CHART OF SOLUTIONS OF THE TWO DEGREES OF FREEDOM DYNAMICAL SYSTEM OF THE COUPLED NON-LINEAR DOUBLE OSCILLATOR FOR VARIOUS SETS OF PARAMETER VALUES OF ITS POTENTIAL

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Abstract: **-** This paper presents the approximate general solution of the axi-symmetric, two-degree of freedom dynamical system of the coupled non-linear double oscillator corresponding to a third order polynomial potential. The general solution is approximated through a set of initial conditions that generate symmetric periodic solutions and a set of initial conditions that generate escape solutions. We give the chart of initial conditions of solutions for various sets of parameter values A and B representing the coefficients of the second degree terms in x and y, respectively, appearing in the polynomial potential, while we retain constant, e.g. b=0.5, the coefficient of the third degree term $x\,y^2$.

Key-Words: **-** Non-linear double oscillator, escape solutions, third order potential, periodic symmetric solutions.

1 Introduction

The problem which we have treated here is that of the coupled non-linear double oscillator. The potential V of this problem is given by the expression:

$$\mathbf{V} = \frac{1}{2}\left(\mathbf{Ax^2} + \mathbf{By^2}\right) - \mathbf{bxy^2}, \qquad (1)$$

with A, B and b being constants. The equations governing the motion are the following ones:

$$\frac{d^2x}{dt^2} = -Ax + by^2, \quad \frac{d^2y}{dt^2} = -By + 2bxy, \qquad (2)$$

with the expression (3)

$$\frac{1}{2}\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + Ax^2 + By^2\right] - bxy^2 = C,$$

being the integral of its energy.

2 Chart of the Problem Solutions

The solutions of the problem we search here are the periodic solutions which are dense in the set of all solutions. To facilitate the task we select to search those periodic solutions which are symmetric with respect to Ox axis in the plane Oxy. More specifically those periodic orbits according to which the particle, under consideration, starts its motion perpendicularly from Ox axis. Thus, the particle will have a certain initial position x on Ox axis, zero initial position y on Oy axis, zero its initial velocity along Ox axis and a certain initial velocity along Oy axis, which is simply computed as follows: For a given value of energy C, and from the integral of energy of the particle, given by the formula (3), if we put y=0 and dx/dt=0, we obtain

$$dy/dt = \sqrt{2C - Ax^2} \qquad (4)$$

The quantity existing under the symbol of the square root in the second member of the relationship (4) is a positive number in the whole area of the plane (x, C) where the motion is permitted, whereas it is annihilated along the zero velocity curve, which is, obviously, computed by the following relationship:

$$C = \frac{1}{2}\left(Ax^2\right) \quad (5)$$

The search of periodic symmetric solutions of the problem is undertaken into a chosen rectangle of the plane (x, C), say for $x \in [-a, a]$ and for $C \in [0, c]$, with $a > 0$, $c > 0$ the values of which depend on the values of the parameters A and B. This search is realized by means of a fine partition of the energy interval of C and by another finer partition of the position interval of x, and subsequently by sweeping along x first, and next along C. More specifically, we proceed as follows: For a given value of the energy C and for a certain initial value of x we integrate the equations of motion (2) until the particle intersects the Ox axis of the plane Oxy. At this point of intersection the particle will possess a velocity along Ox axis. For the same value of energy C and for the next initial value of x, we integrate the equations of motion, until the particle intersects the Ox axis. If at this point of intersection the particle possesses a velocity with opposite sign along Ox axis compared with the respective one found from the former integration, then this means that there exists an intermediate initial value of x, which, after integration, induces the particle velocity along Ox axis to get the zero value. Therefore an intermediate initial value of x corresponds to a periodic symmetric orbit. The multiplicity of a periodic symmetric orbit is determined by the serial number of the intersection of the orbit with Ox axis where we have "perpendicularity", which means velocity along Ox axis equal to zero. For instance, we have a simple periodic symmetric orbit or a periodic symmetric orbit of multiplicity one, every time whenever we attain "perpendicularity" in the first intersection of the orbit with Ox axis; we have a double periodic symmetric orbit or a periodic symmetric orbit of multiplicity two, every time whenever we attain "perpendicularity" in the second intersection of the orbit with Ox axis, and so on.

During the procedure of searching the initial conditions of periodic symmetric solutions of a predetermined n multiplicity in the plane (x, C) we encounter initial conditions leading to orbits without the n^{th} intersection with Ox axis of the plane Oxy, namely escape solutions without n^{th} intersection with Ox axis [1]. In the charts presented below for several values of parameters A and B (the value of parameter b being 0.5) we give, on the one hand, the initial conditions leading to periodic symmetric orbits and on the other hand, the initial conditions leading to escape solutions. The initial conditions leading to escape solutions occupy a large part of the area of permissible motion in the plane (x, C), whereas the remaining part of this area is occupied by initial conditions leading to periodic symmetric solutions. The initial conditions leading to periodic symmetric solutions occupy the region of order for specific values of the parameters A and B or, additionally, regions of chaos for different values of the parameters A and B. On the plane (x, C), for a specific value of C, if we say that we have order or chaos we mean that on the plane $(x, \dot{x})$ for the same value of C the invariant curves present respectively an image of order or chaos.

Example: In the chart of Figure 1 the pixels in magenta color, having as abscissas x = -0.9, -0.66, -0.57, -0.5, -0.2, -0.15, -0.1, 0.32, 0.72, 0.9 and ordinate C=1.2, determine the boundaries of regions named as CHAOS, because the same exactly pixels in Figure 2 demarcate regions inside which the invariant curves present an image of chaos. However, on the plane (x, C) we have the possibility to penetrate into the content of the region CHAOS by computing periodic symmetric solutions of smaller multiplicity.

Example: In Figure 1, in the rectangle in magenta color, into the region CHAOS, we have computed the initial conditions of periodic symmetric solutions of multiplicity 10 which appear in red color. By contrast, on the chart, in general, the initial conditions appearing correspond to periodic symmetric solutions of larger multiplicity, that is 90.

In Figure 1 the initial conditions leading to escape solutions without first, second, third, fourth, fifth, sixth, seventh, eighth, ninth, 90th intersection with Ox axis appear in red, green, blue, cyan, magenta, yellow, navy, purple, wine, dark yellow color, respectively. Additionally, the region ORDER and the regions CHAOS appear to consist of the initial conditions of periodic symmetric solutions, whereas the initial conditions of simple solutions appear in green color. In Table 1 we give the results of computation of the initial conditions of some periodic symmetric solutions appearing in Figure 1. We denote by X01 the initial position of the particle, by CINT the initial constant of energy, by VOUT(1) the final position at half period, by CT the final constant of energy at half period and by TEND the time at half period.

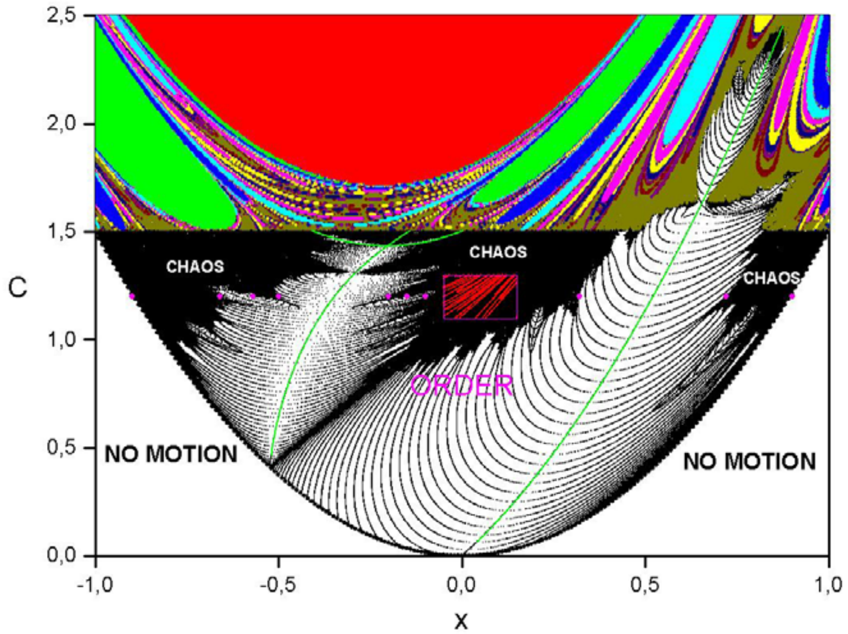

Fig. 1: Chart for A=3, B=1

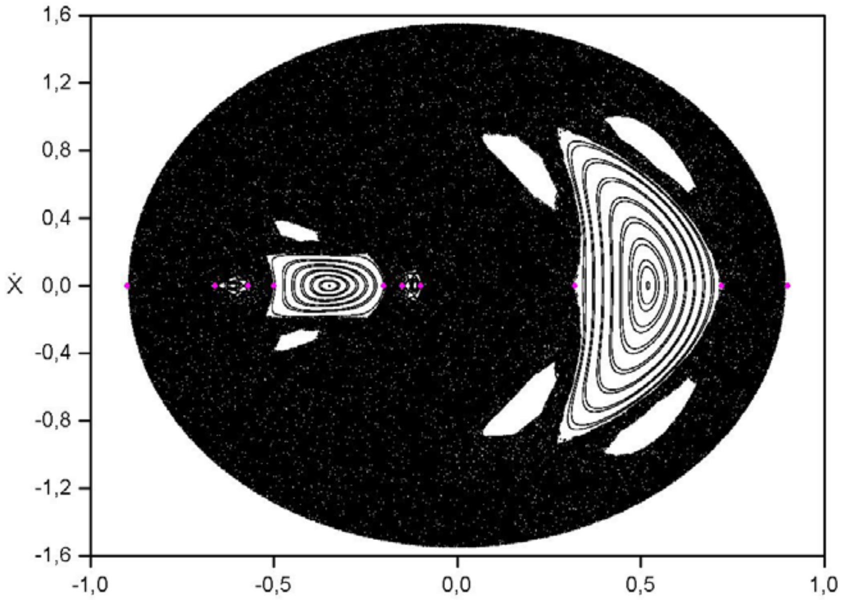

Fig. 2: Invariant curves for A=3, B=1 and C=1.2

Table 1

X01	CINT	VOUT(1)	CT	TEND
0,001	0,001	0,00096	0,001	282,7314
0,002	0,002	0,0019	0,002	282,7195
0,002	0,003	0,002	0,003	282,7081
0,003	0,004	0,0029	0,004	282,6962
-0,056	0,005	-0,056	0,005	282,9642
0,004	0,005	0,00376	0,005	282,6843
0,057	0,005	0,05698	0,005	282,9728
-0,058	0,006	-0,05798	0,006	282,9786
0,06	0,006	0,06	0,006	282,9873
-0,061	0,007	-0,06099	0,007	283,0015
0,005	0,007	0,00481	0,007	282,6613
0,064	0,007	0,06396	0,007	283,0192
-0,064	0,008	-0,064	0,008	283,0257
0,006	0,008	0,00557	0,008	282,6494
0,067	0,008	0,06697	0,008	283,038
-0,066	0,009	-0,06599	0,009	283,043
0,006	0,009	0,00597	0,009	282,6385
0,07	0,009	0,06997	0,009	283,0587
-0,069	0,01	-0,069	0,01	283,0694
0,007	0,01	0,00669	0,01	282,6267
0,073	0,01	0,07296	0,01	283,0814

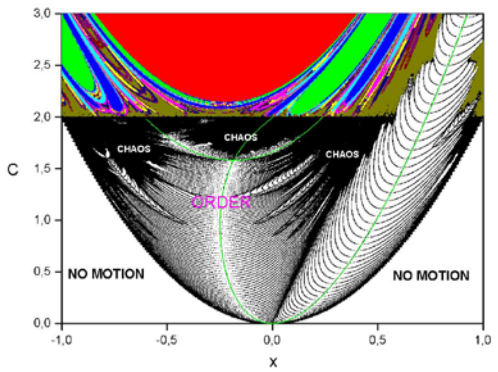

Fig. 3: Chart for A=4, B=1

Subsequently we increase gradually the value for the parameter A, while the parameter B conserves the same value (B=1). In Figures 3, 4, 5 and 6, on the plane (x, C) we present the initial conditions of periodic and symmetric solutions and those of escape solutions for values of A and B as they are referred to in the captions. Especially, in the charts of Figures 3 and 4 the initial conditions of periodic and symmetric solutions of multiplicity 90 have been computed, whereas in the charts of Figures 5 and 6 the initial conditions of periodic and symmetric solutions of multiplicity 60 have been computed.

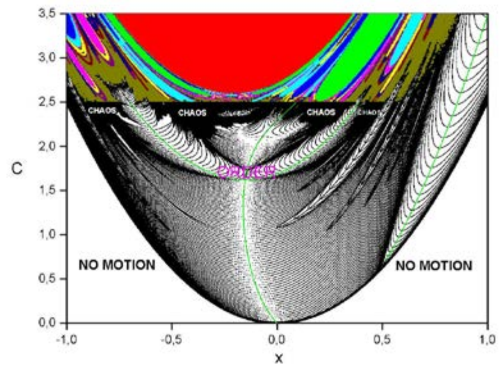

Fig. 4: Chart for A=5, B=1

The initial conditions corresponding to escape solutions without first, second, third, fourth, fifth, sixth, seventh, eighth, ninth intersection with Ox axis appear in red, green, blue, cyan, magenta, yellow, navy, purple, wine color, respectively. The initial conditions corresponding to escape solutions, without 90th intersection with Ox axis in the charts of Figures 3 and 4 and without 60th intersection with Ox axis in the charts of Figures 5 and 6, appear in dark yellow color. Moreover the region ORDER or / and the regions CHAOS formed by the initial conditions of periodic and symmetric solutions appear, likewise the initial conditions of simple solutions appear in green color. We note that, as A increases, the initial conditions of escape solutions occupy smaller and smaller part of the plane (x, C) and more specifically they extend up to the value of C=A/2.

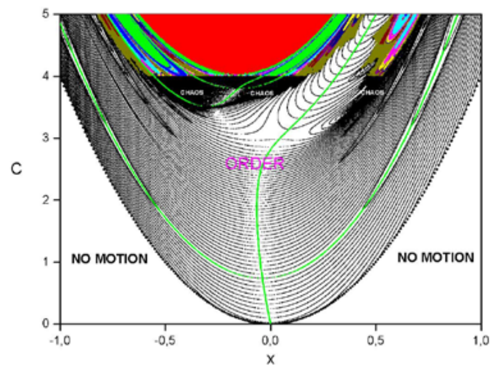

Fig. 5: Chart for A=8, B=1

We, also, observe that the initial conditions of periodic and symmetric solutions for a great value of A, i.e. A=16, do not form a region of CHAOS, but the whole region formed is characterized as a region ORDER.

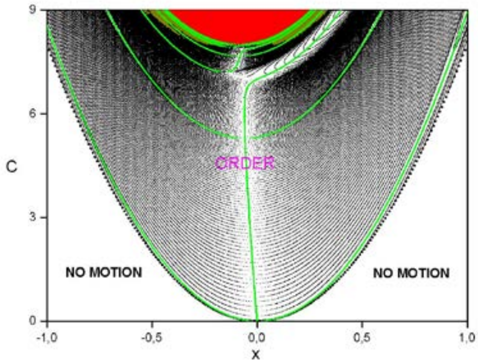

Fig. 6: Chart for A=16, B=1

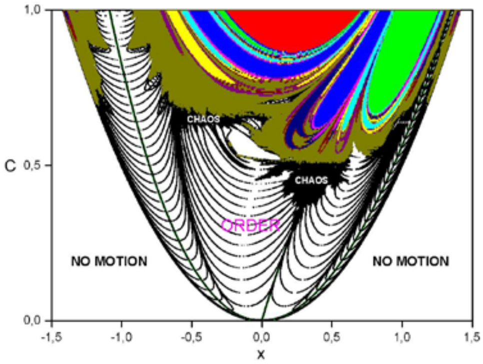

Fig. 7: Chart for A=1, B=1

For the same value of A, the region of the initial conditions in dark yellow color is occupied almost thoroughly by the region of the initial conditions in red color. Finally, we maintain constant the value for the parameter A, for instance A=1, while we increase gradually the value for the parameter B. In Figures 7, 8, 9 and 10 on the plane (x, C) the initial conditions of periodic and symmetric solutions and those of escape solutions appear for values of the parameters A, B as they are referred to in the corresponding captions. In the charts of Figures 7 and 8 the initial conditions of periodic and symmetric solutions of multiplicity 120 and 300, respectively, have been computed. In the charts of Figures 9 and 10 the initial conditions of periodic and symmetric solutions of multiplicity 600 and 900, respectively, have been computed. The initial conditions corresponding to escape solutions

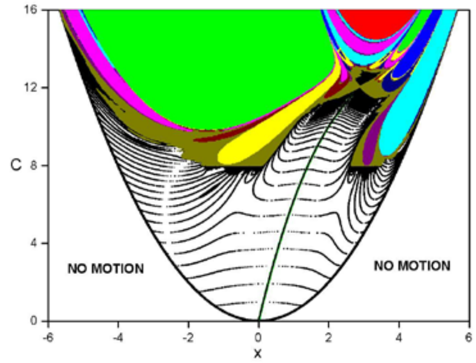

Fig. 8: Chart for A=1, B=4

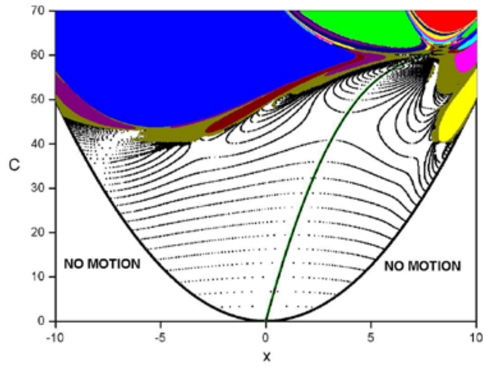

Fig. 9: Chart for A=1, B=9

without first, second, third, fourth, fifth, sixth, seventh, eighth, ninth intersection with Ox axis appear in red, green, blue, cyan, magenta, yellow, navy, purple, wine color, respectively, whereas in dark yellow color the initial conditions corresponding to escape solutions without 120th and 300th intersection with Ox axis appear, in the charts of the Figures 7 and 8, respectively and without 600th and 900th intersection with Ox axis, in the charts of the Figures 9 and 10, respectively. We note that, by increasing B, the initial conditions of escape solutions occupy a quite smaller part of the plane (x, C) and more specifically they extend up to the value of $C=B^2/2$. Furthermore, the initial conditions of periodic and symmetric solutions form regions CHAOS, for B=1; the area of the regions

CHAOS is significantly restricted for greater values of B.

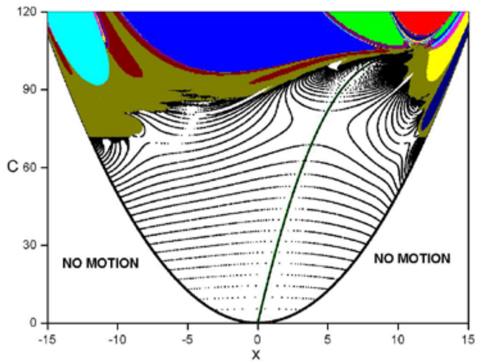

Fig. 10: Chart for A=1, B=12

3 Conclusions

The computation of the initial conditions of periodic symmetric solutions of an appropriate multiplicity, depending on the parameter values, and the initial conditions of the corresponding escape solutions help to set up a chart presenting the region of order for some parameter values or /and regions of chaos for other ones.

References:

[1] Valaris, E.P. and Leftaki, M.A., Escape Solutions of Two-Degree of Freedom Dynamical System of the Coupled Non-Linear Double Oscillator with Third Order Potential, *Wseas Transactions on Mathematics*, Vol. 5, Issue 4, 2006, pp. 435-438.