On Static Hierarchical Two-Dimensional Models of Thermoelastic Piezoelectric Plates with Variable Thickness

GIA AVALISHVILI[1], MARIAM AVALISHVILI[2]
[1]Faculty of Exact and Natural Sciences
Iv. Javakhishvili Tbilisi State University
3, I. Tchavtchavadze Ave., 0179 Tbilisi, Georgia
[2]School of Informatics, Engineering and Mathematics
University of Georgia
77a, M. Kostava Str., 0175 Tbilisi, Georgia
GEORGIA
[1]gavalish@yahoo.com [2]mavalish@yahoo.com

Abstract: - This paper is devoted to the construction and investigation of a hierarchy of two-dimensional models for thermoelastic piezoelectric plate with variable thickness, which may vanish on a part of the lateral boundary. The hierarchical two-dimensional models are constructed for plate consisting of inhomogeneous anisotropic thermoelastic piezoelectric material with regard to magnetic field, when density of surface force, and normal components of electric displacement, magnetic induction and heat flux vectors are given along the upper and the lower face surfaces of the plate. The boundary value problems corresponding to the constructed static two-dimensional models are investigated in suitable weighted Sobolev spaces. The relationship between the constructed two-dimensional models and the original three-dimensional one is investigated, and the convergence of the sequence of vector-functions of three variables restored from the solutions of the constructed two-dimensional problems to the solution of the original three-dimensional boundary value problem is proved and under additional conditions modeling error estimate is obtained.

Key-Words: - thermo-electro-magneto-elasticity, plates, two-dimensional models, boundary value problem, well-posedness, error estimate

1 Introduction

Piezoelectric materials are widely used to build engineering smart flat panels [1]. Inhomogeneous materials, and in particular, functionally graded materials [2] are used to increase the durability and efficiency of engineering constructions undergoing high mechanical and thermal loads. Therefore, construction and investigation of mathematical models of inhomogeneous thermoelastic piezoelectric plates and shells has attracted increasing attention in recent years.

In this paper, thermoelastic piezoelectric plate with variable thickness, which may vanish on a part of the lateral boundary, consisting of inhomogeneous, in particular, functionally graded, anisotropic material is considered. It should be pointed out that two-dimensional models for inhomogeneous anisotropic thermoelastic piezoelectric plates with regard to magnetic field have not been constructed and investigated. The two-dimensional models are mainly obtained for homogeneous piezoelectric plates.

The method of construction of static two-dimensional models for thermoelastic piezoelectric plate used in the present paper is a generalization and extension of the dimensional reduction method suggested by I. Vekua in the classical theory of elasticity for plates with variable thickness [3]. Note that this method unlike classical methods of construction of two-dimensional models is not based on any a-priori assumptions of mechanical or geometrical nature. The classical Kirchoff-Love and Reissner-Mindlin models can be incorporated into the hierarchy obtained by I. Vekua so that the high order models can be considered as generalizations of the well-known engineering plate models. The boundary value problem corresponding to static two-dimensional model obtained by I. Vekua for linearly elastic shallow shell [4] was investigated in in Sobolev spaces [5], and the relationship between the two-dimensional hierarchical models constructed in [3] and the three-dimensional one in static case first was studied in the spaces of classical regular functions in the paper [6]. Later on, various static and dynamical hierarchical models for plates, shells, bars and multistructures were constructed and investigated applying Vekua's reduction method and its generalizations (see [7-15] and references given therein).

In Section 2, by applying variational approach we investigate in Sobolev spaces boundary value problem corresponding to the linear static three-dimensional model of the plate, when on certain parts of the boundary density of surface force, and normal components of electric displacement, magnetic induction and heat flux vectors are given, and on the remaining parts of the boundary displacement vector, temperature, electric and magnetic potentials vanish.

In Section 3, we construct two-dimensional hierarchical models of plate, when density of surface force, and normal components of electric displacement, magnetic induction and heat flux vectors are given along the upper and the lower face surfaces of the plate. The subspaces of the space corresponding to the three-dimensional problem are constructed, which consist of vector-functions, whose components are polynomials with respect to the variable of plate thickness. Note that the constructed subspaces are weighted Sobolev spaces of vector-functions defined on two-dimensional domain, when the thickness of the plate vanishes on a part of the lateral boundary. By projecting the three-dimensional problem on the constructed subspaces hierarchies of static two-dimensional models of the plate are obtained. The constructed two-dimensional models are investigated in suitable spaces, and the existence and uniqueness of the solution of the corresponding boundary value problems are proved. The relationship between the constructed two-dimensional models and the original one is investigated, and it is proved that the sequence of vector-functions of three variables restored from the solutions of the two-dimensional problems converges in the corresponding function spaces to the exact solution of the three-dimensional boundary value problem and under additional regularity conditions modeling error estimate is obtained.

2 Three-Dimensional Model

Throughout this article we denote by $W^{r,2}(D)=H^r(D)$ and $H^r(\hat{\Gamma})$, $r\geq 1$, $r\in\mathbf{R}$, the Sobolev spaces of order r based on the spaces $H^0(D)=L^2(D)$ and $H^0(\hat{\Gamma})=L^2(\hat{\Gamma})$ of square-integrable functions, respectively, where $D\subset\mathbf{R}^p$, $p\in\mathbf{N}$, is a bounded domain with Lipschitz boundary [16] and $\hat{\Gamma}\subset\partial D$ is a Lipschitz surface. We denote by $\mathbf{H}^s(D)=[H^s(D)]^3$, $\mathbf{L}^s(D)=[L^s(D)]^3$, $\mathbf{H}^s(\hat{\Gamma})=[H^s(\hat{\Gamma})]^3$, $\mathbf{L}^s(\hat{\Gamma})=[L^s(\hat{\Gamma})]^3$, $s\geq 1$, $s\in\mathbf{R}$, the corresponding spaces of vector-valued functions. The trace operators we denote by $tr_{\hat{\Gamma}}$: $H^1(D)\to H^{1/2}(\hat{\Gamma})$ and $\mathbf{tr}_{\hat{\Gamma}}:\mathbf{H}^1(D)\to\mathbf{H}^{1/2}(\hat{\Gamma})$.

Let us consider a thermoelastic piezoelectric plate with variable thickness, which may vanish on a part of its boundary, i.e. body with initial configuration $\overline{\Omega}\subset\mathbf{R}^3$, where $\Omega\subset\mathbf{R}^3$ is a bounded domain with Lipschitz boundary $\partial\Omega$ of the following form

$$\Omega=\{x=(x_1,x_2,x_3)\in\mathbf{R}^3;h^-(x_1,x_2)<x_3<h^+(x_1,x_2),\quad (x_1,x_2)\in\omega\},$$

where $\omega\subset\mathbf{R}^2$ is a bounded domain with Lipschitz boundary $\partial\omega$, $h^\pm\in C^0(\overline{\omega})\cap C^{0,1}_{loc}(\omega)$ are continuous on $\overline{\omega}$ and Lipschitz continuous in the interior of the domain ω, $h^+(x_1,x_2)>h^-(x_1,x_2)$, for $(x_1,x_2)\in\omega\cup\tilde{\gamma}$, $\tilde{\gamma}\subset\partial\omega$ is a Lipschitz curve, $h^+(x_1,x_2)=h^-(x_1,x_2)$, for $(x_1,x_2)\in\partial\omega\setminus\tilde{\gamma}$. We denote by Γ^+ and Γ^- the upper and the lower face surfaces of the plate, which are defined by the equations $x_3=h^+(x_1,x_2)$ and $x_3=h^-(x_1,x_2)$, $(x_1,x_2)\in\omega$, respectively, and the lateral boundary, where the thickness of $\overline{\Omega}$ is positive, we denote by

$$\tilde{\Gamma}=\partial\Omega\setminus(\overline{\Gamma^+\cup\Gamma^-})=\{x\in\mathbf{R}^3;h^-(x_1,x_2)<x_3<h^+(x_1,x_2),(x_1,x_2)\in\tilde{\gamma}\}.$$

We assume that the plate is clamped along a part $\Gamma_0=\{x\in\tilde{\Gamma};(x_1,x_2)\in\tilde{\gamma}_0\}$ of the lateral boundary $\tilde{\Gamma}$, $\tilde{\gamma}_0\subset\tilde{\gamma}$ is a Lipschitz curve, and on the remaining part $\Gamma_1=\Gamma\setminus\overline{\Gamma_0}$ surface force with density $\mathbf{g}=(g_i):\Gamma_1\to\mathbf{R}^3$ is given; electric potential φ vanishes along $\Gamma_0^\varphi=\{x\in\tilde{\Gamma};(x_1,x_2)\in\tilde{\gamma}_0^\varphi\}$ of the lateral boundary $\tilde{\Gamma}$, $\tilde{\gamma}_0^\varphi\subset\tilde{\gamma}$ is a Lipschitz curve, and on the remaining part $\Gamma_1^\varphi=\Gamma\setminus\overline{\Gamma_0^\varphi}$ of the boundary the normal component of the electric displacement with density $g^\varphi:\Gamma_1^\varphi\to\mathbf{R}$ is given; magnetic potential ψ vanishes along $\Gamma_0^\psi=\{x\in\tilde{\Gamma};(x_1,x_2)\in\tilde{\gamma}_0^\psi\}$ of the lateral boundary $\tilde{\Gamma}$, $\tilde{\gamma}_0^\psi\subset\tilde{\gamma}$ is a Lipschitz curve, and on the remaining part $\Gamma_1^\psi=\Gamma\setminus\overline{\Gamma_0^\psi}$ of the boundary the normal component of the magnetic induction with density $g^\psi:\Gamma_1^\psi\to\mathbf{R}$ is given;

temperature θ vanishes along $\Gamma_0^\theta = \{x \in \tilde{\Gamma};\ (x_1, x_2) \in \tilde{\gamma}_0^\theta\}$ of the lateral boundary $\tilde{\Gamma}$, $\tilde{\gamma}_0^\theta \subset \tilde{\gamma}$ is a Lipschitz curve, and on the remaining part $\Gamma_1^\theta = \Gamma \setminus \overline{\Gamma_0^\theta}$ of the boundary the normal component of heat flux with density $g^\theta : \Gamma_1^\theta \to \mathbf{R}$ is given.

The static three-dimensional model of the thermoelastic piezoelectric plate Ω with regard to magnetic field in differential form is given by the following boundary value problem [17, 18] for system of partial differential equations:

$$-\sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j} = f_i \quad \text{in } \Omega,\ i = 1,2,3, \quad (1)$$

$$\sum_{j=1}^{3} \frac{\partial D_j}{\partial x_j} = f^\varepsilon \quad \text{in } \Omega, \quad (2)$$

$$\sum_{j=1}^{3} \frac{\partial B_j}{\partial x_j} = 0 \quad \text{in } \Omega, \quad (3)$$

$$-\sum_{i,j=1}^{3} \frac{\partial}{\partial x_i}\left(\eta_{ij} \frac{\partial \theta}{\partial x_j} \right) = f^\theta \quad \text{in } \Omega, \quad (4)$$

$$\mathbf{u} = \mathbf{0} \ \text{ on } \Gamma_0, \ \sum_{j=1}^{3} \sigma_{ij} n_j = g_i \ \text{ on } \Gamma_1, i = 1,2,3, \ (5)$$

$$\varphi = 0 \ \text{ on } \Gamma_0^\varphi, \quad \sum_{i=1}^{3} D_i n_i = g^\varphi \ \text{ on } \Gamma_1^\varphi, \quad (6)$$

$$\psi = 0 \ \text{ on } \Gamma_0^\psi, \quad \sum_{i=1}^{3} B_i n_i = g^\psi \ \text{ on } \Gamma_1^\psi, \quad (7)$$

$$\theta = 0 \ \text{ on } \Gamma_0^\theta, \ -\sum_{i,j=1}^{3} \eta_{ij} \frac{\partial \theta}{\partial x_j} n_i = g^\theta \ \text{ on } \Gamma_1^\theta, \ (8)$$

where $\mathbf{n} = (n_i)_{i=1}^3$ is the unit outward normal vector to Γ, $\mathbf{u} = (u_i) : \Omega \to \mathbf{R}^3$ is the displacement vector-function, $\varphi : \Omega \to \mathbf{R}$ and $\psi : \Omega \to \mathbf{R}$ stand for the electric and magnetic potentials such that electric and magnetic fields $\mathbf{E}$ and $\mathbf{H}$ are gradients of φ and ψ, respectively, $\mathbf{E} = -\mathrm{grad}\varphi$, $\mathbf{H} = -\mathrm{grad}\psi$, $\theta : \Omega \to \mathbf{R}$ is the temperature distribution, $\mathbf{f} = (f_i)_{i=1}^3 : \Omega \to \mathbf{R}^3$ is the density of applied body force, $f^\varepsilon : \Omega \to \mathbf{R}$ is the density of electric charges, and $f^\theta : \Omega \to \mathbf{R}$ is the density of heat sources, $(\sigma_{ij})_{i,j=1}^3$ is the mechanical stress tensor, $\mathbf{D} = (D_j)_{j=1}^3$ is the electric displacement vector, and $\mathbf{B} = (B_j)_{j=1}^3$ is the magnetic induction vector, which are given by the following constitutive equations:

$$\sigma_{ij} = \sum_{p,q=1}^{3} c_{ijpq} e_{pq}(\mathbf{u}) + \sum_{p=1}^{3} \varepsilon_{pij} \frac{\partial \varphi}{\partial x_p} + \sum_{p=1}^{3} b_{pij} \frac{\partial \psi}{\partial x_p} - \lambda_{ij} \theta, \quad i, j = 1,2,3,$$

$$D_i = \sum_{p,q=1}^{3} \varepsilon_{ipq} e_{pq}(\mathbf{u}) - \sum_{j=1}^{3} d_{ij} \frac{\partial \varphi}{\partial x_j} - \sum_{j=1}^{3} a_{ij} \frac{\partial \psi}{\partial x_j} + \mu_i \theta, \quad i = 1,2,3,$$

$$B_i = \sum_{p,q=1}^{3} b_{ipq} e_{pq}(\mathbf{u}) - \sum_{j=1}^{3} a_{ij} \frac{\partial \varphi}{\partial x_j} - \sum_{j=1}^{3} \zeta_{ij} \frac{\partial \psi}{\partial x_j} + m_i \theta, \quad i = 1,2,3,$$

where $e_{ij}(\mathbf{v}) = 1/2\left(\partial v_i/\partial x_j + \partial v_j/\partial x_i\right)$, $i, j = 1,2,3$, $\mathbf{v} = (v_i)_{i=1}^3$, is the strain tensor, $(c_{ijpq})_{i,j,p,q=1}^3$ is the elasticity tensor, $(\varepsilon_{pij})_{i,j,p=1}^3$ are piezoelectric and $(b_{pij})_{i,j,p=1}^3$ are piezomagnetic coefficients, $(\lambda_{ij})_{i,j=1}^3$ is the stress-temperature tensor, $(d_{ij})_{i,j=1}^3$ and $(\zeta_{ij})_{i,j=1}^3$ are the permittivity and permeability tensors, $(a_{ij})_{i,j=1}^3$ are the coupling coefficients connecting electric and magnetic fields, $(\mu_i)_{i=1}^3$ and $(m_i)_{i=1}^3$ are coefficients characterizing the relation between thermal, electric and magnetic fields, $(\eta_{ij})_{i,j=1}^3$ is the thermal conductivity tensor.

We assume that the elasticity tensor, piezoelectric and piezomagnetic coefficients, and the stress-temperature tensor satisfy the following symmetry conditions

$$\begin{aligned} &c_{ijpq} = c_{ijqp} = c_{jipq}, \quad \varepsilon_{pij} = \varepsilon_{pji}, \\ &b_{pij} = b_{pji}, \quad \lambda_{ij} = \lambda_{ji}, \quad i, j, p, q = 1,2,3. \end{aligned} \quad (9)$$

By multiplying the equations (1) by arbitrary continuously differentiable functions $v_i : \overline{\Omega} \to \mathbf{R}$ $(i = 1,2,3)$, which vanish on Γ_0, the equation (2) by a continuously differentiable function $\overline{\varphi} : \overline{\Omega} \to \mathbf{R}$, such that $\overline{\varphi} = 0$ on Γ_0^φ, the equation (3) by a continuously differentiable function $\overline{\psi} : \overline{\Omega} \to \mathbf{R}$, which vanishes on Γ_0^ψ, and the equation (4) by a continuously differentiable function $\overline{\theta} : \overline{\Omega} \to \mathbf{R}$, such that $\overline{\theta} = 0$ on Γ_0^θ, by integrating on Ω, using Green's formula, and taking into account constitutive equations for $(\sigma_{ij})_{i,j=1}^{3}$, $\mathbf{D}$, $\mathbf{B}$ and the symmetry conditions (9) we obtain the following integral equations:

$$c(\mathbf{u},\mathbf{v})+\varepsilon(\varphi,\mathbf{v})+b(\psi,\mathbf{v})-\lambda(\theta,\mathbf{v})=L^{\mathbf{u}}(\mathbf{v}), \tag{10}$$

$$-\varepsilon(\overline{\varphi},\mathbf{u})+d(\varphi,\overline{\varphi})+a(\psi,\overline{\varphi}) -\mu(\theta,\overline{\varphi})=L^{\varphi}(\overline{\varphi}), \tag{11}$$

$$-b(\overline{\psi},\mathbf{u})+a(\varphi,\overline{\psi})+\zeta(\psi,\overline{\psi}) -m(\theta,\overline{\psi})=L^{\psi}(\overline{\psi}), \tag{12}$$

$$\eta(\theta,\overline{\theta})=L^{\theta}(\overline{\theta}), \tag{13}$$

where

$$c(\mathbf{u},\mathbf{v})=\int_{\Omega}\sum_{i,j,p,q=1}^{3}c_{ijpq}e_{pq}(\mathbf{u})e_{ij}(\mathbf{v})dx,$$

$$\varepsilon(\varphi,\mathbf{v})=\int_{\Omega}\sum_{i,j,p=1}^{3}\varepsilon_{pij}\frac{\partial\varphi}{\partial x_p}e_{ij}(\mathbf{v})dx,$$

$$b(\psi,\mathbf{v})=\int_{\Omega}\sum_{i,j,p=1}^{3}b_{pij}\frac{\partial\psi}{\partial x_p}e_{ij}(\mathbf{v})dx,$$

$$\lambda(\theta,\mathbf{v})=\int_{\Omega}\sum_{i,j=1}^{3}\lambda_{ij}\theta e_{ij}(\mathbf{v})dx,$$

$$d(\varphi,\overline{\varphi})=\int_{\Omega}\sum_{i,j=1}^{3}d_{ij}\frac{\partial\varphi}{\partial x_j}\frac{\partial\overline{\varphi}}{\partial x_i}dx,$$

$$a(\psi,\overline{\varphi})=\int_{\Omega}\sum_{i,j=1}^{3}a_{ij}\frac{\partial\psi}{\partial x_j}\frac{\partial\overline{\varphi}}{\partial x_i}dx,$$

$$\mu(\theta,\overline{\varphi})=\int_{\Omega}\sum_{i=1}^{3}\mu_i\theta\frac{\partial\overline{\varphi}}{\partial x_i}dx,$$

$$\zeta(\psi,\overline{\psi})=\int_{\Omega}\sum_{i,j=1}^{3}\zeta_{ij}\frac{\partial\psi}{\partial x_j}\frac{\partial\overline{\psi}}{\partial x_i}dx,$$

$$m(\theta,\overline{\psi})=\int_{\Omega}\sum_{i=1}^{3}m_i\theta\frac{\partial\overline{\psi}}{\partial x_i}dx,$$

$$\eta(\theta,\overline{\theta})=\int_{\Omega}\sum_{i,j=1}^{3}\eta_{ij}\frac{\partial\theta}{\partial x_j}\frac{\partial\overline{\theta}}{\partial x_i}dx,$$

$$L^{\mathbf{u}}(\mathbf{v})=\int_{\Omega}\sum_{i=1}^{3}f_iv_idx+\int_{\Gamma_1}\sum_{i=1}^{3}g_i tr_{\Gamma_1}(v_i)d\Gamma,$$

$$L^{\varphi}(\overline{\varphi})=\int_{\Omega}f^{\varepsilon}\overline{\varphi}dx-\int_{\Gamma_1^{\varphi}}g^{\varphi}tr_{\Gamma_1^{\varphi}}(\overline{\varphi})d\Gamma,$$

$$L^{\psi}(\overline{\psi})=-\int_{\Gamma_1^{\psi}}g^{\psi}tr_{\Gamma_1^{\psi}}(\overline{\psi})d\Gamma,$$

$$L^{\theta}(\overline{\theta})=\int_{\Omega}f^{\theta}\overline{\theta}dx-\int_{\Gamma_1^{\theta}}g^{\theta}tr_{\Gamma_1^{\theta}}(\overline{\theta})d\Gamma.$$

Note that the integral equations (10)-(13) are equivalent to the boundary value problem (1)-(8) in the spaces of twice continuously differentiable functions. Because functions $\mathbf{v}$, $\overline{\varphi}$ and $\overline{\psi}$ are independent of each other on the basis of the integral equations we obtain the following variational formulation of the boundary value problem (1)-(8): Find

$$\mathbf{u}\in\mathbf{V}(\Omega)=\{\mathbf{v}\in\mathbf{H}^1(\Omega);\ \mathbf{tr}_{\Gamma}(\mathbf{v})=\mathbf{0}\ \text{ on }\ \Gamma_0\},$$

$$\varphi\in V^{\varphi}(\Omega)=\{\overline{\varphi}\in H^1(\Omega);\ tr_{\Gamma}(\overline{\varphi})=0\ \text{ on }\ \Gamma_0^{\varphi}\},$$

$$\psi\in V^{\psi}(\Omega)=\{\overline{\psi}\in H^1(\Omega);\ tr_{\Gamma}(\overline{\psi})=0\ \text{ on }\ \Gamma_0^{\psi}\},$$

$$\theta\in V^{\theta}(\Omega)=\{\overline{\theta}\in H^1(\Omega);\ tr_{\Gamma}(\overline{\theta})=0\ \text{ on }\ \Gamma_0^{\theta}\},$$

which satify the equations

$$A((\mathbf{u},\varphi,\psi),(\mathbf{v},\overline{\varphi},\overline{\psi}))=L(\mathbf{v},\overline{\varphi},\overline{\psi}) +\lambda(\theta,\mathbf{v})+\mu(\theta,\overline{\varphi})+m(\theta,\overline{\psi}), \tag{14}$$

$$\eta(\theta,\overline{\theta})=L^{\theta}(\overline{\theta}), \tag{15}$$

for all $\mathbf{v}\in\mathbf{V}(\Omega)$, $\overline{\varphi}\in V^{\varphi}(\Omega)$, $\overline{\psi}\in V^{\psi}(\Omega)$, $\overline{\theta}\in V^{\theta}(\Omega)$, where

$$\begin{aligned}A((\mathbf{u},\varphi,\psi),(\mathbf{v},\overline{\varphi},\overline{\psi}))&=c(\mathbf{u},\mathbf{v})+d(\varphi,\overline{\varphi})\\&+a(\psi,\overline{\varphi})+a(\varphi,\overline{\psi})+\zeta(\psi,\overline{\psi})+\varepsilon(\varphi,\mathbf{v})\\&-\varepsilon(\overline{\varphi},\mathbf{u})+b(\psi,\mathbf{v})-b(\overline{\psi},\mathbf{u}),\end{aligned}$$

$$L(\mathbf{v},\overline{\varphi},\overline{\psi})=L^{\mathbf{u}}(\mathbf{v})+L^{\varphi}(\overline{\varphi})+L^{\psi}(\overline{\psi}).$$

For the problem (14), (15), which is equivalent to the boundary value problem (1)-(8) in the spaces of classical smooth enough function, we have the following existence, uniqueness and continuous dependence theorem.

Theorem 1. *Suppose that $\Omega\subset\mathbf{R}^3$ is a bounded domain with Lipschitz boundary, the curves $\tilde{\gamma}_0,\tilde{\gamma}_0^{\varphi}$, $\tilde{\gamma}_0^{\psi},\tilde{\gamma}_0^{\theta}$ have positive lengths and the parameters characterizing thermo-mechanical and electro-magnetic properties of the body Ω are such that c_{ijpq}, ε_{pij}, b_{pij}, d_{ij}, ζ_{ij}, a_{ij}, λ_{ij}, μ_i, m_i, $\eta_{ij}\in L^{\infty}(\Omega)$, $i,j,p,q=1,2,3$, satisfy symmetry conditions (9) and the following positive definiteness conditions*

$$\sum_{i,j,p,q=1}^{3}c_{ijpq}\xi_{ij}\xi_{pq}\geq\alpha_c\sum_{i,j=1}^{3}(\xi_{ij})^2, \tag{16}$$

$$\sum_{i,j=1}^{3}\eta_{ij}\xi_j\xi_j \geq \alpha_\eta \sum_{i=1}^{3}(\xi_i)^2, \tag{17}$$

$$\sum_{i,j=1}^{3} d_{ij}\xi_j\xi_i + \sum_{i,j=1}^{3} a_{ij}\bar{\xi}_j\xi_i + \sum_{i,j=1}^{3} a_{ij}\xi_j\bar{\xi}_i + \sum_{i,j=1}^{3}\zeta_{ij}\bar{\xi}_j\bar{\xi}_i \geq \alpha\sum_{i=1}^{3}\left((\xi_i)^2+(\bar{\xi}_i)^2\right), \tag{18}$$

for all $\xi_{ij}\in\mathbf{R}$, $\xi_{ij}=\xi_{ji}$, $\xi_i,\bar{\xi}_i\in\mathbf{R}$, *and for almost all* $x\in\Omega$, *where* $\alpha_c,\alpha_\eta,\alpha$ *are positive constants. If* $\mathbf{f}\in\mathbf{L}^{6/5}(\Omega)$, $\mathbf{g}\in\mathbf{L}^{4/3}(\Gamma_1)$, $f^{\varepsilon}\in L^{6/5}(\Omega)$, $g^{\varphi}\in L^{4/3}(\Gamma_1^{\varphi})$, $g^{\psi}\in L^{4/3}(\Gamma_1^{\psi})$, $f^{\theta}\in L^{6/5}(\Omega)$, $g^{\theta}\in L^{4/3}(\Gamma_1^{\theta})$, *then the problem (14), (15) possesses a unique solution* $(\mathbf{u},\varphi,\psi,\theta)\in \mathbf{V}(\Omega)\times V^{\varphi}(\Omega)\times V^{\psi}(\Omega)\times V^{\theta}(\Omega)$ *and the mapping* $(\mathbf{f},\mathbf{g},f^{\varepsilon},g^{\varphi},g^{\psi},f^{\theta},g^{\theta})\to(\mathbf{u},\varphi,\psi,\theta)$ *is linear and continuous from the space* $\mathbf{L}^{6/5}(\Omega)\times\mathbf{L}^{4/3}(\Gamma_1)\times L^{6/5}(\Omega)\times L^{4/3}(\Gamma_1^{\varphi})\times L^{4/3}(\Gamma_1^{\psi})\times L^{6/5}(\Omega)\times L^{4/3}(\Gamma_1^{\theta})$ *to the space* $\mathbf{V}(\Omega)\times V^{\varphi}(\Omega)\times V^{\psi}(\Omega)\times V^{\theta}(\Omega)$.

3 Two-Dimensional Models

In this section we construct and investigate static hierarchical models of the thermoelastic piezoelectric plate with variable thickness, which was considered in Section 2. In order to construct the hierarchy of two-dimensional models let us consider the subspaces $\mathbf{V_N}(\Omega)$ of $\mathbf{V}(\Omega)$, $\mathbf{N}=(N_1,N_2,N_3)$, consisting of vector-functions whose components are polynomials with respect to the variable x_3,

$$\mathbf{v_N}=(v_{\mathbf{N}i}),\ \ v_{\mathbf{N}i}=\sum_{r_i=0}^{N_i}\frac{1}{h}\left(r_i+\frac{1}{2}\right)\overset{r_i}{v}_{\mathbf{N}i}\,P_{r_i}(z),$$

$$\overset{r_i}{v}_{\mathbf{N}i}\in L^2(\omega),\ \ 0\leq r_i\leq N_i,\ \ i=1,2,3,$$

where $z=\dfrac{x_3-\bar{h}}{h}$, $h=\dfrac{h^+-h^-}{2}$, $\bar{h}=\dfrac{h^++h^-}{2}$, P_r is the Legendre polynomial of order $r\in\mathbf{N}\cup\{0\}$. We also consider the subspaces $V_{N_\varphi}^{\varphi}(\Omega)$, $V_{N_\psi}^{\psi}(\Omega)$, and $V_{N_\theta}^{\theta}(\Omega)$ of $V^{\varphi}(\Omega)$, $V^{\psi}(\Omega)$, and $V^{\theta}(\Omega)$, respectively, which consist of the following functions

$$\bar{\varphi}_{N_\varphi}=\sum_{r_\varphi=0}^{N_\varphi}\frac{1}{h}\left(r_\varphi+\frac{1}{2}\right)\overset{r_\varphi}{\bar{\varphi}}_{N_\varphi}\,P_{r_\varphi}(z),\ \overset{r_\varphi}{\bar{\varphi}}_{N_\varphi}\in L^2(\omega),$$

$$\bar{\psi}_{N_\psi}=\sum_{r_\psi=0}^{N_\psi}\frac{1}{h}\left(r_\psi+\frac{1}{2}\right)\overset{r_\psi}{\bar{\psi}}_{N_\psi}\,P_{r_\psi}(z),\ \overset{r_\psi}{\bar{\psi}}_{N_\psi}\in L^2(\omega),$$

$$\bar{\theta}_{N_\theta}=\sum_{r_\theta=0}^{N_\theta}\frac{1}{h}\left(r_\theta+\frac{1}{2}\right)\overset{r_\theta}{\bar{\theta}}_{N_\theta}\,P_{r_\theta}(z),\ \overset{r_\theta}{\bar{\theta}}_{N_\theta}\in L^2(\omega),$$

where $r_\varphi=0,...,N_\varphi$, $r_\psi=0,...,N_\psi$, $r_\theta=0,...,N_\theta$.

Since the functions h^+ and h^- are Lipschitz continuous in ω due to Rademacher's theorem [19], h^+ and h^- are differentiable almost everywhere in ω and $\partial_\alpha h^{\pm}\in L^{\infty}(\omega^*)$ for all subdomains ω^*, $\overline{\omega^*}\subset\omega$, $\alpha=1,2$. Therefore, the positiveness of h in ω implies that for any vector-function $\mathbf{v_N}=(v_{\mathbf{N}i})_{i=1}^3\in\mathbf{V_N}(\Omega)$ the corresponding functions $\overset{r_i}{v}_{\mathbf{N}i}\in H^1(\omega^*)$, for all ω^*, $\overline{\omega^*}\subset\omega$, i.e. $\overset{r_i}{v}_{\mathbf{N}i}\in H^1_{loc}(\omega)$, $0\leq r_i\leq N_i$, $i=1,2,3$. Similarly, for all functions $\bar{\varphi}_{N_\varphi}\in V_{N_\varphi}^{\varphi}(\Omega)$, $\bar{\psi}_{N_\psi}\in V_{N_\psi}^{\psi}(\Omega)$, $\bar{\theta}_{N_\theta}\in V_{N_\theta}^{\theta}(\Omega)$, the functions $\overset{r_\varphi}{\bar{\varphi}}_{N_\varphi}$, $\overset{r_\psi}{\bar{\psi}}_{N_\psi}$, $\overset{r_\theta}{\bar{\theta}}_{N_\theta}$ of two space variables in the expressions of $\bar{\varphi}_{N_\varphi},\bar{\psi}_{N_\psi}$, $\bar{\theta}_{N_\theta}$ belong to $H^1(\omega^*)$, $\overline{\omega^*}\subset\omega$, i.e. $\overset{r_\varphi}{\bar{\varphi}}_{N_\varphi}$, $\overset{r_\psi}{\bar{\psi}}_{N_\psi}$, $\overset{r_\theta}{\bar{\theta}}_{N_\theta}\in H^1_{loc}(\omega)$, $r_\varphi=0,...,N_\varphi$, $r_\psi=0,...,N_\psi$, $r_\theta=0,...,N_\theta$. Moreover, the norms $\|\,.\,\|_{\mathbf{H}^1(\Omega)}$ and $\|\,.\,\|_{H^1(\Omega)}$ in the spaces $\mathbf{H}^1(\Omega)$ and $H^1(\Omega)$ define weighted norms $\|\,.\,\|_*$ and $\|\,.\,\|_{\varphi*}$, $\|\,.\,\|_{\psi*}$, $\|\,.\,\|_{\theta*}$ of vector-functions $\vec{v}_{\mathbf{N}}=(\overset{r_i}{v}_{\mathbf{N}i})\in[H^1_{loc}(\omega)]^{N_{1,2,3}}$, $N_{1,2,3}=N_1+N_2+N_3+3$, and $\vec{\bar{\varphi}}_{N_\varphi}=(\overset{r_\varphi}{\bar{\varphi}}_{N_\varphi})\in[H^1_{loc}(\omega)]^{N_\varphi+1}$, $\vec{\bar{\psi}}_{N_\varphi}=(\overset{r_\psi}{\bar{\psi}}_{N_\psi})\in[H^1_{loc}(\omega)]^{N_\psi+1}$, $\vec{\bar{\theta}}_{N_\theta}=(\overset{r_\theta}{\bar{\theta}}_{N_\theta})\in[H^1_{loc}(\omega)]^{N_\theta+1}$, such that $\|\vec{v}_{\mathbf{N}}\|_*=\|\mathbf{v_N}\|_{\mathbf{H}^1(\Omega)}$ and $\|\vec{\bar{\varphi}}_{N_\varphi}\|_{\varphi*}=\|\bar{\varphi}_{N_\varphi}\|_{H^1(\Omega)}$, $\|\vec{\bar{\psi}}_{N_\psi}\|_{\psi*}=\|\bar{\psi}_{N_\psi}\|_{H^1(\Omega)}$, $\|\vec{\bar{\theta}}_{N_\theta}\|_{\theta*}=\|\bar{\theta}_{N_\theta}\|_{H^1(\Omega)}$. Using the properties of the Legendre polynomials, we can obtain explicit expressions of the norms $\|\,.\,\|_*$ and $\|\,.\,\|_{\varphi*}$, $\|\,.\,\|_{\psi*}$, $\|\,.\,\|_{\theta*}$. In particular, $\|\,.\,\|_*$ is given by the following expression:

$$\|\vec{v}_{\mathbf{N}}\|_*^2=\sum_{i=1}^{3}\sum_{r_i=0}^{N_i}\left(r_i+\frac{1}{2}\right)\left[\left\|\sum_{s_i=r_i}^{N_i}\left(s_i+\frac{1}{2}\right)(1-(-1)^{r_i+s_i})\right.\right.$$

$$h^{-3/2}\overset{s_i}{v}_{\mathbf{N}i}\Bigg\|^2_{L^2(\omega)}+\|h^{-1/2}\overset{r_i}{v}_{\mathbf{N}i}\|^2_{L^2(\omega)}+\sum_{\alpha=1}^{2}\Bigg\|\sum_{s_i=r_i+1}^{N_i}(s_i+\frac{1}{2})$$
$$(\partial_\alpha h^+-(-1)^{r_i+s_i}\partial_\alpha h^-)h^{-3/2}\overset{s_i}{v}_{\mathbf{N}i}-h^{-1/2}\partial_\alpha\overset{r_i}{v}_{\mathbf{N}i}$$
$$+(r_i+1)h^{-3/2}\partial_\alpha h\overset{r_i}{v}_{\mathbf{N}i}\Bigg\|^2_{L^2(\omega)}\Bigg].$$

For components $\overset{r_i}{v}_{\mathbf{N}i}$ and $\overset{r_\varphi}{\overline{\varphi}}_{N_\varphi}$, $\overset{r_\psi}{\overline{\psi}}_{N_\psi}$, $\overset{r_\theta}{\overline{\theta}}_{N_\theta}$ of $\vec{v}_{\mathbf{N}}$ and $\vec{\overline{\varphi}}_{N_\varphi}$, $\vec{\overline{\psi}}_{N_\psi}$, $\vec{\overline{\theta}}_{N_\theta}$, which possess the properties $\|\vec{v}_{\mathbf{N}}\|_*<\infty$ and $\|\vec{\overline{\varphi}}_{N_\varphi}\|_{\varphi*}<\infty$, $\|\vec{\overline{\psi}}_{N_\psi}\|_{\psi*}<\infty$, $\|\vec{\overline{\theta}}_{N_\theta}\|_{\theta*}<\infty$ we can define the traces on $\tilde{\gamma}$. Indeed, the corresponding vector-function of three space variables $\mathbf{v}_{\mathbf{N}}=(v_{\mathbf{N}i})_{i=1}^3$ and functions $\overline{\varphi}_{N_\varphi}$, $\overline{\psi}_{N_\psi}$, $\overline{\theta}_{N_\theta}$ belong to the space $\mathbf{V}_{\mathbf{N}}(\Omega)\subset\mathbf{H}^1(\Omega)$ and $V^\varphi_{N_\varphi}(\Omega),V^\psi_{N_\psi}(\Omega),V^\theta_{N_\theta}(\Omega)\subset H^1(\Omega)$, respectively. Consequently, applying the trace operator $tr:H^1(\Omega)\to H^{1/2}(\Gamma)$, we define the traces of $\overset{r_i}{v}_{\mathbf{N}i}$, $\overset{r_\varphi}{\overline{\varphi}}_{N_\varphi}$, $\overset{r_\psi}{\overline{\psi}}_{N_\psi}$, $\overset{r_\theta}{\overline{\theta}}_{N_\theta}$ on $\tilde{\gamma}$,

$$tr_{\tilde{\gamma}}(\overset{r_i}{v}_{\mathbf{N}i})=\int_{h^-}^{h^+}tr_{\tilde{\Gamma}}(v_{\mathbf{N}i})P_{r_i}(z)dx_3,\ r_i=0,..,N_i,i=1,2,3,$$
$$tr_{\tilde{\gamma}}(\overset{r_\varphi}{\overline{\varphi}}_{N_\varphi})=\int_{h^-}^{h^+}tr_{\tilde{\Gamma}}(\overline{\varphi}_{N_\varphi})P_{r_\varphi}(z)dx_3,\ r_\varphi=0,...,N_\varphi,$$
$$tr_{\tilde{\gamma}}(\overset{r_\psi}{\overline{\psi}}_{N_\psi})=\int_{h^-}^{h^+}tr_{\tilde{\Gamma}}(\overline{\psi}_{N_\psi})P_{r_\psi}(z)dx_3,\ r_\psi=0,...,N_\psi,$$
$$tr_{\tilde{\gamma}}(\overset{r_\theta}{\overline{\theta}}_{N_\theta})=\int_{h^-}^{h^+}tr_{\tilde{\Gamma}}(\overline{\theta}_{N_\theta})P_{r_\theta}(z)dx_3,\ r_\theta=0,...,N_\theta.$$

Since the vector-functions $\mathbf{v}_{\mathbf{N}}\in\mathbf{V}_{\mathbf{N}}(\Omega)$ and the functions $\overline{\varphi}_{N_\varphi}\in V^\varphi_{N_\varphi}(\Omega)$, $\overline{\psi}_{N_\psi}\in V^\psi_{N_\psi}(\Omega)$, $\overline{\theta}_{N_\theta}\in V^\theta_{N_\theta}(\Omega)$ are uniquely defined by the functions $\overset{r_i}{v}_{\mathbf{N}i}$, $\overset{r_\varphi}{\overline{\varphi}}_{N_\varphi}$, $\overset{r_\psi}{\overline{\psi}}_{N_\psi}$, $\overset{r_\theta}{\overline{\theta}}_{N_\theta}$ of two space variables, considering the original three-dimensional problem (14), (15) on these subspaces, we obtain the following hierarchy of two-dimensional boundary value problems: Find $\vec{u}_{\mathbf{N}}\in\vec{V}_{\mathbf{N}}(\omega)$, $\vec{\varphi}_{N_\varphi}\in\vec{V}^\varphi_{N_\varphi}(\omega)$, $\vec{\psi}_{N_\psi}\in\vec{V}^\psi_{N_\psi}(\omega)$, $\vec{\theta}_{N_\theta}\in\vec{V}^\theta_{N_\theta}(\omega)$, which satisfy the equations

$$A_{\mathbf{N}N_\varphi N_\psi}((\vec{u}_{\mathbf{N}},\vec{\varphi}_{N_\varphi},\vec{\psi}_{N_\psi}),(\vec{v}_{\mathbf{N}},\vec{\overline{\varphi}}_{N_\varphi},\vec{\overline{\psi}}_{N_\psi}))$$
$$=L_{\mathbf{N}N_\varphi N_\psi}(\vec{v}_{\mathbf{N}},\vec{\overline{\varphi}}_{N_\varphi},\vec{\overline{\psi}}_{N_\psi})+\lambda_{N_\theta\mathbf{N}}(\vec{\theta}_{N_\theta},\vec{v}_{\mathbf{N}}) \tag{19}$$
$$+\mu_{N_\theta N_\varphi}(\vec{\theta}_{N_\theta},\vec{\overline{\varphi}}_{N_\varphi})+m_{N_\theta N_\psi}(\vec{\theta}_{N_\theta},\vec{\overline{\psi}}_{N_\psi}),$$
$$\eta_{N_\theta}(\vec{\theta}_{N_\theta},\vec{\overline{\theta}}_{N_\theta})=L^\theta_{N_\theta}(\vec{\overline{\theta}}_{N_\theta}), \tag{20}$$

for all $\vec{v}_{\mathbf{N}}\in\vec{V}_{\mathbf{N}}(\omega)$, $\vec{\overline{\varphi}}_{N_\varphi}\in\vec{V}^\varphi_{N_\varphi}(\omega)$, $\vec{\overline{\psi}}_{N_\psi}\in\vec{V}^\psi_{N_\psi}(\omega)$, $\vec{\overline{\theta}}_{N_\theta}\in\vec{V}^\theta_{N_\theta}(\omega)$, where

$$\vec{V}_{\mathbf{N}}(\omega)=\{\vec{v}_{\mathbf{N}}=(\overset{r_i}{v}_{\mathbf{N}i})\in[H^1_{loc}(\omega)]^{N_{1,2,3}};\|\vec{v}_{\mathbf{N}}\|_*<\infty,$$
$$tr_{\tilde{\gamma}}(\overset{r_i}{v}_{\mathbf{N}i})=0\ \text{ on }\tilde{\gamma}_0,r_i=0,...,N_i,i=1,2,3\},$$
$$\vec{V}^\varphi_{N_\varphi}(\omega)=\{\vec{\overline{\varphi}}_{N_\varphi}=(\overset{r_\varphi}{\overline{\varphi}}_{N_\varphi})\in[H^1_{loc}(\omega)]^{N_\varphi+1};$$
$$\left\|\vec{\overline{\varphi}}_{N_\varphi}\right\|_{\varphi*}<\infty,\ tr_{\tilde{\gamma}}(\overset{r_\varphi}{\overline{\varphi}}_{N_\varphi})=0\ \text{ on }\tilde{\gamma}^\varphi_0,r_\varphi=0,...,N_\varphi\},$$
$$\vec{V}^\psi_{N_\psi}(\omega)=\{\vec{\overline{\psi}}_{N_\psi}=(\overset{r_\psi}{\overline{\psi}}_{N_\psi})\in[H^1_{loc}(\omega)]^{N_\psi+1};$$
$$\left\|\vec{\overline{\psi}}_{N_\psi}\right\|_{\psi*}<\infty,\ tr_{\tilde{\gamma}}(\overset{r_\psi}{\overline{\psi}}_{N_\psi})=0\ \text{ on }\tilde{\gamma}^\psi_0,r_\psi=0,...,N_\psi\},$$
$$\vec{V}^\theta_{N_\theta}(\omega)=\{\vec{\overline{\theta}}_{N_\theta}=(\overset{r_\theta}{\overline{\theta}}_{N_\theta})\in[H^1_{loc}(\omega)]^{N_\theta+1};$$
$$\left\|\vec{\overline{\theta}}_{N_\theta}\right\|_{\theta*}<\infty,\ tr_{\tilde{\gamma}}(\overset{r_\theta}{\overline{\theta}}_{N_\theta})=0\ \text{ on }\tilde{\gamma}^\theta_0,r_\theta=0,...,N_\theta\},$$

the bilinear forms $A_{\mathbf{N}N_\varphi N_\psi}(.,.)$, $\lambda_{N_\theta\mathbf{N}}(.,.)$, $\mu_{N_\theta N_\varphi}(.,.)$, $m_{N_\theta N_\psi}(.,.)$ and $\eta_{N_\theta}(.,.)$ are the restrictions of the bilinear forms $A(.,.)$, $\lambda(.,.)$, $\mu(.,.)$, $m(.,.)$ and $\eta(.,.)$, respectively, on the subspaces $\mathbf{V}_{\mathbf{N}}(\Omega)$, $V^\varphi_{N_\varphi}(\Omega)$, $V^\psi_{N_\psi}(\Omega)$, $V^\theta_{N_\theta}(\Omega)$, which are considered as the bilinear forms with respect to the vector-functions $\vec{v}_{\mathbf{N}}$, $\vec{\overline{\varphi}}_{N_\varphi}$, $\vec{\overline{\psi}}_{N_\psi}$, $\vec{\overline{\theta}}_{N_\theta}$ of two variables, i.e.

$$A_{\mathbf{N}N_\varphi N_\psi}((\vec{\tilde{v}}_{\mathbf{N}},\vec{\tilde{\varphi}}_{N_\varphi},\vec{\tilde{\psi}}_{N_\psi}),(\vec{v}_{\mathbf{N}},\vec{\overline{\varphi}}_{N_\varphi},\vec{\overline{\psi}}_{N_\psi}))$$
$$=A_{\mathbf{N}N_\varphi N_\psi}((\tilde{\mathbf{v}}_{\mathbf{N}},\tilde{\varphi}_{N_\varphi},\tilde{\psi}_{N_\psi}),(\mathbf{v}_{\mathbf{N}},\overline{\varphi}_{N_\varphi},\overline{\psi}_{N_\psi})),$$
$$\lambda_{N_\theta\mathbf{N}}(\vec{\overline{\theta}}_{N_\theta},\vec{v}_{\mathbf{N}})=\lambda(\overline{\theta}_{N_\theta},\mathbf{v}_{\mathbf{N}}),$$
$$\mu_{N_\theta N_\varphi}(\vec{\overline{\theta}}_{N_\theta},\vec{\overline{\varphi}}_{N_\varphi})=\mu(\overline{\theta}_{N_\theta},\overline{\varphi}_{N_\varphi}),$$
$$m_{N_\theta N_\psi}(\vec{\overline{\theta}}_{N_\theta},\vec{\overline{\psi}}_{N_\psi})=m(\overline{\theta}_{N_\theta},\overline{\psi}_{N_\psi}),$$
$$\eta_{N_\theta}(\vec{\tilde{\theta}}_{N_\theta},\vec{\overline{\theta}}_{N_\theta})=\eta(\tilde{\theta}_{N_\theta},\overline{\theta}_{N_\theta}),$$

for all vector-functions $\vec{\tilde{v}}_{\mathbf{N}},\vec{v}_{\mathbf{N}}\in\vec{V}_{\mathbf{N}}(\omega)$, $\vec{\tilde{\varphi}}_{N_\varphi},\vec{\overline{\varphi}}_{N_\varphi}\in\vec{V}^\varphi_{N_\varphi}(\omega)$, $\vec{\tilde{\psi}}_{N_\psi},\vec{\overline{\psi}}_{N_\psi}\in\vec{V}^\psi_{N_\psi}(\omega)$, $\vec{\tilde{\theta}}_{N_\theta},\vec{\overline{\theta}}_{N_\theta}\in\vec{V}^\theta_{N_\theta}(\omega)$, corresponding to $\tilde{\mathbf{v}}_{\mathbf{N}}, \mathbf{v}_{\mathbf{N}} \in \mathbf{V}_{\mathbf{N}}(\Omega), \tilde{\varphi}_{N_\varphi}, \bar{\varphi}_{N_\varphi} \in V^{\varphi}_{N_\varphi}(\Omega)$, $\tilde{\psi}_{N_\psi}, \bar{\psi}_{N_\psi} \in V^{\psi}_{N_\psi}(\Omega)$, $\tilde{\theta}_{N_\theta}, \bar{\theta}_{N_\theta} \in V^{\theta}_{N_\theta}(\Omega)$, respectively. The linear forms $L_{\mathbf{N}N_\varphi N_\psi}(.)$ and $L^{\theta}_{N_\theta}(.)$ are defined by the linear forms $L(.)$ and $L^{\theta}(.)$, and they are given by the following expressions:

$$L_{\mathbf{N}N_\varphi N_\psi}(\vec{v}_{\mathbf{N}}, \vec{\bar{\varphi}}_{N_\varphi}, \vec{\bar{\psi}}_{N_\psi}) = \sum_{i=1}^{3}\sum_{r_i=0}^{N_i}\left(r_i+\frac{1}{2}\right)\left[\int_{\omega}\frac{1}{h}\overset{r_i}{v}_{\mathbf{N}i}\left(\overset{r_i}{f}_i + g_i^{+}\lambda_{+} + g_i^{-}\lambda_{-}(-1)^{r_i}\right)d\omega + \int_{\gamma_1}\frac{1}{h}tr_{\tilde{\gamma}}(\overset{r_i}{v}_{\mathbf{N}i})\overset{r_i}{g}_i\, d\gamma_1\right]$$

$$+\sum_{r_\varphi=0}^{N_\varphi}\left(r_\varphi+\frac{1}{2}\right)\left[\int_{\omega}\frac{1}{h}\overset{r_\varphi}{\bar{\varphi}}_{N_\varphi}\left(\overset{r_\varphi}{f}^{\varepsilon} + g^{\varphi+}\lambda_{+} + g^{\varphi-}\lambda_{-}(-1)^{r_\varphi}\right)d\omega + \int_{\gamma_1^{\varphi}}\frac{1}{h}tr_{\tilde{\gamma}}(\overset{r_\varphi}{\bar{\varphi}}_{N_\varphi})\overset{r_\varphi}{g}^{\varphi}\, d\gamma_1^{\varphi}\right]$$

$$+\sum_{r_\psi=0}^{N_\psi}\left(r_\psi+\frac{1}{2}\right)\left[\int_{\omega}\frac{1}{h}\overset{r_\psi}{\bar{\psi}}_{N_\psi}\left(g^{\psi+}\lambda_{+} + g^{\psi-}\lambda_{-}(-1)^{r_\psi}\right)d\omega + \int_{\gamma_1^{\psi}}\frac{1}{h}tr_{\tilde{\gamma}}(\overset{r_\psi}{\bar{\psi}}_{N_\psi})\overset{r_\psi}{g}^{\psi}\, d\gamma_1^{\psi}\right],$$

$$L^{\theta}_{N_\theta}(\vec{\bar{\theta}}_{N_\theta}) = \sum_{r_\theta=0}^{N_\theta}\left(r_\theta+\frac{1}{2}\right)\left[\int_{\omega}\frac{1}{h}\overset{r_\theta}{\bar{\theta}}_{N_\theta}\left(\overset{r_\theta}{f}^{\theta} + g^{\theta+}\lambda_{+} + g^{\theta-}\lambda_{-}(-1)^{r_\theta}\right)d\omega + \int_{\gamma_1^{\theta}}\frac{1}{h}tr_{\tilde{\gamma}}(\overset{r_\theta}{\bar{\theta}}_{N_\theta})\overset{r_\theta}{g}^{\theta}\, d\gamma_1^{\theta}\right],$$

where $\gamma_1 = \tilde{\gamma}\setminus\tilde{\gamma}_0$, $\gamma_1^{\varphi} = \tilde{\gamma}\setminus\tilde{\gamma}_0^{\varphi}$, $\gamma_1^{\psi} = \tilde{\gamma}\setminus\tilde{\gamma}_0^{\psi}$, $\gamma_1^{\theta} = \tilde{\gamma}\setminus\tilde{\gamma}_0^{\theta}$, $\lambda_{\pm} = \sqrt{1+(\partial_1 h^{\pm})^2+(\partial_2 h^{\pm})^2}$, $\overset{r}{v} = \int_{h^-}^{h^+} vP_r(z)dx_3$, for all functions $v \in L^2(\Omega)$, $r \in \mathbf{N}\cup\{0\}$, $g_i^{\pm}, g^{\varphi\pm}$, $g^{\psi\pm}$, $g^{\theta\pm}$ are restrictions of $g_i, g^{\varphi}, g^{\psi}, g^{\theta}$ on the upper Γ^{+} and the lower Γ^{-} faces of the plate, respectively.

Note that the constructed hierarchy of problems (19), (20) are two-dimensional hierarchical models of inhomogeneous anisotropic thermoelastic piezoelectric plate with regard to magnetic field. For the obtained static models (19), (20) the following theorem is valid.

Theorem 2. *Suppose that $\Omega \subset \mathbf{R}^3$ is a bounded domain with Lipschitz boundary, the curves $\tilde{\gamma}_0, \tilde{\gamma}_0^{\varphi}$, $\tilde{\gamma}_0^{\psi}, \tilde{\gamma}_0^{\theta}$ have positive lengths and the parameters characterizing thermo-mechanical and electro-magnetic properties of the body Ω are such that c_{ijpq}, ε_{pij}, b_{pij}, d_{ij}, ζ_{ij}, a_{ij}, λ_{ij}, μ_i, m_i, $\eta_{ij} \in L^{\infty}(\Omega)$, $i, j, p, q = 1,2,3$, they satisfy symmetry conditions (9) and the positive definiteness conditions (16)-(18). If $h^{-1/6}\overset{r_i}{f}_i \in L^{6/5}(\omega)$, $h^{-1/4}\overset{r_i}{g}_i \in L^{4/3}(\gamma_1)$ $(r_i = 0,...,N_i)$, $\lambda_{+}g_i^{+}, \lambda_{-}g_i^{-} \in L^{4/3}(\omega)$ $(i=1,2,3)$, $h^{-1/6}\overset{r_\varphi}{f}^{\varepsilon} \in L^{6/5}(\omega)$, $h^{-1/4}\overset{r_\varphi}{g}^{\varphi} \in L^{4/3}(\gamma_1^{\varphi})$ $(r_\varphi = 0,...,N_\varphi)$, $\lambda_{+}g^{\varphi+}, \lambda_{-}g^{\varphi-} \in L^{4/3}(\omega)$, $h^{-1/4}\overset{r_\psi}{g}^{\psi} \in L^{4/3}(\gamma_1^{\psi})$ $(r_\psi = 0,...,N_\psi)$, $\lambda_{+}g^{\psi+}, \lambda_{-}g^{\psi-} \in L^{4/3}(\omega)$, $h^{-1/6}\overset{r_\theta}{f}^{\theta} \in L^{6/5}(\omega)$, $h^{-1/4}\overset{r_\theta}{g}^{\theta} \in L^{4/3}(\gamma_1^{\theta})$ $(r_\theta = 0,...,N_\theta)$, $\lambda_{+}g^{\theta+}, \lambda_{-}g^{\theta-} \in L^{4/3}(\omega)$, then for each $N_i \in \mathbf{N}\cup\{0\}$ $(i=1,2,3)$, $N_\varphi \in \mathbf{N}\cup\{0\}$, $N_\psi \in \mathbf{N}\cup\{0\}$, $N_\theta \in \mathbf{N}\cup\{0\}$, the two-dimensional problem (19), (20) possesses a unique solution.*

So, we have reduced the three-dimensional boundary value problem (14), (15) to the hierarchy of two-dimensional ones (19), (20) and have investigated the existence and uniqueness of the solution of the obtained problems. In order to justify the constructed two-dimensional models we estimate the difference between the exact solution $(\mathbf{u}, \varphi, \psi, \theta)$ of the three-dimensional problem and the vector-function $(\mathbf{u}_{\mathbf{N}}, \varphi_{N_\varphi}, \psi_{N_\varphi}, \theta_{N_\theta}) \in \mathbf{V}_{\mathbf{N}}(\Omega) \times V^{\varphi}_{N_\varphi}(\Omega) \times V^{\psi}_{N_\psi}(\Omega) \times V^{\theta}_{N_\theta}(\Omega)$, corresponding to the solution $(\vec{u}_{\mathbf{N}}, \vec{\varphi}_{N_\varphi}, \vec{\psi}_{N_\psi}, \vec{\theta}_{N_\theta}) \in \vec{V}_{\mathbf{N}}(\omega) \times \vec{V}^{\varphi}_{N_\varphi}(\omega) \times \vec{V}^{\psi}_{N_\psi}(\omega) \times \vec{V}^{\theta}_{N_\theta}(\omega)$ of the two-dimensional problem (19), (20). Convergence result and an estimate of the rate of convergence are given in the next theorem, but before we formulate it, let us introduce the following anisotropic weighted Sobolev space

$$H^{1,1,s}_{h^{\pm}}(\Omega) = \{v;\, h^{k-1}\partial_3^{k-1}\partial_i^{r} v \in L^2(\Omega),\, h^{k-1}\partial_\alpha h^{\pm}\partial_3^{k} v \in L^2(\Omega),\ 1 \le k \le s,\ r = 0,1,\ i = 1,2,3, \alpha = 1,2\},$$

which is a Hilbert space equipped with the norm

$$\| v \|_{H^{1,1,s}_{h^{\pm}}(\Omega)} = \left(\sum_{k=1}^{s}\left(\sum_{r=0}^{1}\sum_{i=1}^{3}\| h^{k-1}\partial_3^{k-1}\partial_i^{r} v \|_{L^2(\Omega)} + \sum_{\alpha=1}^{2}\| h^{k-1}\partial_\alpha h^{+}\partial_3^{k} v \|_{L^2(\Omega)} + \sum_{\alpha=1}^{2}\| h^{k-1}\partial_\alpha h^{-}\partial_3^{k} v \|_{L^2(\Omega)}\right)\right)^{1/2}.$$

The following theorem is valid.

Theorem 3. *Suppose that* $\Omega \subset \mathbf{R}^3$ *is a bounded domain with Lipschitz boundary, the curves* $\tilde{\gamma}_0, \tilde{\gamma}_0^{\varphi}$, $\tilde{\gamma}_0^{\psi}, \tilde{\gamma}_0^{\theta}$ *have positive lengths, the parameters* c_{ijpq}, ε_{pij}, b_{pij}, d_{ij}, ζ_{ij}, a_{ij}, λ_{ij}, μ_i, m_i, $\eta_{ij} \in L^{\infty}(\Omega)$, $i,j,p,q = 1,2,3$, *satisfy symmetry conditions (9) and the positive definiteness conditions (16)-(18). If* $\mathbf{f} \in \mathbf{L}^{6/5}(\Omega)$, $\mathbf{g} \in \mathbf{L}^{4/3}(\Gamma_1)$, $f^{\varepsilon} \in L^{6/5}(\Omega)$, $g^{\varphi} \in L^{4/3}(\Gamma_1^{\varphi})$, $g^{\psi} \in L^{4/3}(\Gamma_1^{\psi})$, $f^{\theta} \in L^{6/5}(\Omega)$, $g^{\theta} \in L^{4/3}(\Gamma_1^{\theta})$, *then the two-dimensional problem (19), (20) possesses a unique solution and sequence of vector-functions* $(\mathbf{u}_{\mathbf{N}}, \varphi_{N_\varphi}, \psi_{N_\varphi}, \theta_{N_\theta}) \in \mathbf{V}_{\mathbf{N}}(\Omega) \times V_{N_\varphi}^{\varphi}(\Omega) \times V_{N_\psi}^{\psi}(\Omega)$ $\times V_{N_\theta}^{\theta}(\Omega)$, *which correspond to the solutions* $(\vec{u}_{\mathbf{N}}, \vec{\varphi}_{N_\varphi}, \vec{\psi}_{N_\psi}, \vec{\theta}_{N_\theta}) \in \vec{V}_{\mathbf{N}}(\omega) \times \vec{V}_{N_\varphi}^{\varphi}(\omega) \times \vec{V}_{N_\psi}^{\psi}(\omega) \times \vec{V}_{N_\theta}^{\theta}(\omega)$ *of (19), (20), tends to the solution* $(\mathbf{u}, \varphi, \psi, \theta)$ *in the space* $\mathbf{V}(\Omega) \times V^{\varphi}(\Omega) \times V^{\psi}(\Omega) \times V^{\theta}(\Omega)$, *as* $N_{\min} = \min\limits_{1 \le i \le 3}\{N_i, N_\varphi, N_\psi, N_\theta\} \to \infty$. *Moreover, if* $\mathbf{u} \in (H^{1,1,s_{\mathbf{u}}}(\Omega))^3, \varphi \in H^{1,1,s_\varphi}(\Omega), \psi \in H^{1,1,s_\psi}(\Omega)$, $\theta \in H^{1,1,s_\theta}(\Omega)$, $s_{\mathbf{u}}, s_\varphi, s_\psi, s_\theta \in \mathbf{N}$, $s_{\mathbf{u}}, s_\varphi, s_\psi, s_\theta \ge 2$, *then*

$$\| \mathbf{u} - \mathbf{u}_{\mathbf{N}} \|_{\mathbf{H}^1(\Omega)} + \| \varphi - \varphi_{N_\varphi} \|_{H^1(\Omega)} + \| \psi - \psi_{N_\psi} \|_{H^1(\Omega)}$$

$$+ \| \theta - \theta_{N_\theta} \|_{H^1(\Omega)} \le \frac{1}{N_{\min}^{s-1}} \delta(\Omega, \Gamma_0, \Gamma_0^{\varphi}, \Gamma_0^{\psi}, \Gamma_0^{\theta}, \vec{N}),$$

where $\delta(\Omega, \Gamma_0, \Gamma_0^{\varphi}, \Gamma_0^{\psi}, \Gamma_0^{\theta}, \vec{N}) \to 0$, *as* $N_{\min} \to \infty$, $\vec{N} = (\mathbf{N}, N_\varphi, N_\psi, N_\theta)$.

4 Conclusion

We studied boundary value problem with mixed boundary conditions for displacement vector-field, electric and magnetic potentials, and temperature corresponding to the linear dynamical three-dimensional model for inhomogeneous anisotropic thermoelastic piezoelectric plate with variable thickness with regard to magnetic field. We obtained variational formulation of the three-dimensional problem in the corresponding Sobolev spaces, which is equivalent to the original differential formulation in the spaces of sufficiently smooth functions. We formulated theorem on the existence and uniqueness of the solution of the three-dimensional boundary value problem, and the continuous dependence of the solution on the given data in suitable function spaces. We constructed a hierarchy of two-dimensional static models and investigated the existence and uniqueness of solutions of the corresponding boundary value problems in suitable weighted Sobolev spaces. Moreover, we proved that the constructed hierarchical two-dimensional models for thermoelastic piezoelectric plate approximate the original three-dimensional model and obtained estimate of the approximation error. Note that the lower order models of the constructed hierarchy can be used as engineering models of thermoelastic piezoelectric plates.

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