

# Some Researches regarding Stress Intensity Factors in Crack Closure Problems

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*Abstract:* - In the paper is presented the influence of the most used parameter regarding the cyclic fatigue life Stress Intensity Factor in the field of Linear Elastic Fracture Mechanics. Also, the researches of this study are focused on the crack closure influence on modifying the stress factors of effective stress that conduct to the increasing the material cyclic fatigue life. Using the orthotropic environments by means of the BEM method, it is shown the definition of the contact problem with small distortions and displacements and will be demonstrated its capabilities carrying important advantages in front of other means in this domain, especially in terms of studying crack closure problems.

*Key-Words:* - stress intensity factor, closure, cyclic fatigue, contact problem.

## 1 Introduction

The rising expectations in the design of mechanical elements generate a need to incorporate, in more accurate ways, aspects that were previously solely approximated, or not even taken into consideration. Such is the case of the crack fatigue and propagation problems, both relevant when estimating the lifespan of a mechanical element that is subject to alternating loads, or that has initial cracks of certain extension.

In Linear Elastic Fracture Mechanics (LEFM), the most used parameter in terms of determining the cyclic fatigue life or the unstable nature of a process of monotonic loads is the Stress Intensity Factor (SIF). Many work studies are dedicated to the presentation of this parameter's values in different situations and to the specific programs developed in order to obtain it both in finite elements and in boundary elements.

However, the majority of such studies focus on cases in which the crack lips are almost completely open and smooth, respectively with a null crack friction coefficient. This case, that can result very relevant when it comes to predominant one mode problems or in metals, becomes less relevant in mixed mode problems, especially in the anisotropic materials and composites. Due to the increasing use of this types of materials – like concrete, and especially fiber

composites – this problem becomes one of unique importance and of great essence, if we take into account (1) the dramatic reduction that the consideration of such factors might lead to for the stress intensity factor and for the predicted cyclic fatigue life, and (2) the possible lack of crack propagation in situations in which a simple calculation of an open crack factor indicated a crack propagation. This is mainly the case of mode II cracks with increased friction between the crack lips. Given that the main purpose of this paper is to study the crack closure influence on modifying the stress factors of effective stress, and consequently increasing the material cyclic fatigue life, it is necessarily to include various aspects that have not been yet taken into consideration in programs of contact mechanics, such as the nodes with possible tensile stress on the contact area, or elements with a  $-1/2$ -type singularity, similar to the ones present at the crack tips.

## 2 Formulation of the BEM in 2-D linear elastically multidomain problems

The first equation of the BEM, in its direct formulation, is the well-known Somigliana's identity,

which expresses the displacement vector  $u_i(Q)$  of a point  $Q$  of a domain  $\Omega$  as a function of the displacements  $u_i(P)$  and tractions  $t_i(P)$  of the

$$C_{ik} u_i(Q) = \int_{\partial\Omega} U_{ik}(Q,P) t_i(P) d\partial\Omega - \int_{\partial\Omega} T_{ik}(Q,P) u_i(P) d\partial\Omega + \int_{\partial\Omega} U_{ik}(Q,P) X_i(P) d\Omega \quad (2.1)$$

where  $U_{ik}$  is the Kelvin fundamental solution of the Navier's equations,  $T_{ik}$  are tractions corresponding to those fundamental solution of the Navier's equations,  $T_{ik}$  are the tractions corresponding to those fundamental solutions (the expressions for the orthotropic case are included in the Appendix), and  $C_{ik}$  can be expressed as:<sup>9</sup>

$$C'_{ik} = \frac{1}{4\pi(1-\nu)} \times \begin{bmatrix} 2(1-\nu)(\pi + \alpha_1 - \alpha_2) \\ + \frac{1}{2(\sin 2\alpha_1 - \sin 2\alpha_2)} \\ \sin^2 \alpha_1 - \sin^2 \alpha_2 \\ 2(1-\nu)(\pi + \alpha_1 - \alpha_2) \\ - \frac{1}{2}(\sin 2\alpha_1 - \sin \alpha_2) \end{bmatrix} \quad (2.2b)$$

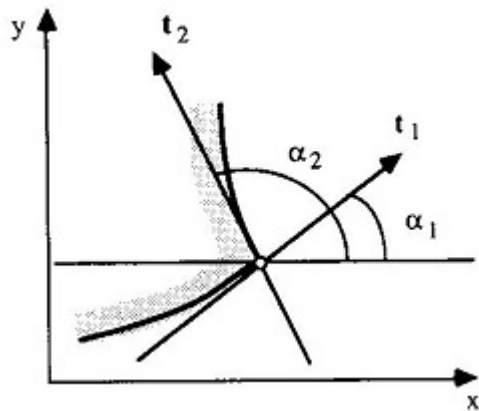


Fig.1 Geometrical mean of  $\alpha_1$  and  $\alpha_2$

where  $U_{ik}$  is the fundamental solution of the Navier equation,  $T_{ik}$  are the tractions corresponding to the mentioned fundamental solution, included in the Appendix on the orthotropic case, and  $C_{ik}$  can be expressed as formula 2 for isotropic materials, where

boundary points of this dominium and the body forces  $X_i$

$$C_{ik} = \begin{cases} \delta_{ik} \rightarrow Q \in \Omega \\ C'_{ik} \rightarrow Q \in \partial\Omega \\ 0 \rightarrow Q \notin QU\partial\Omega \end{cases} \quad (2.2a)$$

with

$\alpha_1$  and  $\alpha_2$  have the geometrical meaning shown in Fig.1,  $\delta_{ik}$  is the Kronecker tensor,  $r$  the radio vector joining the points  $P$  and  $Q$ ,  $n$  the outward normal to the boundary at point  $P$  and  $\nu$  the Poisson coefficient [for plane stress, this value must be modified by the

$$\text{well-known expression } \nu^* = \frac{\nu}{1+\nu}$$

Under some circumstances, the domain integral in (1) can be rewritten as the sum of two boundary integrals, in such a way that is possible to express the displacement of any point of the domain  $\Omega$  in terms of only boundary integrals. In this work, however, no body forces have been considered, hence such integral disappears, and the equation (1) is directly expressed based on the boundary integral function.

If a boundary discretization with  $Ne$  elements is used, and the displacements and tractions are approximated inside each element in terms of nodal values, in the standard form of BEM, as (formula (2.3)

$$\begin{aligned}
 u_i^j &= \sum_{k=1}^{Nnj} (u_i^j)_k \varphi_k \\
 t_i^j &= \sum_{k=1}^{Nnj} (t_i^j)_k \varphi_k \quad (2.3)
 \end{aligned}$$

where  $Nnj$  is the number of nodes of the element  $j$ , and  $\varphi_k$  the shape function for 2-D continuous elements, then the eq. (2.1) can be approximated by

$$C_{ik} u_i(Q) = \sum_{j=1}^{Ne} \int_{\delta Q_j} U_{ik}(Q,P) \left[ \sum_{m=1}^{Nnj} (t_i^j)_m \varphi_m \right] d\delta Q_j - \sum_{j=1}^{Ne} \int_{\delta Q_j} T_{ik}(Q,P) \left[ \sum_{m=1}^{Nnj} (u_{ij})_m \varphi_m \right] d\delta Q_j \quad (2.4)$$

With  $N_{nj}$  the number of nodes of element  $j$ , and  $\varphi_k$  the interpolation functions for 2-D continuous elements, then equation (1) can be approximated by (formula 2.4)

For example, in the case of linear elements (two nodes per element), equation (4) can be rewritten as (formula 2.5a, 2.5b)

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} u_1(k) \\ u_2(k) \end{bmatrix} + \sum_{j=1}^{Ne} \begin{bmatrix} A_{111}^{kj} & A_{211}^{kj} & A_{112}^{kj} & A_{212}^{kj} \\ A_{121}^{kj} & A_{221}^{kj} & A_{122}^{kj} & A_{222}^{kj} \end{bmatrix} \times \begin{bmatrix} (u_i^j)_1 \\ (u_i^j)_2 \\ (u_i^j)_3 \\ (u_i^j)_4 \end{bmatrix} = \sum_{j=1}^{Ne} \begin{bmatrix} B_{111}^{kj} & B_{211}^{kj} & B_{112}^{kj} & B_{212}^{kj} \\ B_{121}^{kj} & B_{221}^{kj} & B_{122}^{kj} & B_{222}^{kj} \end{bmatrix} \begin{bmatrix} (t_i^j)_1 \\ (t_i^j)_2 \\ (t_i^j)_3 \\ (t_i^j)_4 \end{bmatrix} \quad (2.5)$$

If this expression is applied to each of the nodes and the corresponding boundary conditions are also included, it is possible to compute an algebraic linear system with  $[2\sum_j (Nnj - 1)]$  equations and unknowns, corresponding to the displacements and tractions of the boundary nodes.

If the collocation point is not one of the nodes of the element along which the integrals in (5) are computed, a standard Gauss-Legendre quadrature is used. On the other hand, when it is placed from a node inside the adjacent element, singular integrands appear in the integrals of (5). In this case, the constants  $B$  are computed by using a quadrature with logarithmic weight function, while the constants  $A$  are computed, together with the free term  $C_{ik}$ , by imposing a rigid body condition to the studied body.

At each node two equations and six unknowns (two displacements, and two tractions for each of the elements to which the node belongs) can then be established. Most of the times, these tractions are expressed in local coordinates being necessary to transform the traction vector based on these coordinates.

Ultimately, once the coefficient and independent term vector matrix is assembled, and the boundary conditions are applied, an algebraic system is obtained in the form (2.6).

$$K_x = f \quad (2.6)$$

in which the unknowns,  $x$ , correspond to boundary displacements and/or tractions. The solution of this system is performed by any standard method, depending on its size.

Once the unknown displacements and tractions have been obtained, the displacements of any internal point are also obtained by (1), while the stresses may be computed by applying the stress operator to it.

Focusing solely on the contact problem formulation between to elastic solids, with their interface initially in a full contact, and normal for both solids. This is the only case of interest for this context. The non-traction condition for the mentioned point and with the data (tipology of zone) described in Fig. 2 is expressed as

$$u_N \leq 0 \quad (2.7)$$

Where  $u_N$  is the projection of relative displacement between equivalent points (equal to the post-contact position) above normal.

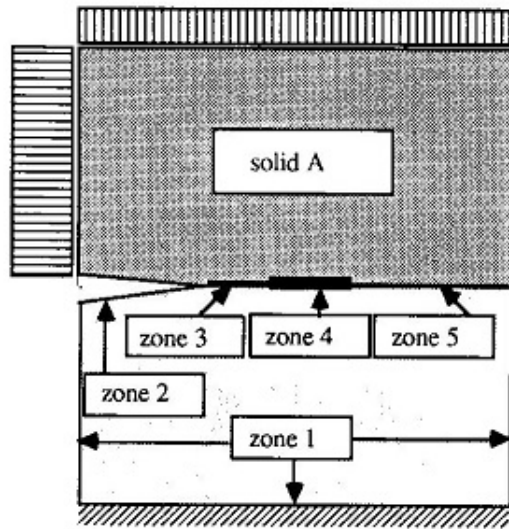


Fig.2 Typology of zone.

The static boundary conditions, in the unilateral case proposed in this work and based on the Coulomb's Law of Friction like the one used here, can be expressed as formula 2.7

Besides that, the compatibility and equilibrium conditions are to be met between the two solids, in the points in which contact has been established. For this, the following different areas are defined in terms of the global boundary of each solid (Fig.2.)

- No contact area (area no. 1) – the area that shall never establish contact
- Candidate to contact area (area no. 2) – the area that still has not established a contact, that might establish one at a specific load level.
- Slip area (area no. 3) –  $|r = \mu\sigma_N|$
- Adherent area (area no. 4) –  $|r < \mu\sigma_N|$
- Welding area (area no. 5) – the contact area in which both solids are considered welded, thus recognizing the traction stresses.

The contact problem between two solids, or better said between two domains of one body, as in this case, consists therefore in approaching the BEM equations to each contact solid, including implicitly or explicitly (in this case the second option was chosen) the boundary conditions (compatibility and equilibrium) in the contact area for each load level, as well as the boundary conditions in the other areas for each one of the aforementioned solids.

### 3 Equation system structure and problem solving, incremental algorithm

Given the non-linear character of the contact problem, and independent of the solving method chosen: incremental, iterative or incremental-iterative, it is necessary to build and solve several times a linear equation system, in order for the major execution time would to match this process. Therefore, it is very important to choose the corresponding algorithm to use in order to reduce this time as much as possible.

When selecting the system's basic unknowns, two possibilities arise. The first one is selecting explicitly the necessary unknowns, in order for the problem to be solved just by merely applying the integral equations. In other words, instance, both the boundary conditions and the compatibility and equilibrium equations in the contact area are included by default without appearing in the final system. In this way, the number of equations is reduced, but it is necessary to proceed with building the constants of integration for each step, given that these basic unknowns alternate in each iteration once the contact conditions modify in all steps. This involves the necessity to archive the corresponding contact area-related constants given that recalculating the latter would result totally inefficient.

The second possibility points out to the selection of a vector with unchangeable unknowns throughout the entire process, so that the only equations to modify would be the contact area conditions, which now get to be included by default. The number of equations increases substantially, especially if the contact area dimension is increased compared to the rest of the boundary – nevertheless, the assembly is very easy due to the contact condition form. Moreover, it is possible from the beginning to identify the unknown vector and the equation order and, finally, no additional archiving is required for the constants of integration, since all of them can be found in the system's matrix.

In conclusion, both proceedings are in fact equivalent in terms of execution time and requested memory, even though the one adopted in this case is easier to schedule in respect to the assembly and a bit more complex than the first one in terms of solving.

The first step to complete (if necessarily), regardless of the chosen process, uses to be the compression of the unknowns belonging to the nodes outside the

contact area. For this, a boundary element standard process must be independently applied for each of the two solids, taking into consideration the following for each of them (formula 3.1)

$$\begin{aligned}
 &K_{LL}^S \cdot x_L^S + K_{LC}^S \cdot x_C^S = V_L^S \\
 &\left[ K_{CC}^S - K_{CL}^S \cdot (K_{LL}^S)^{-1} \cdot K_{LC}^S \right] \cdot x_C^S = V_C^S - K_{CL}^S (K_{LL}^S)^{-1} V_L^S \\
 &\quad (3.1) \\
 &K_{CL}^S \cdot x_L^S + K_{CC}^S \cdot x_C^S = V_C^S
 \end{aligned}$$

with S=A, B;  $x_L$  unknowns to eliminated, and  $x_C$  the corresponding to the candidate to contact zone. Equations (3.1) can also be expressed as

$$\bar{K}^S \cdot x_C^S = \bar{V}^S \quad (3.4)$$

Each one of the  $K^S$  matrixes is a  $2n \times 6n$  with  $n$  the number of nodes of the contact area. In fact, there are two integral equations for each solid, for each node as collocation point, and each node of the contact area with 6 unknowns (2 displacements and 2 tractions in each of its previous and subsequent elements), identified as  $u_1, u_2, \sigma_{ant}, \tau_{ant}, \sigma_{pos}, \tau_{pos}$ . Finally, it is necessary to add to the previous equations 8 equations for each contact node corresponding to the contact conditions (the  $K^{AB}$  matrix) and node type-dependent. For example, for a 44 node, those would be expressed as (formula 3.5)

$$\begin{aligned}
 u_1^A &= u_1^B \quad u_2^A = u_2^B \quad \sigma_{ant}^A = \sigma_{pos}^A \\
 \tau_{ant}^A &= \tau_{pos}^A \quad \sigma_{pos}^B = \sigma_{ant}^B \quad \tau_{pos}^B = \tau_{ant}^B \quad (3.5) \\
 \tau_{ant}^A &= \tau_{pos}^B \quad \tau_{ant}^B = \tau_{pos}^A
 \end{aligned}$$

The implemented structure for the matrix can be seen in Figure 5, with matrixes  $K^A, K^{AB}, K^B$  as the only matrixes archived.

With regard to the solving process, the Gauss elimination process is used, but with a pretriangulation of matrixes  $K^A$  and  $K^B$ , the ones kept unaltered throughout the entire process, with pivot on the rows. This said, in each incremental step, this can be performed by simply solving a very easy and multiple zeroed  $6n \times 6n$  equation system ( $4n \times 4n$  on the first assembly alternative, when the mandatory traction continuity is taken into consideration).

Solving a contact problem with friction requires knowing the history of the entire process, given its irreversible character. This implies the necessity to follow an incremental process for the solution. On the other hand, in a contact problem without friction, with an unknown contact area, an iterative process can be followed for its computation, and in order to establish the contact stress distribution. Finally, in a contact problem without friction and with an a priori known contact area, a single load process enables determining its stress distribution.

The only general procedure is, therefore, an incremental process, the one used in this work. The following selection consists of allowing the user to establish the value of load increment, with an additional iterative process for each increment, in order to compute the new contact area or the slip-adherence changes occurred, or analyze a node-node incremental process with a known contact area, leaving thus the iterative process exclusively for the slip to adherence steps or vice versa, respectively for the friction problems. The latter is much easier than the first, relieves the user from making any choice throughout the incremental process, and the sole bondage consists of establishing consistent discretizations in the candidate-to-contact area of both solids ultimately, given a particular contact area situation, divided into its corresponding areas, a new load increment is applied, during which the load process is considered proportional and can generally be expressed as

$$\Delta Q_n = f_n (Q - Q_{n-1}) \quad (3.6)$$

with  $Q$  the total load to apply at the end of the process,  $Q_{n-1}$  the load applied until then, and  $f_n$  the scale factor to compute, corresponding to the minimum scale factor that modifies the conditions of a particular discretization node, respectively

$$f_n = \min_n^\alpha \quad (3.7)$$

with  $f_n^\alpha$  the scaled factor necessary in order to modify each contact condition  $\alpha$ . Once  $f_n^\alpha$  is determined, the displacements and tractions distribution is equivalent to

$$\Delta u^k = f_n \bar{u}_k \quad \Delta t_k = f_n \bar{t}_k \quad (3.8)$$

$\Delta u^k = f_n \bar{u}_k$  with  $u^k$  and  $t^k$  the displacements and tractions of each node  $k$  obtained by applying the load  $Q - Q_{n-1}$  with the boundary conditions of the  $n - 1$  iteration.

The incompatibility conditions that might occur are the following:

- a. Traction occurrence on adherence and slipping areas. Transition to the corresponding free node.
- b. Slip condition incompatibility (the tangential slip direction and the tangential stress must be opposite). Transition to the adherence node.
- c. Superior tangential stress during friction. The node starts slipping.

## 4 Conclusions

A complete definition of the contact problem with small distortions and displacements was shown between the orthotropic environments by means of the BEM method, and its capabilities have been demonstrated, carrying important advantages in front of other means in this domain, especially in terms of studying crack closure problems.

Including singular elements facilitates in a simple way the determination of SIFs, including in cases of friction and in areas subject to compression, a null value being noticed in such cases, as expected.

It has been confirmed the necessity to incorporate the crack closure effects when establishing the SIFs strengths if such thing occurs, and the great importance of the friction coefficient on the mode II factors. On the other hand, the friction coefficient essentially does not affect the actual crack length and the SIFs in mode I.

No major influence has been noticed with regard to the non-isotropic properties on SIF values, at least when the orthotropic axes are aligned with the crack and its corresponding load, even though the mode II ones alter on high level.

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