \mathbf{Z}$ such that $d = \xi_i$; denote $d^- = \xi_{i-1}$ and $d^+ = \xi_{i+1}$. Discuss $c, d \in \Xi$ and $c^+ < d^-$; by definition, put $]c, d[= \{c \le \xi_s \le d,\ \xi_s \in \Xi\}$. The set $]c, d[$ is called the grid segment.

Suppose $a, b \in \Xi$, , $a = \xi_0$, $b = \xi_M$, $M \in \mathbf{Z}$, $a^+ < b^-$. Let $C\,]a, b[$ be the linear finite-dimensional space of functions $u(t)$ defined for $t \in]a, b[$ and $\|u\|_{C\,]a,b[} = \max_{t\in\,]a,b[} |u(t)|$.

Let f be a function defined on Ξ and such that

$$f(t) \ge C_0 \quad \forall t \in]a, b[\,, \qquad C_0 = const > 0. \quad (2)$$

By definition, put

$$\varepsilon^* = \max_{\xi\in\,]a,b[}\ \max_{t\in\{\xi,\xi^+\}} f(t)(\xi^+ - \xi), \quad (3)$$

$$\varepsilon^{**} = (b-a)\|f\|_{C\,]a,b[}\,. \quad (4)$$

Lemma 1 *If $\varepsilon \in (\varepsilon^*, \varepsilon^{**})$ and conditions (2) – (4) are fulfilled, then there exist the unique natural integer $K = K(f, \varepsilon, \Xi)$ and the grid $\widetilde{X} \subset]a, b[$,*

$$\widetilde{X} = \widetilde{X}(f, \varepsilon, \Xi):$$

$$a = \widetilde{x}_0 < \widetilde{x}_1 < \ldots < \widetilde{x}_K \le \widetilde{x}_{K+1} = b \quad (5)$$

such that

$$\max_{t\in\,]\widetilde{x}_s,\,\widetilde{x}_{s+1}[} f(t)(\widetilde{x}_{s+1} - \widetilde{x}_s) \le \varepsilon <$$

$$< \max_{t\in\,]\widetilde{x}_s,\,\widetilde{x}^+_{s+1}[} f(t)(\widetilde{x}^+_{s+1} - \widetilde{x}_s) \quad (6)$$

$$\forall s \in \{0, 1, \ldots, K-1\},$$

$$\max_{t\in\,]\widetilde{x}_K,\,b[} f(t)(b - \widetilde{x}_K) \le \varepsilon, \quad \widetilde{X} \subset \Xi. \quad (7)$$

The proof of Lemma 1 is given by mathematical induction as to parameter s; the induction is the source of the algorithm for the construction of grid (5) with properties (6) – (7) (see [24], see also an illustrative example there);the grid is called *the adaptive grid.*

Summation of relations (6) leads to inequality

$$\sum_{s=0}^{K-1} \max_{t\in\,]\widetilde{x}_s,\,\widetilde{x}_{s+1}[} f(t)(\widetilde{x}_{s+1} - \widetilde{x}_s) \le K\varepsilon <$$

$$< \sum_{s=0}^{K-1} \max_{t\in\,]\widetilde{x}_s,\,\widetilde{x}^+_{s+1}[} f(t)(\widetilde{x}^+_{s+1} - \widetilde{x}_s). \quad (8)$$

2.2 Pseudo-equidistant grid

By definition, put

$$\overline{\varepsilon}^* = \max_{\xi\in\,]a,b^-[} (\xi^+ - \xi)\|f\|_{C\,]a,b[}. \quad (9)$$

Under the condition of

$$\varepsilon \in (\overline{\varepsilon}^*, \varepsilon^{**}) \quad (10)$$

we discuss values[1]

$$N = N(f, \varepsilon, \Xi) = \lceil \varepsilon^{**}/\varepsilon \rceil + 1,\ 4 \le N < M, \quad (11)$$

and

$$h = h(f, \varepsilon, \Xi) = \frac{b-a}{N}. \quad (12)$$

Suppose there exists a grid

$$\overline{X} = \overline{X}(f, \varepsilon, \Xi):$$

$$a = \overline{x}_0 < \overline{x}_1 < \ldots < \overline{x}_N = b,\ \overline{X} \subset \Xi, \quad (13)$$

where

$$h/p \le \overline{x}_{s+1} - \overline{x}_s \le ph, \quad p = const, \quad p \ge 1, \quad (14)$$
$$s \in \{0, 1, \ldots, N-1\}.$$

In the next we add a knot $\overline{x}_{N+1} \in \Xi$ to the grid $\overline{X}$, where $\overline{x}_{N+1} > \overline{x}_N$ and

$$\overline{x}_{N+1} - \overline{x}_N \le ph. \quad (15)$$

Suppose that

$$\varepsilon < h\|f\|_{C]a,b[} \le \varepsilon\Big(1 + \frac{1}{N}\Big). \quad (16)$$

Using (12) and (16), we get

$$(b-a)\|f\|_{C\,]a,b[} - \varepsilon < N\varepsilon \le$$
$$\le (b-a)\|f\|_{C\,]a,b[}. \quad (17)$$

Taking into account inequalities (14) – (15) and formulas (11) – (12), we obtain

$$\max_{t\in\,]\overline{x}_s,\,\overline{x}_{s+1}[} f(t)\,(\overline{x}_{s+1} - \overline{x}_s) \le \max_{t\in\,]\overline{x}_s,\,\overline{x}_{s+1}[} f(t)ph =$$

$$= \max_{t\in\,]\overline{x}_s,\,\overline{x}_{s+1}[} f(t)p\frac{b-a}{\lceil\varepsilon^{**}/\varepsilon\rceil + 1} \le$$

$$\le \max_{t\in\,]\overline{x}_s,\,\overline{x}_{s+1}[} f(t)p\frac{b-a}{\varepsilon^{**}/\varepsilon} =$$

$$\max_{t\in\,]\overline{x}_s,\,\overline{x}_{s+1}[} f(t)p\,\varepsilon(b-a)/\varepsilon^{**};$$

thus by (4) we have

$$\max_{t\in\,]\overline{x}_s,\,\overline{x}_{s+1}[} f(t)\,(\overline{x}_{s+1} - \overline{x}_s) \le p\,\varepsilon, \quad (18)$$

$$s \in \{0, 1, \ldots, N\}.$$

Grid (13) with properties (14) – (17) is named *pseudo-equidistant grid with mesh width h.*

[1]For value r the expression $\lceil r\rceil$ is integer number k with property $0 \le k - r < 1$.

2.3 Relative quantity of knots

Let's suppose that function $f(t)$ is continuous on segment $[a, b]$, and

$$f(t) \geq C_0 > 0 \quad \forall t \in [a, b]. \tag{19}$$

Consider the sequence of grids $\Xi(\lambda)$,

$$\Xi(\lambda) : \ldots < \xi_{-1}(\lambda) < \xi_0(\lambda) < \xi_1(\lambda) < \ldots, \tag{20}$$

depending on parameter $\lambda > 0$ such that $a, b \in \Xi(\lambda)$.

By definition, put

$$]a, b\,[_\lambda = \Xi(\lambda) \cap [a, b], \quad h_\lambda = \max_{\xi \in\,]a, b^-\,[} (\xi^+ - \xi).$$

Theorem 2 *If function $f(t)$ is continuous and satisfies condition (19), and the sequence of grids (20) such that*

$$\lim_{\lambda \to +0} h_\lambda = 0, \tag{21}$$

then the relation

$$\lim_{\varepsilon \to +0} \lim_{\lambda \to +0} \frac{N}{K} = \frac{\|f\|_{C[a,b]}}{\frac{1}{b-a} \int_a^b f(t)dt} \tag{22}$$

is true.

Relation (22) follows from (8) and (17) by simple processing and passing to the limit.

3 Flows and their approximations

Let $\mathcal{F}$ be a metrized field[2]; the appropriate metric is denoted by $|\cdot|$ and it has the following properties: a) $|f| \geq 0 \ \forall f \in \mathcal{F}$, and $|f| = 0 \iff f = 0$, b) the relations $|f + g| \leq |f| + |g|$ and c) $|fg| = |f||g|$ are right $\forall f, g \in \mathcal{F}$.

Consider linear normed space $\mathcal{M}$ over field $\mathcal{F}$; let $\|\cdot\|$ be a norm in the space.

Denote by $\mathbf{C}_\mathcal{M}\,]a, b\,[$ the linear finite-dimensional space of abstract functions $U(t)$, $t \in\,]a, b\,[$, with values of the functions[3] in space $\mathcal{M}$. Let

$$\|U\|_{\mathbf{C}_\mathcal{M}\,]a,b\,[} = \max_{t \in\,]a,b\,[} \|U(t)\|$$

be a norm in space $\mathbf{C}_\mathcal{M}\,]a, b\,[$. The element $U(t)$ of space $\mathbf{C}_\mathcal{M}\,]a, b\,[$ is called the *general flow*. Later we need to use abstract functions defined on segment $[c, d]$ of the real axis such that their range of values is situated in $\mathcal{M}$; for them the differentiation is introduced in the usual way. Therefore we also discuss the linear spaces $\mathbf{C}_\mathcal{M}[c, d]$, $\mathbf{C}^i_\mathcal{M}[c, d]$, $i = 1, 2$, of continuous and of continuously differentiated abstract functions.

Let $U(t)$ be a function defined on grid (1). By definition, put

$$D_\Xi U(\xi) = \frac{U(\xi^+) - U(\xi)}{\xi^+ - \xi},$$

$$D^2_\Xi U(\xi) = \frac{D_\Xi U(\xi) - D_\Xi U(\xi^-)}{\xi - \xi^-}, \quad \xi \in \Xi.$$

Let $\widehat{X}$ be a subset of grid Ξ such that

$$\widehat{X} : \quad a = \widehat{x}_0 < \widehat{x}_1 < \widehat{x}_2 < \ldots < \widehat{x}_{\widehat{K}} < \widehat{x}_{\widehat{K}+1} = b.$$

Let

$$\widetilde{U}(t) = U(\widehat{x}_j) + \frac{U(\widehat{x}_{j+1}) - U(\widehat{x}_j)}{\widehat{x}_{j+1} - \widehat{x}_j}(t - \widehat{x}_j)$$

$$\forall t \in [\widehat{x}_j, \widehat{x}_{j+1}), \ j \in \{0, 1, \ldots, \widehat{K}\}$$

be a piecewise linear interpolation of function $U(t)$, defined on segment $]a, b\,[$.

It is easy to obtain the next assertion.

Theorem 3 *If $y, z \in]a, b[$, $y^+ < z^-$ and $t \in]y, z[$ then the functions $U(t)$ and $\widetilde{U}(t)$ satisfy the relations*

$$\|U(t) - \widetilde{U}(t)\| \leq$$

$$\leq 2 \min\{t - y, z - t\} \max_{\xi \in]y,\, z^-[} \|D_\Xi U(\xi)\|, \tag{23}$$

$$\|U(t) - \widetilde{U}(t)\| \leq$$

$$(z - y) \max_{\xi \in]y,\, z^-[} \|D_\Xi U(\xi)\|, \tag{24}$$

$$\|U(t) - \widetilde{U}(t)\| \leq$$

$$\leq (z - y)^2 \max_{\xi \in]y^+,\, z^-[} \|D^2_\Xi U(\xi)\|, \quad t \in\,]y, z[\,. \tag{25}$$

Theorem 4 *If $t \in\,]\widehat{x}_j, \widehat{x}_{j+1}[$, then inequalities*

$$\|U(t) - \widetilde{U}(t)\| \leq$$

$$\leq (\widehat{x}_{j+1} - \widehat{x}_j) \max_{\xi \in]\widehat{x}_j,\, \widehat{x}^-_{j+1}[} \|D_\Xi U(\xi)\|, \tag{26}$$

$$\|U(t) - \widetilde{U}(t)\| \leq$$

$$\leq (\widehat{x}_{j+1} - \widehat{x}_j)^2 \max_{\xi \in]\widehat{x}^+_j,\, \widehat{x}^-_{j+1}[} \|D^2_\Xi U(\xi)\| \tag{27}$$

hold.

[2]The field of real numbers, the field of complex numbers and the field of p-adic numbers are metrized fields (i.e. fields with evaluation).

[3]The expression "abstract function" is often replaced by the word "function"; that doesn't lead to confusion because in all cases when we discuss an abstract function with values in the space $\mathcal{M}$, we denote it with capital letter or semiboldface type.

If $U \in \mathbf{C}^1_{\mathcal{M}}[a,b]$, $y,z \in [a,b]$, $t \in [y,z]$ *then*

$$\|U(t) - \widetilde{U}(t)\| \leq$$

$$\leq \max_{\xi\in[\widehat{x}_j,\widehat{x}_{j+1}]} \|U'(\xi)\|(z-y), \quad (28)$$

and if $U \in \mathbf{C}^2_{\mathcal{M}}[a,b]$, *then*

$$\|U(t) - \widetilde{U}(t)\| \leq$$

$$\leq \max_{\zeta\in[y,z]} \|U''(\zeta)\|(z-y)^2. \quad (29)$$

Proof. The evaluations (26) – (27) follow from inequalities (23) – (25), and the relations (28) – (29) come out by passage to the limit in formulas (26) – (27) under condition $\max_{\xi\in]y,\,z^-[}(\xi^+ - \xi) \to +0$.[4]

4 On number of grid knots

4.1 A grid of adaptive type

Theorem 5 *Suppose that*

$$\|D_\Xi U(t)\| \geq C_0 \ \forall t \in \Xi, \ C_0 = const > 0. \quad (30)$$

If $\eta > 0$, *and grid* $\widehat{X}$ *coincides with grid* $\widetilde{X}(\|D_\Xi U(t)\|, \eta, \Xi)$, *then*

1) the quantity of knots $K'_{U,\Xi}(\eta) = K(\|D_\Xi U(t)\|, \eta, \Xi)$ *of the grid satisfy relations*

$$\sum_{s=0}^{K-1} \max_{t\in]\widetilde{x}_s,\,\widetilde{x}_{s+1}[} \|D_\Xi U(t)\|(\widetilde{x}_{s+1} - \widetilde{x}_s)/\eta \leq$$

$$\leq K'_{U,\Xi}(\eta) <$$

$$< \sum_{s=0}^{K-1} \max_{t\in]\widetilde{x}_s,\,\widetilde{x}^+_{s+1}[} \|D_\Xi U(t)\|(\widetilde{x}^+_{s+1} - \widetilde{x}_s)/\eta, \quad (31)$$

2) inequality

$$\|U(t) - \widetilde{U}(t)\| \leq \eta \qquad \forall t \in]a,b[\quad (32)$$

is true,

3) if there are sequences of grids (20) with condition (21) and function $U \in \mathbf{C}^1_{\mathcal{M}}[a,b]$, *for which* $\|U'(t)\| \geq C_0 > 0 \ \forall t \in [a,b]$, *then relation*

$$\lim_{\eta'\to+0} \lim_{\lambda\to+0} K'_{U,\Xi(\lambda)}(\eta')\eta' = \int_a^b \|U'(t)\|dt \quad (33)$$

is fulfilled.

[4]Evaluations (23) – (29) aren't precise, but that isn't actual in discussed case.

Proof: Formula (31) follows from relation (8), where it needs to put $f(t) = \|D_\Xi U(t)\|$. Under condition (30) inequality (32) follows from (23) and (6), where $f(t) = \|D_\Xi U(t)\|$, $\varepsilon = \eta$. Finally, formula (33) follows from (31) by passing to the limit.

Theorem 6 *Suppose that a condition*

$$\|D^2_\Xi U(t)\| \geq C_0 \ \forall t \in]y,z[\ , \ C_0 = const > 0 \quad (34)$$

is fulfilled. If $\eta > 0$ *and grid* $\widehat{X}$ *coincides with grid* $\widetilde{X}(\sqrt{\|D^2_\Xi U(t)\|}, \eta, \Xi)$, *then*

1) quantity of knots $K''_{U,\Xi}(\eta) = K(\sqrt{\|D^2_\Xi U(t)\|}, \eta, \Xi)$ *of the grid satisfies relations*

$$\sum_{s=0}^{K-1} \max_{t\in]\widetilde{x}_s,\,\widetilde{x}_{s+1}[} \sqrt{\|D^2_\Xi U(t)\|}(\widetilde{x}_{s+1} - \widetilde{x}_s)/\eta \leq K''_{U,\Xi}(\eta) <$$

$$< \sum_{s=0}^{K-1} \max_{t\in]\widetilde{x}_s,\,\widetilde{x}^+_{s+1}[} \sqrt{\|D^2_\Xi U(t)\|}(\widetilde{x}^+_{s+1} - \widetilde{x}_s)/\eta, \quad (35)$$

2) the inequality

$$\|U(t) - \widetilde{U}(t)\| \leq \eta^2 \qquad \forall t \in]a,b[\quad (36)$$

is true,

3) if there is a sequence of grids (20) with property (21), then for arbitrary function $U \in \mathbf{C}^2_{\mathcal{M}}[a,b]$, *for which* $\|U''(t)\| \geq c > 0 \ \forall t \in [a,b]$, *we have*

$$\lim_{\eta'\to+0} \lim_{\lambda\to+0} K''_{U,\Xi(\lambda)}(\eta')\eta' = \int_a^b \sqrt{\|U''(t)\|}dt. \quad (37)$$

Proof of this Theorem is analogous to the proof of Theorem 5.

4.2 Pseudo-equidistant grid

Theorem 7 *If grid* $\widehat{X}$ *coincides with grid* $\overline{X}(\|D_\Xi U\|, \eta/p, \Xi)$, *then*

1) the number $N'_{U,\Xi}(\eta) = N(\|D_\Xi U\|, \eta/p, \Xi)$ *of inner knots of the grid satisfies the relation*

$$p(b-a)\|D_\Xi U\|_{\mathbf{C}_{\mathcal{M}}]a,b[}/\eta - 1 < N'_{U,\Xi}(\eta) \leq$$

$$\leq p(b-a)\|D_\Xi U\|_{\mathbf{C}_{\mathcal{M}}]a,b[}/\eta, \quad (38)$$

2) inequality

$$\|U(t) - \widetilde{U}(t)\| \leq \eta \qquad \forall t \in]a,b[\quad (39)$$

is right.

Proof: Considering grid $\widehat{X} = \overline{X}(\|D_\Xi U\|, \eta/p, \Xi)$, we apply formula (17); as a result we get the relation (38). The inequality (39) follows from relations (26) and (18) if $f(t) = \|D_\Xi U(t)\|$ and $\varepsilon = \eta/p$.

Theorem 8 *If grid* $\widehat{X}$ *coincides with grid* $\overline{X}(\sqrt{\|D^2_\Xi U\|}, \eta/p, \Xi)$, *then*

1) the quantity $N''_{U,\Xi}(\eta) = N(\sqrt{\|D^2_\Xi U\|}, \eta/p, \Xi)$ *of inner knots of that grid satisfies relation*

$$p(b-a)\| \, \|D^2_\Xi U(t)\|^{1/2}\|_{\mathbf{C}_\mathcal{M}]a,\,b[} \,/\eta - 1 < N''_{U,\Xi}(\eta) \leq$$

$$\leq p(b-a)\| \, \|D^2_\Xi U(t)\|^{1/2}\|_{\mathbf{C}_\mathcal{M}]a,\,b[} \,/\eta, \qquad (40)$$

2) inequality

$$\|U(t) - \widetilde{U}(t)\| \leq \eta^2 \qquad \forall t \in \,]a, b[\qquad (41)$$

is true.

Proof. By analogy with the proof of previous theorem we apply formula (17) to $\widehat{X} = \overline{X}(\sqrt{\|D^2_\Xi U\|}, \eta/p, \Xi)$; it follows relation (40). The inequality (41) can be obtained by application of relations (18) and (27), where $f = \sqrt{\|D^2_\Xi U\|}$ and $\varepsilon = \eta/p$.

4.3 Comparative characteristics of the quantity of knots

Theorem 9 *Suppose that abstract function* $U(t)$ *is approximated by function* $\widetilde{U}(t)$ *with evaluation* $\|\widetilde{U}(t) - U(t)\| \leq \eta$ *under two choices of grid: in the first variant the pseudo-equidistant grid is used such that* $\widehat{X} = \overline{X}(\|D_\Xi U\|, \eta/p, \Xi)$, *and in the second variant the adaptive grid* $\widehat{X} = \widetilde{X}(\|D_\Xi U\|, \eta, \Xi)$ *is used; then we have*

$$\frac{p(b-a)\|D_\Xi U\|_{\mathbf{C}_\mathcal{M}]a,\,b[} - \eta}{\sum_{s=0}^{K-1} \max_{t\in]\widetilde{x}_s,\,\widetilde{x}^+_{s+1}[} \|D_\Xi U(t)\|(\widetilde{x}^+_{s+1} - \widetilde{x}_s)} <$$

$$< \frac{N'_{U,\Xi}(\eta)}{K'_{U,\Xi}(\eta)} \leq$$

$$\leq \frac{p(b-a)\|D_\Xi U\|_{\mathbf{C}_\mathcal{M}]a,\,b[}}{\sum_{s=0}^{K-1} \max_{t\in]\widetilde{x}_s,\,\widetilde{x}_{s+1}[} \|D_\Xi U(t)\|(\widetilde{x}_{s+1} - \widetilde{x}_s)}. \qquad (42)$$

Formula (42) follows from inequalities (31) – (32) and (38) – (39).

Theorem 10 *Consider the family of grids (20) – (21). Let* $U(t)$, $t \in [a, b]$, *be a continuously differentiable function with property*

$$\|U'\|_{\mathbf{C}_\mathcal{M}[a,b]} \neq 0; \qquad (43)$$

then

$$\lim_{\eta\to+0}\lim_{\lambda\to+0} \frac{K'_{U,\Xi}(\eta)}{N'_{U,\Xi}(\eta)} = \frac{\frac{1}{b-a}\int_a^b \|U'(t)\|dt}{p\|U'\|_{\mathbf{C}_\mathcal{M}[a,b]}}. \qquad (44)$$

Proof: Under the conditions of (43) we can discuss ratio $\frac{K'_{U,\Xi}(\eta)}{N'_{U,\Xi}(\eta)}$; the passing to the limit in (42) gives the correlation (44).

Theorem 11 *Suppose the construction of approximations* $\widetilde{U}(t)$ *of discrete function* $U(t)$ *with evaluation* $\|\widetilde{U}(t) - U(t)\| \leq \eta^2$ *is accompanied by two variants of grids: in the first variant the pseudo-equidistant grid* $\widehat{X} = \overline{X}(\sqrt{\|D^2_\Xi U\|}, \eta/p, \Xi)$ *is utilized, and in the second variant the adaptive grid* $\widehat{X} = \widetilde{X}(\sqrt{\|D^2_\Xi U\|}, \eta, \Xi)$ *is applied; then we have*

$$\frac{p(b-a)\| \, \|D^2_\Xi U(t)\|^{1/2}\|_{\mathbf{C}_\mathcal{M}]a,\,b[} - \eta}{\sum_{s=0}^{K-1} \max_{t\in]\widetilde{x}_s,\,\widetilde{x}^+_{s+1}[} \sqrt{\|D^2_\Xi U(t)\|}(\widetilde{x}^+_{s+1} - \widetilde{x}_s)} <$$

$$< \frac{N''_{U,\Xi}(\eta)}{K''_{U,\Xi}(\eta)} \leq$$

$$\leq \frac{p(b-a)\| \, \|D^2_\Xi U\|^{1/2}\|_{\mathbf{C}_\mathcal{M}]a,\,b[}}{\sum_{s=0}^{K-1} \max_{t\in]\widetilde{x}_s,\,\widetilde{x}_{s+1}[} \sqrt{\|D^2_\Xi U(t)\|}(\widetilde{x}_{s+1} - \widetilde{x}_s)}. \qquad (45)$$

Evaluation (45) follows from inequalities (35) – (36) and (40) – (41).

Theorem 12 *Let's discuss a family of grid (20) with property (21). Suppose an abstract function* $U(t)$ *is twice continuously differentiated on the segment* $[a, b]$ *and has a property*

$$\|U''\|_{\mathbf{C}_\mathcal{M}[\mathbf{a},\mathbf{b}]} \neq 0; \qquad (46)$$

then

$$\lim_{\eta\to+0}\lim_{\lambda\to+0} \frac{N''_{U,\Xi}(\eta)}{K''_{U,\Xi}(\eta)} = \frac{\frac{1}{b-a}\int_a^b \sqrt{\|U''(t)\|}dt}{p\|\sqrt{\|U''(t)\|}\|_{\mathbf{C}_\mathcal{M}[\mathbf{a},\mathbf{b}]}}. \qquad (47)$$

Under condition (46) the relation (47) follows by the passing to the limit in (45).

5 Wavelet support

We considered the construction of embedded grids and evaluation of approximations before. In this section we suppose that embedded grids have been constructed; here we discuss calibration relations, which are wavelet support in the next discussion.

5.1 Embedded grid

Let m be a natural number; by definition, put

$$J_m = \{0, 1, \ldots, m\}, \qquad J'_m = \{-1, 0, 1, \ldots, m\}.$$

Consider the functions $\{\omega_j(t)\}_{j\in J'_{M-1}}$ as elements of the space $C]a, b[$:

$$\omega_j(\xi_s) = \delta_{s,j+1}, \qquad s \in J_M.$$

Let $g^{(i)}$, $i \in J'_{M-1}$ be the linear functionals defined by relations

$$\langle g^{(i)}, u\rangle = u(\xi_{i+1}) \qquad \forall u \in C]a, b[. \tag{48}$$

The system $\{\omega_j\}_{j\in J'_{M-1}}$ is the basis of the space $C]a, b[$; we have

$$\langle g^{(i)}, \omega_j\rangle = \delta_{i,j} \qquad \forall\, i, j \in J'_{M-1}.$$

In further we discuss a set $]c, d[$ as an empty set if $c > d$.

Suppose $5 \le K < M$. Consider an injective map κ of the set J_K to the set J_M such that

$$\kappa(0) = 0, \quad \kappa(i) < \kappa(i + 1), \quad \kappa(K) = M. \tag{49}$$

Let $J^* \subset J_M$ be the set defined by the formula

$$J^* = \kappa J_K. \tag{50}$$

In view of (49) – (50) the revised map κ^{-1} defined on the set J^* uniquely: $\forall r \in J^* \quad \kappa^{-1} : \ r \longrightarrow s, \quad s \in J_K, \qquad J_K = \kappa^{-1}J^*$.

Let

$$\widehat{X} : \quad a = \widehat{x}_0 < \widehat{x}_1 < \ldots < \widehat{x}_K = b$$

be a new grid with knots $\widehat{x}_i = \xi_{\kappa(i)}$, $i \in J_K$.

Sometimes we discuss additional knots ξ_{-1} and $\widehat{x}_{-1}$ with property $\xi_{-1} = \widehat{x}_{-1} < a$.

5.2 Calibration relations

Consider functions $\widehat{\omega}_j(t)$, $j \in J'_{K-1}$ defined by relations

$$\widehat{\omega}_i(t) = (t - \xi_{\kappa(i)})(\xi_{\kappa(i+1)} - \xi_{\kappa(i)})^{-1}$$

$$\text{for} \quad t \in]\xi^{\,+}_{\kappa(i)}, \xi_{\kappa(i+1)}[, \ i \in J_{K-1}, \tag{51}$$

$$\widehat{\omega}_i(t) = (\xi_{\kappa(i+2)} - t)(\xi_{\kappa(i+2)} - \xi_{\kappa(i+1)})^{-1}$$

$$\text{for} \quad t \in]\xi_{\kappa(i+1)}, \xi^{\,-}_{\kappa(i+2)}[, \ i \in J'_{K-2}; \tag{52}$$

$$\widehat{\omega}_i(t) = 0 \quad \text{for} \quad t \in]a, b[\setminus]\xi^{\,+}_{\kappa(i)}, \xi^{\,-}_{\kappa(i+2)}[. \tag{53}$$

It is clear to see that

$$\widehat{\omega}_i(\xi_{\kappa(i+1)}) = 1 \quad \forall i \in J'_{K-1}. \tag{54}$$

Splines $\widehat{\omega}_i$ could be written as linear combinations of splines ω_j:

$$\widehat{\omega}_r(t) = \sum_{q\in J'_{M-1}} p_{r,q}\omega_j(t) \quad \forall t \in]a, b[, \quad r \in J'_{K-1}; \tag{55}$$

formulas (55) are called *calibration relations*.

Applying the functionals $g^{(j)}$ to (55) and taking into account relations (48), we have

$$p_{-1,j} = \widehat{\omega}_{-1}(\xi_{j+1})$$

$$\forall j \in \{\kappa(0) - 1, \kappa(0), \ldots, \kappa(1) - 2\}, \tag{56}$$

$$p_{i,j} = \widehat{\omega}_i(\xi_{j+1})$$

$$\forall j \in \{\kappa(i), \ldots, \kappa(i + 2) - 2\} \quad \forall i \in J_{K-2}, \tag{57}$$

$$p_{K-1,j} = \widehat{\omega}_{K-1}(\xi_{j+1})$$

$$\forall j \in \{\kappa(K - 1), \ldots, \kappa(K) - 1\}; \tag{58}$$

the numbers $p_{r,s}$, $r \in J'_{K-1}$, $s \in J'_{M-1}$, which are absent in these formulas, are equal to zero.

Consider functionals

$$\langle \widehat{g}^{(i)}, u\rangle = u(\widehat{x}_{i+1}) \ \forall u \in C]a, b[, \ i \in J'_{K-1}. \tag{59}$$

By (51) – (54) and (59) we have

$$\langle \widehat{g}^{(i)}, \widehat{\omega}_j\rangle = \delta_{i,j} \qquad \forall i, j \in J'_{K-1}. \tag{60}$$

Using the relations (48) – (50) and (59), for arbitrary $u \in C]a, b[$ and $i \in J'_{K-1}$ we have

$$\langle \widehat{g}^{(i)}, u\rangle = u(\widehat{x}_{i+1}) = \langle g^{\kappa(i+1)-1}, u\rangle;$$

thus the equalities

$$\widehat{g}^{(i)} = g^{(\kappa(i+1)-1)} \ \ i \in J'_{K-1} \tag{61}$$

are true. By (61) we have

$$\widehat{g}^{(\kappa^{-1}(j+1)-1)} = g^{(j)} \qquad \forall j + 1 \in J^*. \tag{62}$$

5.3 Matrix of restriction

Discuss matrix $P = (p_{i,j})_{i\in J'_{K-1}, j\in J'_{M-1}}$; here $p_{i,j} = \langle g^{(j)}, \widehat{\omega}_i\rangle$. The matrix P is called *a restriction matrix*. We introduce the ascending ordered subsets of set **Z**:

$$J^0 = \{-1, \ldots, \kappa(1) - 2\},$$

$$J^1(r) = \{\kappa(r), \ldots, \kappa(r+1) - 1\} \qquad \forall r \in J_{K-1},$$

$$J^2(r) = \{\kappa(r+1), \ldots, \kappa(r+2)-2\} \qquad \forall r \in J_{K-2},$$

$$J(r) = J^1(r) \bigcup J^2(r) \quad \forall r \in J_{K-2},$$

$$J(K-1) = J^1(K-1).$$

The ascending ordered set will be discussed as empty if its first element is more then last one.

Theorem 13 *The coefficients $p_{r,q}$ of the calibration relations (55) might be written in next form*

$$p_{-1,q} = \frac{\xi_{\kappa(1)} - \xi_{q+1}}{\xi_{\kappa(1)} - \xi_{\kappa(0)}} \qquad q \in J^0, \tag{63}$$

$$p_{r,q} = \frac{\xi_{q+1} - \xi_{\kappa(r)}}{\xi_{\kappa(r+1)} - \xi_{\kappa(r)}}$$

$$q \in J^1(r), \quad r \in J_{K-1}, \tag{64}$$

$$p_{r,q} = \frac{\xi_{\kappa(r+2)} - \xi_{q+1}}{\xi_{\kappa(r+2)} - \xi_{\kappa(r+1)}}$$

$$q \in J^2(r), \quad r \in J_{K-2}, \tag{65}$$

with elements $p_{r,q}$ unmentioned in formulas (63) – (65) equal to zero.

Proof. First of all we note that relations (64) and (65) aren't converse to each other, because for data r the sets $J^1(r)$ and $J^2(r)$ aren't intersects. It's clear to see that formulas (63) – (65) follow from relations (56) – (58) by correlations (51) – (54).

Corrolary 5.1 *The formula*

$$\mathbf{p_{i,j}} = \delta_{\mathbf{i},\kappa^{-1}(\mathbf{j}+1)-1} \qquad \forall \mathbf{i} \in \mathbf{J'_{K-1}}, \qquad \mathbf{j}+1 \in \mathbf{J}^*. \tag{66}$$

is right.

Proof. Using the relations $p_{i,j} = \langle g^{(j)}, \widehat{\omega}_i\rangle$, formula (62) and property (60), we have

$$p_{i,j} = \langle g^{(j)}, \widehat{\omega}_i\rangle = \langle \widehat{g}^{(\kappa^{-1}(j+1)-1)}, \widehat{\omega}_i\rangle,$$

for all $i \in J'_{K-1}$; hence we get relation (66).

5.4 Matrix of prolongation

Consider matrix $Q = (q_{s,j})_{s\in J'_{K-1}, j\in J'_{M-1}}$ with elements

$$q_{s,j} = \langle \widehat{g}^{(s)}, \omega_j\rangle; \tag{67}$$

the matrix Q is called *the matrix of prolongation*.

Taking into account the formulas (59), (67), we obtain the next assertions.

Theorem 14 *In the matrix Q*

1) if $j+1 \notin J^*$, then column $q^{(j)} = (q_{sj})_{s\in J'_{K-1}}$ is zero column;

2) if $j+1 \in J^*$, then column $q^{(j)}$ contains a unit on the s_0-th place, where $\kappa(s_0+1) = j+1$; the other elements of the column are equal to zero.

Theorem 15 *The matrix Q is left inverse matrix for the matrix P^T:*

$$QP^T = I;$$

here I is the identity matrix of size $K+1 \times K+1$.

Theorem 16 *Elements $[P^TQ]_{i,j}$, $i,j \in J'_{M-1}$, of a matrix production P^TQ are defined by formulas*

$$[P^TQ]_{i,j} = 0 \qquad \textit{for } i \in J'_{M-1},\ j+1 \in J_M \backslash J^*,$$

$$[P^TQ]_{i,j} = p_{\kappa^{-1}(j+1)-1,i} \qquad \textit{for } i \in J'_{M-1},\ j+1 \in J^*.$$

Corrolary 5.2 *If $i+1,\ j+1 \in J^*$, then*

$$[P^TQ]_{i,j} = \delta_{i,j}.$$

6 General flows and their reconstruction

Consider linear spaces

$$\mathcal{S} = \mathcal{S}(X, \varphi, \mathcal{M}) = \{\mathbf{u} \mid \mathbf{u}(t) =$$

$$= \sum_{j\in J'_{M-1}} \mathbf{C}_j\omega_j(t)\ \forall \mathbf{C}_s \in \mathcal{M}\ \forall j \in J'_{M-1}, t \in]a,b[\},$$

$$\widehat{\mathcal{S}} = \mathcal{S}(\widehat{X}, \varphi, \mathcal{M}) = \{\mathbf{u} \mid \mathbf{u}(t) =$$

$$= \sum_{i\in J'_{K-1}} \mathbf{A}_i\widehat{\omega}_i(t) \quad \forall \mathbf{A}_s \in \mathcal{M}\ \forall s \in J'_{K-1}, t \in]a,b[\}.$$

Taking into account calibration relations (55), we have $\widehat{\mathcal{S}} \subset \mathcal{S} \subset \mathbf{C}_{\mathcal{M}}]\mathbf{a},\mathbf{b}[$.

Suppose there is the next equivalence

$$\sum_{j\in J'_{M-1}} \mathbf{C}_j\omega_j(t) \equiv \mathbf{0} \quad \forall t \in]a,b[\iff$$

$$\iff \mathbf{C}_j = \mathbf{0} \quad \forall j \in J'_{M-1}. \tag{68}$$

If (68) is fulfilled then we say that *the system* $\{\omega_j\}_{j\in J'_{M-1}}$ *is linear independent over space* $\mathcal{S}$.

Let $\mathcal{P}$ be an operation of projection for space $\mathcal{S}$ on space $\widehat{\mathcal{S}}$ defined by formula

$$\mathcal{P}\mathbf{u} = \sum_{s\in J'_{K-1}} \sum_{j\in J'_{M-1}} \mathbf{C}_j\langle \widehat{g}^{(s)}, \omega_j\rangle \widehat{\omega}_s$$

$$\forall \mathbf{u} = \sum_{j\in J'_{M-1}} \mathbf{C}_j\omega_j \in \mathcal{S}. \qquad (69)$$

By definition, put

$$\langle \widehat{g}^{(s)}, \mathbf{u}\rangle = \sum_{j\in J'_{M-1}} \mathbf{C}_j\langle \widehat{g}^{(s)}, \omega_j\rangle;$$

by (69) we have

$$\mathcal{P}\mathbf{u}(t) = \langle \widehat{g}^{(k-1)}, \mathbf{u}\rangle \widehat{\omega}_{k-1}(t) + \langle \widehat{g}^{(k)}, \mathbf{u}\rangle \widehat{\omega}_k(t)$$

$$\forall t \in t \in]\widehat{x}_k, \widehat{x}_{k+1}[, \quad k \in J_{K-1}.$$

The operation $\mathcal{P}$ defines direct decomposition of linear space $\mathcal{S}$:

$$\mathcal{S} = \mathcal{S} + \mathcal{W}. \qquad (70)$$

The space $\mathcal{S}$ is named *the initial space*; linear spaces $\widehat{\mathcal{S}}$ and $\mathcal{W}$ are called *the main space* and *the wavelet space* respectively.

Let $\mathbf{C} = (\mathbf{C}_{-1}, \mathbf{C}_0, \mathbf{C}_1, \ldots, \mathbf{C}_{M-1})^T$ be the initial flow of elements from space $\mathcal{M}$. By definition, put

$$\mathbf{u} = \sum_{s\in J'_{M-1}} \mathbf{C}_s\omega_s. \qquad (71)$$

Using relation (70), we get the second representation of the element $\mathbf{u}$:

$$\mathbf{u} = \widehat{\mathbf{u}} + \mathbf{w}, \qquad (72)$$

where

$$\widehat{\mathbf{u}} = \sum_{i\in J'_{K-1}} \mathbf{A}_i\widehat{\omega}_i, \qquad \mathbf{w} = \sum_{j\in J'_{M-1}} \mathbf{B}_j\omega_j,$$

$$\mathbf{B}_j,\ \mathbf{C}_s\ \in \mathcal{M} \quad \forall j, s \in J'_{M-1},$$

$$\mathbf{A}_i = \langle \widehat{g}^{(i)}, \mathbf{u}\rangle \quad \forall i \in J'_{K-1}. \qquad (73)$$

By (71) – (72) we have

$$\sum_{j\in J'_{M-1}} \mathbf{C}_j\omega_j = \sum_{i\in J'_{K-1}} \mathbf{A}_i \sum_{j\in J'_{M-1}} p_{i,j}\omega_j +$$

$$+ \sum_{j\in J'_{M-1}} \mathbf{B}_j\omega_j,$$

whence taking into account the linear independence of the system $\{\omega_j\}_{j\in J'_{M-1}}$ over space $\mathcal{S}$, we get *the formulas of reconstruction*

$$\mathbf{C}_j = \sum_{i\in J'_{K-1}} p_{i,j}\mathbf{A}_i + \mathbf{B}_j \qquad \forall j \in J'_{M-1}. \quad (74)$$

7 Formulas of decomposition

Using representation (73), we rewrite formulas (74) in the form

$$\mathbf{C}_j = \sum_{i\in J'_{K-1}} \langle \widehat{g}^{(i)}, \mathbf{u}\rangle p_{i,j} + \mathbf{B}_j \qquad \forall j \in J'_{M-1}$$

and taking into account (71), we have

$$\mathbf{C}_j = \sum_{i\in J'_{K-1}} \sum_{s\in J'_{M-1}} \mathbf{C}_s\langle \widehat{g}^{(i)}, \omega_s\rangle p_{i,j} + \mathbf{B}_j$$

$$\forall j \in J'_{M-1};$$

now we get

$$\mathbf{B}_j = \mathbf{C}_j - \sum_{s\in J'_{M-1}} \Big(\sum_{i\in J'_{K-1}} q_{i,s}p_{i,j}\Big)\mathbf{C}_s. \qquad (75)$$

Substituting (71) in (73), we have

$$\mathbf{A}_i = \langle \widehat{g}^{(i)}, \sum_{s\in J'_{M-1}} \mathbf{C}_s\omega_s\rangle \qquad \forall i \in J'_{K-1};$$

therefore

$$\mathbf{A}_i = \sum_{s\in J'_{M-1}} q_{i,s}\mathbf{C}_s \qquad \forall i \in J'_{K-1}. \qquad (76)$$

The formulas (75) – (76) are called *the formulas of decomposition.*

Using the vectors

$$\mathbf{A} = (\mathbf{A}_{-1}, \mathbf{A}_0, \ldots, \mathbf{A}_{K-1})^T,$$

$$\mathbf{B} = (\mathbf{B}_{-1}, \mathbf{B}_0, \ldots, \mathbf{B}_{M-1})^T,$$

we rewrite formulas (74) and (75) – (76) in matrix form: the formulas of decomposition (75) – (76) take the form

$$\mathbf{A} = Q\mathbf{C}, \qquad \mathbf{B} = \mathbf{C} - P^TQ\mathbf{C},$$

and the formulas of reconstruction (74) are represented as

$$\mathbf{C} = P^T\mathbf{A} + \mathbf{B}.$$

Using obtained assertions (see Theorems 13 and 14) for the elements of matrices P and Q, we get the following propositions.

Theorem 17 *The formulas of decomposition have the following properties*

$$\mathbf{A}_i = \mathbf{C}_{\kappa(i+1)-1} \qquad \forall i \in J'_{K-1}, \qquad (77)$$

$$\mathbf{B}_q = 0 \qquad \forall q + 1 \in J^*, \qquad (78)$$

$$\mathbf{B}_q = \mathbf{C}_q - \sum_{j\in J'_{K-1}} \langle g^{(q)}, \widehat{\omega}_j\rangle \mathbf{C}_{\kappa(j+1)-1} \qquad (79)$$

$$\forall q + 1 \in J_M \backslash J^*.$$

Theorem 18 *The wavelet flow satisfies the next relations: for $q+1 \in J_M \backslash J^*$ the equalities*

$$\mathbf{B}_q = \mathbf{C}_q - (\widehat{x}_{i+1} - \widehat{x}_i)^{-1}\Big[(\widehat{x}_{i+1} - \xi_{q+1})\mathbf{C}_{\kappa(i)-1} +$$

$$+(\xi_{q+1} - \widehat{x}_i)\mathbf{C}_{\kappa(i+1)-1}\Big]$$

are fulfilled; here

$$\widehat{x}_i < \xi_{q+1} < \widehat{x}_{i+1}. \qquad (80)$$

The formula (79) can be written in the form

$$\mathbf{B}_q = \mathbf{C}_q - p_{i-1,q}\mathbf{C}_{\kappa(i)-1} - p_{i,q}\mathbf{C}_{\kappa(i+1)-1},$$

where i satisfies relation (80).

The formulas (78) – (79) demonstrate that the space of wavelet flows $\mathcal{B}$ is

$$\mathcal{B} = \{\mathbf{B} \mid \mathbf{B} = (\mathbf{B}_{-1}, \mathbf{B}_0, \ldots, \mathbf{B}_{M-1})$$

$$\forall \mathbf{B}_{j-1} \in \mathcal{M},\ j \in J_M \backslash J^*; \mathbf{B}_{\,i-1} = 0\ \forall i \in J^*\}.$$

The relation (77) indicates that the construction of the main flow is reduced to values of initial flow on the embedded grid. If the embedded grid is adaptive, then the deviation of the main flow from the initial flow is defined by Theorem 5, and if the constructed grid is the pseudo-equidistant grid, then the mentioned deviation is given by Theorem 7.

8 Applications

We give important examples of applications for results mentioned above.

8.1 Spaces of matrices

The transmission of matrix flow across communication lines is very relevant; it is usually associated with large volumes of transmitted information, and therefore the selection of the main part of this flow is actual. The main part, apparently, should be transferred in the first place, and the non-main part (wavelet part) can be transferred in the second place or not at all.

In practice, matrices with rational elements are most frequently used; in this case it is possible to consider a linear normed space $\mathcal{R}_{p\times q}$ of $p \times q$-matrices with real rational elements.

Let $\mathcal{F}$ be the field of real rational numbers; we put $\mathcal{M} = \mathcal{R}_{p\times q}$. Since the original grid Ξ consists of rational numbers, and the condition of linear independence (68) is fulfilled, then the previous results can be applied to the occasion. The same way the case of a linear normed space $\mathcal{M} = \mathcal{C}_{p\times q}$ of matrices with complex rational elements is discussed; condition (68) is also valid here.

Completion of rational numbers leads to new linear spaces of matrices of sizes $p \times q$. As it is known, the completion in the standard metric (absolute value) will lead us to the space of matrices with real elements $\mathcal{M} = \mathbf{R}_{p\times q}$ and to the space of matrices with complex elements $\mathcal{M} = \mathbf{C}_{p\times q}$ over fields of real and complex numbers, respectively.

On the other hand, the completion of real rational numbers with respect to the p-adic metric leads to a linear space $\mathcal{M} = \mathcal{A}_{p\times q}$ matrices, which elements are p-adic numbers; here we need to discuss the field $\mathcal{F}$ of p-adic numbers.

It is obvious that in all these cases the condition (68) is valid, so that the results obtained here are applicable.

8.2 Spaces of infinite vectors

Linear spaces of infinite-dimensional vectors are also interested because restrictions relative to number of nonzero component are absent (it is possible to discuss spaces of infinite matrices, spaces of polynomials with arbitrary degrees and so on).

LetV^∞ be a linear normed space of vectors $\mathbf{v} = (v_0, v_1, v_2, \ldots)$ with rational components; that space could be discussed over field $\mathcal{F}$ of the rational numbers. Condition (68) is right, thus we can apply the obtained results for $\mathcal{M} = V^\infty$. Supplements of set of rational numbers with standard metric and with p-adic metric give the new linear spaces of infinite-dimensional vectors. Condition (68) is correct for mentioned spaces, therefore obtained results could be applied to the spaces.

9 Conclusion

The results give the opportunity to obtain the main flow in wavelet decomposition for flows of elements from linear normed spaces; sometimes it is very important to have decomposition of flows of matrices or flows of p-adic numbers. The results also demonstrate a large economy of computer memory in the case of usage of adaptive algorithms for the construction of the main flow. Now it is simple to obtain formulas of decomposition and reconstruction.

The transmission of matrix flow across communication lines is very essential for computer technic and TV-services; it is associated with large volumes of transmitted information, and therefore the selection of the main part of this flow is actual. The main part of the transmission should be transferred in the first place, and the non-main part (wavelet part) can be transferred in the second place or not at all. Such strategy might be realized by using the results obtained here: the last part of the paper is devoted to wavelet decomposition of different spaces of matrices of fixed order with numerical elements (rational, real, complex and p-adic elements) and to spaces of infinite-dimension vectors.

Acknowledgements

This research was partially supported by RFBR, grant 15-01-08847.

Acknowledgements: The research was partially supported by RFBR (grant No. 15-01-08847).

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