The Impact Induced 3:1 Internal Resonance in Nonlinear Doubly Curved Shallow Panels with Rectangular Platform*

Y.A. ROSSIKHIN and M.V. SHITIKOVA
Voronezh State University of Architecture
and Civil Engineering
Research Center on Dynamics of Structures
20-letiya Oktyabrya Str. 84, Voronezh 394006
RUSSIAN FEDERATION
yar@vgasu.vrn.ru, mvs@vgasu.vrn.ru

MUHAMMED SALIH KHALID J. M.
Voronezh State University of Architecture
and Civil Engineering
on leave from Ministry of Education
IRAQ
Khalid_bus@yahoo.com

Abstract: Geometrically non-linear vibrations of doubly curved shallow panels with rectangular platform under the low-velocity impact by an elastic sphere are investigated. It is assumed that the target is simply supported and partial differential equations are obtained in terms of its transverse displacement and Airy's stress function. The local bearing of the target and impactor's materials is neglected with respect to the shell deflection in the contact region. The equations of motion are reduced to a set of infinite nonlinear ordinary differential equations of the second order in time and with cubic and quadratic nonlinearities in terms of the generalized displacements. The internal resonance three-to-one induced during the process of impact is investigated by the method of multiple time scales. The time dependence of the contact force is determined.

Key–Words: Doubly curved shallow panel rectangular in base, Method of multiple time scales, Impact induced resonance, Internal resonance three-to-one

1 Introduction

Doubly curved panels are widely used in aeronautics, aerospace and civil engineering and are subjected to dynamic loads that can cause vibration amplitude of the order of the shell thickness, giving rise to significant non-linear phenomena [1]–[4].

A review of the literature devoted to dynamic behaviour of curved panels and shells could be found in Amabili and Paidoussis [5], as well as in [3], wherein it has been emphasized that free vibrations of doubly curved shallow shells were studied in the majority of papers either utilizing a slightly modified version of the Donnell's theory taking into account the double curvature [1, 6] or the nonlinear first-order theory of shells [7, 8].

Large-amplitude vibrations of doubly curved shallow shells with rectangular base, simply supported at the four edges and subjected to harmonic excitation were investigated in [3], while chaotic vibrations were analyzed in [4]. It has been revealed that such an important nonlinear phenomenon as the occurrence of internal resonances in the problems considered in [3] and [4] is of fundamental importance in the study of curved shells.

In spite of the fact that the impact theory is substantially developed, there is a limited number of papers devoted to the problem of impact over geometrically nonlinear shells. Literature review on this subject could be found in Kistler and Wass [9].

An analysis to predict the transient response of a thin, curved laminated plate subjected to low velocity transverse impact by a rigid object was carried out by Ramkumar and Thakar [10], in so doing the contact force history due to the impact phenomenon was assumed to be a known input to the analysis. The coupled governing equations, in terms of the Airy stress function and shell deformation, are solved using Fourier series expansions for the variables.

The review of papers dealing with the impact response of curved panels and shells shows that a finite element method and such commercial finite element software as ABAQUS and its modifications are the main numerical tools adopted by many researchers [11]–[26].

Thus, the nonlinear impact response of laminated composite cylindrical and doubly curved shells was analyzed using a modified Hertzian contact law in [11] via a finite element model, which was developed based on Sander's shell theory involving shear defor-

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mation effects and nonlinearity due to large deflection. The nonlinear time dependent equations were solved using an iterative scheme and Newmark's method. Numerical results for the contact force and center deflection histories were presented for various impactor conditions, shell geometry and boundary conditions.

Later large deflection dynamic responses of laminated composite cylindrical shells under impact have been analyzed in [12] by the geometrically nonlinear finite element method based on a generalized Sander's shell theory with the first order shear deformation and the von Karman large deflection assumption.

Nonlinear dynamic response for shallow spherical moderate thick shells with damage under low velocity impact has been studied in [13] by using the orthogonal collocation point method and the Newmark method to discrete the unknown variable function in space and in time domain, respectively, and the whole problem is solved by the iterative method. Further this approach was generalized for investigating dynamic response of elasto-plastic laminated composite shallow spherical shell under low velocity impact [15], and for functionally graded shallow spherical shell under low velocity impact in thermal environment [16].

The nonlinear transient response of laminated composite shell panels subjected to low velocity impact in hygrothermal environments was investigated in [17] using finite element method considering doubly curved thick shells involving large deformations with Green-Lagrange strains. The analysis was carried out using quadratic eight-noded isoparametric element. A modified Hertzian contact law was incorporated into the finite element program to evaluate the impact force. The nonlinear equation was solved using the Newmark average acceleration method in conjunction with an incremental modified Newton-Raphson scheme. A parametric study was carried out to investigate the effects of the curvature and side to thickness ratios of simply supported composite cylindrical and spherical shell panels.

The impact behaviour and the impact-induced damage in laminated composite cylindrical shell subjected to transverse impact by a foreign object were studied in [18, 19] using three-dimensional nonlinear transient dynamic finite element formulation. Non-linear system of equations resulting from nonlinear strain displacement relation and non-linear contact loading was solved using Newton-Raphson incremental-iterative method. Some example problems of graphite/epoxy cylindrical shell panels were considered with variation of impactor and laminate parameters and influence of geometrical non-linear effect on the impact response and the resulting damage was investigated.

The Sander's shallow shell theory in conjunction with the Reissner-Mindlin shear deformation theory was employed in [20] to develop a finite element analysis procedure to study the impact response of doubly curved laminated composite shells, in so doing the nine-noded quadratic isoparametric elements of Lagrangian family were utilized. Modified Hertzian contact law is used to calculate the contact force. Numerical results were obtained for cylindrical and spherical shells to investigate the effects of various parameters, such as radius to span ratio, span to thickness ratio, boundary condition and stacking sequence on the impact behavior of the target structure [21, 22].

A 4-noded 48 degree-of-freedom doubly curved quadrilateral shell finite element based on Kirchhoff-Love shell theory was used in [24] for the nonlinear finite element analysis to predict the damage of laminated composite cylindrical and spherical shell panels subjected to low velocity impact. The large displacement stiffness matrix was formed using Green's strain tensor based on total Lagrangian approach with further utilization of an iterative scheme for solving resulting nonlinear algebraic equation by Newton-Raphson method. The load due to low velocity impact was treated as an equivalent quasi-static load and Hertzian law of contact was used for finding the peak contact force.

Recently a new approach has been proposed for the analysis of the impact interactions of nonlinear doubly curved shallow shells with rectangular base under the low-velocity impact by an elastic sphere [27]. It has been assumed that the shell is simply supported and partial differential equations have been obtained in terms of shell's transverse displacement and Airy's stress function. The local bearing of the shell and impactor's materials has been neglected with respect to the shell deflection in the contact region. The equations of motion have been reduced to a set of infinite nonlinear ordinary differential equations of the second order in time and with cubic and quadratic nonlinearities in terms of the generalized displacements. Assuming that only two natural modes of vibrations dominate during the process of impact and applying the method of multiple time scales, the set of equations has been obtained, which allows one to find the time dependence of the contact force and to determine the contact duration and the maximal contact force.

In the present paper, the approach proposed by Rossikhin et al. [27] has been generalized for studying the influence of the impact-induced three-to-one internal resonance on the low velocity impact response of a nonlinear doubly curved shallow shell with rectangular platform. Such an additional nonlinear phenomenon as the internal resonance could be examined only via analytical treatment, since any of existing numerical procedures could not catch this subtle phenomenon.

2 Problem Formulation and Governing Equations

Assume that an elastic or rigid sphere of mass M moves along the z-axis towards a thin-walled doubly curved shell with thickness h, curvilinear lengths a and b, principle curvatures k_x and k_y and rectangular base, as shown in Fig. 1. Impact occurs at the moment $t = 0$ with the velocity εV_0 at the point N with Cartesian coordinates x_0, y_0.

According to Donnell's nonlinear shallow shell theory, the equations of motion could be obtained in terms of lateral deflection w and Airy's stress function ϕ [28]

$$\frac{D}{h}\left(\frac{\partial^4 w}{\partial x^4} + 2\,\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) = \frac{\partial^2 w}{\partial x^2}\,\frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 w}{\partial y^2}\,\frac{\partial^2 \phi}{\partial x^2} - 2\,\frac{\partial^2 w}{\partial x \partial y}\,\frac{\partial^2 \phi}{\partial x \partial y} + k_y\,\frac{\partial^2 \phi}{\partial x^2} + k_x\,\frac{\partial^2 \phi}{\partial y^2} + \frac{F}{h} - \rho \ddot{w}, \quad (1)$$

$$\frac{1}{E}\left(\frac{\partial^4 \phi}{\partial x^4} + 2\,\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4}\right) = -\frac{\partial^2 w}{\partial x^2}\,\frac{\partial^2 w}{\partial y^2} + \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - k_y\,\frac{\partial^2 w}{\partial x^2} - k_x\,\frac{\partial^2 w}{\partial y^2}, \quad (2)$$

where $D = \frac{Eh^3}{12(1-\nu^2)}$ is the cylindrical rigidity, ρ is the density, E and ν are the elastic modulus and Poisson's ratio, respectively, t is time, $F = P(t)\delta(x-x_0)\delta(y-y_0)$ is the contact force, $P(t)$ is yet unknown function, δ is the Dirac delta function, x and y are Cartesian coordinates, overdots denote time-derivatives, $\phi(x,y)$ is the stress function which is the potential of the in-plane force resultants

$$N_x = h\,\frac{\partial^2 \phi}{\partial y^2}, \quad N_y = h\,\frac{\partial^2 \phi}{\partial x^2}, \quad N_{xy} = -h\,\frac{\partial^2 \phi}{\partial x \partial y}. \quad (3)$$

The equation of motion of the sphere is written as

$$M\ddot{z} = -P(t) \quad (4)$$

subjected to the initial conditions

$$z(0) = 0, \quad \dot{z}(0) = \varepsilon V_0, \quad (5)$$

where $z(t)$ is the displacement of the sphere, in so doing

$$z(t) = w(x_0, y_0, t). \quad (6)$$

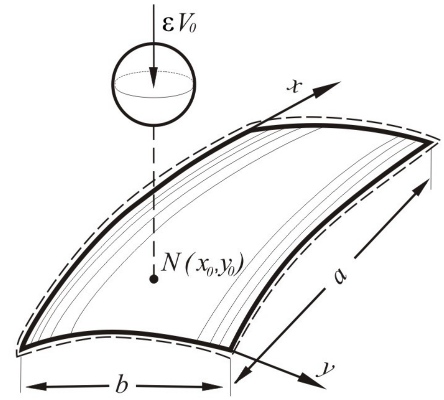

Figure 1: Geometry of the doubly curved shallow shell

Considering a simply supported shell with movable edges, the following conditions should be imposed at each edge:
at $x = 0,\ a$

$$w = 0, \quad \int_0^b N_{xy} dy = 0, \quad N_x = 0, \quad M_x = 0, \quad (7)$$

and at $y = 0,\ b$

$$w = 0, \quad \int_0^a N_{xy} dx = 0, \quad N_y = 0, \quad M_y = 0, \quad (8)$$

where M_x and M_y are the moment resultants.

The suitable trial function that satisfies the geometric boundary conditions is

$$w(x,y,t) = \sum_{p=1}^{\tilde{p}} \sum_{q=1}^{\tilde{q}} \xi_{pq}(t)\ \sin\left(\frac{p\pi x}{a}\right) \sin\left(\frac{q\pi y}{b}\right), \quad (9)$$

where p and q are the number of half-waves in x and y directions, respectively, and $\xi_{pq}(t)$ are the generalized coordinates. Moreover, $\tilde{p}$ and $\tilde{q}$ are integers indicating the number of terms in the expansion.

Substituting (9) in (6) and using (4), we obtain

$$P(t) = -M \sum_{p=1}^{\tilde{p}} \sum_{q=1}^{\tilde{q}} \ddot{\xi}_{pq}(t) \sin\left(\frac{p\pi x_0}{a}\right) \sin\left(\frac{q\pi y_0}{b}\right). \quad (10)$$

In order to find the solution of the set of equations (1) and (2), it is necessary first to obtain the solution of Eq. (2). For this purpose, let us substitute (9) in the right-hand side of Eq. (2) and seek the solution of the equation obtained in the form

$$\phi(x,y,t)=\sum_{m=1}^{\tilde{m}}\sum_{n=1}^{\tilde{n}} A_{mn}(t)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right), \tag{11}$$

where $A_{mn}(t)$ are yet unknown functions.

Substituting (9) and (11) in Eq. (2) and using the orthogonality conditions of sines within the segments $0 \le x \le a$ and $0 \le y \le b$, we have

$$A_{mn}(t) = \frac{E}{\pi^2} K_{mn}\xi_{mn}(t) + \frac{4E}{a^3b^3}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^{-2} \times \sum_k\sum_l\sum_p\sum_q B_{pqklmn}\xi_{pq}(t)\xi_{kl}(t), \tag{12}$$

where

$$B_{pqklmn} = pqkl B^{(2)}_{pqklmn} - p^2l^2 B^{(1)}_{pqklmn},$$

$$B^{(1)}_{pqklmn}=\int_0^a\int_0^b \sin\left(\frac{p\pi x}{a}\right)\sin\left(\frac{q\pi y}{b}\right)\sin\left(\frac{k\pi x}{a}\right) \times\sin\left(\frac{l\pi y}{b}\right)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)dxdy,$$

$$B^{(2)}_{pqklmn}=\int_0^a\int_0^b \cos\left(\frac{p\pi x}{a}\right)\cos\left(\frac{q\pi y}{b}\right)\cos\left(\frac{k\pi x}{a}\right) \times\cos\left(\frac{l\pi y}{b}\right)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)dxdy,$$

$$K_{mn} = \left(k_y\,\frac{m^2}{a^2} + k_x\,\frac{n^2}{b^2}\right)^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^{-2}.$$

Substituting then (9)–(12) in Eq. (1) and using the orthogonality condition of sines within the segments $0 \le x \le a$ and $0 \le y \le b$, we obtain an infinite set of coupled nonlinear ordinary differential equations of the second order in time for defining the generalized coordinates

$$\ddot{\xi}_{mn}(t) + \Omega^2_{mn}\xi_{mn}(t) + \frac{8\pi^2E}{a^3b^3\rho} \times \sum_p\sum_q\sum_k\sum_l B_{pqklmn}\left(K_{kl}-\frac{1}{2}K_{mn}\right) \times\ \xi_{pq}(t)\xi_{kl}(t) + \frac{32\pi^4E}{a^6b^6\rho} \times \sum_r\sum_s\sum_i\sum_j\sum_k\sum_l\sum_p\sum_q B_{rsijmn} \times B_{pqklij}\xi_{rs}(t)\xi_{pq}(t)\xi_{kl}(t) + \frac{4M}{ab\rho h}\sin\left(\frac{m\pi x_0}{a}\right)\sin\left(\frac{n\pi y_0}{b}\right) \times\sum_p\sum_q \ddot{\xi}_{pq}(t)\sin\left(\frac{p\pi x_0}{a}\right)\sin\left(\frac{q\pi y_0}{b}\right)=0, \tag{13}$$

where Ω_{mn} is the natural frequency of the mnth mode of the shell vibration defined as

$$\Omega^2_{mn} = \frac{E}{\rho}\left[\frac{\pi^4h^2}{12(1-\nu^2)}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2 + K_{mn}\right].$$

The last term in each equation from (13) describes the influence of the coupled impact interaction of the target with the impactor of the mass M applied at the point with the coordinates x_0, y_0.

It is known [29, 30] that during nonstationary excitation of thin bodies not all possible modes of vibration would be excited. Moreover, the modes which are strongly coupled by any of the so-called internal resonance conditions are initiated and dominate in the process of vibration, in so doing the types of modes to be excited are dependent on the character of the external excitation.

Thus, in order to study the additional nonlinear phenomenon induced by the coupled impact interaction due to equation (13), we suppose that only two natural modes of vibrations are excited during the process of impact, namely, $\Omega_{\alpha\beta}$ and $\Omega_{\gamma\delta}$. Then the set of equations (13) is reduced to the following two nonlinear differential equations:

$$p_{11}\ddot{\xi}_{\alpha\beta}+p_{12}\ddot{\xi}_{\gamma\delta} + \Omega^2_{\alpha\beta}\xi_{\alpha\beta} + p_{13}\xi^2_{\alpha\beta} + p_{14}\xi^2_{\gamma\delta} +p_{15}\xi_{\alpha\beta}\xi_{\gamma\delta}+p_{16}\xi^3_{\alpha\beta}+p_{17}\xi_{\alpha\beta}\xi^2_{\gamma\delta}=0, \tag{14}$$

$$p_{21}\ddot{\xi}_{\alpha\beta}+p_{22}\ddot{\xi}_{\gamma\delta} + \Omega^2_{\gamma\delta}\xi_{\gamma\delta} + p_{23}\xi^2_{\gamma\delta} + p_{24}\xi^2_{\alpha\beta} +p_{25}\xi_{\alpha\beta}\xi_{\gamma\delta}+p_{26}\xi^3_{\gamma\delta}+p_{27}\xi^2_{\alpha\beta}\xi_{\gamma\delta}=0, \tag{15}$$

where

$$p_{11} = 1 + \frac{4M}{\rho hab}\,s_1^2, \quad p_{22} = 1 + \frac{4M}{\rho hab}\,s_2^2,$$

$$p_{12} = p_{21} = \frac{4M}{\rho hab}\,s_1s_2,$$

$$s_1 = \sin\left(\frac{\alpha\pi x_0}{a}\right)\sin\left(\frac{\beta\pi y_0}{b}\right),$$

$$s_2 = \sin\left(\frac{\gamma\pi x_0}{a}\right)\sin\left(\frac{\delta\pi y_0}{b}\right),$$

$$p_{13} = \frac{8\pi^2E}{a^3b^3\rho}\,B_{\alpha\beta\alpha\beta\alpha\beta}\,\frac{1}{2}\,K_{\alpha\beta},$$

$$p_{14} = \frac{8\pi^2E}{a^3b^3\rho}\,B_{\gamma\delta\gamma\delta\alpha\beta}\left(K_{\gamma\delta}-\frac{1}{2}\,K_{\alpha\beta}\right),$$

$$p_{15} = \frac{8\pi^2E}{a^3b^3\rho}\Big[B_{\gamma\delta\alpha\beta\alpha\beta}\,\frac{1}{2}\,K_{\alpha\beta}$$

$$+ B_{\alpha\beta\gamma\delta\alpha\beta}\left(K_{\gamma\delta} - \frac{1}{2}\,K_{\alpha\beta}\right)\bigg],$$

$$p_{23} = \frac{8\pi^2 E}{a^3b^3\rho}\,B_{\alpha\beta\alpha\beta\gamma\delta}\left(K_{\alpha\beta} - \frac{1}{2}\,K_{\gamma\delta}\right),$$

$$p_{24} = \frac{8\pi^2 E}{a^3b^3\rho}\,B_{\gamma\delta\gamma\delta\gamma\delta}\,\frac{1}{2}\,K_{\gamma\delta},$$

$$p_{25} = \frac{8\pi^2 E}{a^3b^3\rho}\bigg[B_{\alpha\beta\gamma\delta\gamma\delta}\,\frac{1}{2}\,K_{\gamma\delta} + B_{\gamma\delta\alpha\beta\gamma\delta}\left(K_{\alpha\beta} - \frac{1}{2}\,K_{\gamma\delta}\right)\bigg],$$

$$p_{16} = \frac{32\pi^2 E}{a^3b^3\rho}\sum_i\sum_j B_{\alpha\beta ij\alpha\beta}B_{\alpha\beta\alpha\beta ij},$$

$$p_{26} = \frac{32\pi^2 E}{a^3b^3\rho}\sum_i\sum_j B_{\gamma\delta ij\gamma\delta}B_{\gamma\delta\gamma\delta ij},$$

$$p_{17} = \frac{32\pi^2 E}{a^3b^3\rho}\sum_i\sum_j \left(B_{\alpha\beta ij\alpha\beta}B_{\gamma\delta\gamma\delta ij} + B_{\gamma\delta ij\alpha\beta}B_{\alpha\beta\gamma\delta ij} + B_{\gamma\delta ij\alpha\beta}B_{\gamma\delta\alpha\beta ij}\right),$$

$$p_{27} = \frac{32\pi^2 E}{a^3b^3\rho}\sum_i\sum_j \left(B_{\alpha\beta ij\gamma\delta}B_{\gamma\delta\gamma\delta ij} + B_{\gamma\delta ij\gamma\delta}B_{\alpha\beta\gamma\delta ij} + B_{\gamma\delta ij\gamma\delta}B_{\gamma\delta\alpha\beta ij}\right).$$

3 Method of Solution

In order to solve a set of two nonlinear equations (14) and (15), we apply the method of multiple time scales [31] via the following expansions:

$$\xi_{ij}(t,\varepsilon) = \varepsilon X^1_{ij}(T_0,T_2) + \varepsilon^2 X^2_{ij}(T_0,T_2) + \varepsilon^3 X^3_{ij}(T_0,T_2), \quad (16)$$

where $ij = \alpha\beta$ or $\gamma\delta$, and $T_n = \varepsilon^n t$ are new independent variables, among them: $T_0 = t$ is a fast scale characterizing motions with the natural frequencies, and $T_2 = \varepsilon^2 t$ is a slow scale characterizing the modulation of the amplitudes and phases of the modes with nonlinearity. The dependence of $\xi_{ij}(t,\varepsilon)$ on T_1 is suppressed because secular terms appear at third order and not at second order [31].

Considering that

$$\frac{d^2}{dt^2}\,\xi_{ij} = \varepsilon\left(D_0^2 X^1_{ij}\right) + \varepsilon^3\left(D_0^2 X^3_{ij} + 2D_0D_2X^1_{ij}\right),$$

where $D_i^n = \partial^n/\partial T_i^n$ ($n = 1,2,\;\; i = 0,1$), and substituting the proposed solution (16) in (14) and (15), after equating the coefficients at like powers of ε to zero, we are led to a set of recurrence equations to various orders:

to order ε

$$p_{11}D_0^2X_1^1 + p_{12}D_0^2X_2^1 + \Omega_1^2X_1^1 = 0, \quad (17)$$

$$p_{21}D_0^2X_1^1 + p_{22}D_0^2X_2^1 + \Omega_2^2X_2^1 = 0; \quad (18)$$

to order ε^2

$$p_{11}D_0^2X_1^2 + p_{12}D_0^2X_2^2 + \Omega_1^2X_1^2 = -p_{13}(X_1^1)^2 - p_{14}(X_2^1)^2 - p_{15}X_1^1X_2^1, \quad (19)$$

$$p_{21}D_0^2X_1^2 + p_{22}D_0^2X_2^2 + \Omega_2^2X_2^2 = -p_{23}(X_1^1)^2 - p_{24}(X_2^1)^2 - p_{25}X_1^1X_2^1, \quad (20)$$

to order ε^3

$$p_{11}D_0^2X_1^3 + p_{12}D_0^2X_2^3 + \Omega_1^2X_1^3 = -2p_{11}D_0D_2X_1^1 - 2p_{12}D_0D_2X_2^1 - 2p_{13}X_1^1X_1^2 - 2p_{14}X_2^1X_2^2 - p_{15}\left(X_1^1X_2^2 + X_1^2X_2^2\right) - p_{16}\left(X_1^1\right)^3 - p_{17}X_1^1\left(X_2^1\right)^2, \quad (21)$$

$$p_{21}D_0^2X_1^3 + p_{22}D_0^2X_2^3 + \Omega_2^2X_2^3 = -2p_{21}D_0D_2X_1^1 - 2p_{22}D_0D_2X_2^1 - 2p_{23}X_2^1X_2^2 - 2p_{14}X_1^1X_1^2 - p_{25}\left(X_1^1X_2^2 + X_1^2X_2^2\right) - p_{26}\left(X_2^1\right)^3 - p_{27}\left(X_1^1\right)^2X_2^1, \quad (22)$$

where for simplicity is it denoted $\Omega_1 = \Omega_{\alpha\beta}$, $\Omega_2 = \Omega_{\gamma\delta}$, $X_1^1 = X^1_{\alpha\beta}$, $X_2^1 = X^1_{\gamma\delta}$, $X_1^3 = X^3_{\alpha\beta}$, and $X_2^3 = X^3_{\gamma\delta}$.

3.1 Solution of Equations at Order of ε

Following Rossikhin et al. [27], we seek the solution of (17) and (18) in the form:

$$X_1^1 = A_1\left(T_2\right)e^{i\omega_1T_0} + A_2\left(T_2\right)e^{i\omega_2T_0} + \mathrm{cc}, \quad (23)$$

$$X_2^1 = \alpha_1A_1\left(T_2\right)e^{i\omega_1T_0} + \alpha_2A_2\left(T_2\right)e^{i\omega_2T_0} + \mathrm{cc}, \quad (24)$$

where $A_1(T_2)$ and $A_2(T_2)$ are unknown complex functions, cc is the complex conjugate part to the preceding terms, and $\bar{A}_1(T_2)$ and $\bar{A}_2(T_2)$ are their complex conjugates, ω_1 and ω_2 are unknown frequencies of the coupled process of impact interaction of the impactor and the target, and α_1 and α_2 are yet unknown coefficients.

Substituting (23) and (24) in (17) and (18) and gathering the terms with $e^{i\omega_1T_0}$ and $e^{i\omega_2T_0}$ yields

$$\left(\Omega_1^2 - p_{11}\omega_1^2 - p_{12}\alpha_1\omega_1^2\right)A_1e^{i\omega_1T_0} + \left(\Omega_1^2 - p_{11}\omega_2^2 - p_{12}\alpha_2\omega_2^2\right)A_2e^{i\omega_2T_0} + \mathrm{cc} = 0, \quad (25)$$

$$\left(\alpha_1\Omega_2^2 - p_{21}\omega_1^2 - p_{22}\alpha_1\omega_1^2\right) A_1 e^{i\omega_1 T_0} + \left(\alpha_2\Omega_2^2 - p_{21}\omega_2^2 - p_{22}\alpha_2\omega_2^2\right) A_2 e^{i\omega_2 T_0} + \text{cc} = 0. \tag{26}$$

In order to satisfy equations (25) and (26), it is a need to vanish to zero each bracket in these equations. As a result, from four different brackets we have

$$\alpha_1 = -\frac{p_{11}\omega_1^2 - \Omega_1^2}{p_{12}\omega_1^2}, \tag{27}$$

$$\alpha_1 = -\frac{p_{21}\omega_1^2}{p_{22}\omega_1^2 - \Omega_2^2}, \tag{28}$$

$$\alpha_2 = -\frac{p_{11}\omega_2^2 - \Omega_1^2}{p_{12}\omega_2^2}, \tag{29}$$

$$\alpha_2 = -\frac{p_{21}\omega_2^2}{p_{22}\omega_2^2 - \Omega_2^2}. \tag{30}$$

Since the left-hand side parts of relationships (27) and (28), as well as (29) and (30) are equal, then their right-hand side parts should be equal as well. Now equating the corresponding right-hand side parts of (27), (28) and (29), (30) we are led to one and the same characteristic equation for determining the frequencies ω_1 and ω_2:

$$\left(\Omega_1^2 - p_{11}\omega^2\right)\left(\Omega_2^2 - p_{22}\omega^2\right) - p_{12}^2\omega^4 = 0, \tag{31}$$

whence it follows that

$$\omega^2_{1,2} = \frac{\left(p_{22}\Omega_1^2 + p_{11}\Omega_2^2\right) \pm \sqrt{\Delta}}{2\left(p_{11}p_{22} - p_{12}^2\right)}, \tag{32}$$

$$\Delta = \left(p_{22}\Omega_1^2 - p_{11}\Omega_2^2\right)^2 + 4\,\Omega_1^2\Omega_2^2 p_{12}^2.$$

Reference to relationships (32) shows that as the impactor mass $M \to 0$, the frequencies ω_1 and ω_2 tend to the natural frequencies of the shell vibrations Ω_1 and Ω_2, respectively. Coefficients s_1 and s_2 depend on the numbers of the natural modes involved in the process of impact interaction, $\alpha\beta$ and $\gamma\delta$, and on the coordinates of the contact force application $x_0,\ y_0$, resulting in the fact that their particular combinations could vanish coefficients s_1 and s_2 and, thus, coefficients $p_{12} = p_{21} = 0$. Such cases should be considered separately.

3.2 Solution of Equations at Order of ε^2

Now substituting (23) and (24) in (19) and (20), we obtain

$$\begin{aligned} &p_{11}D_0^2X_1^2 + p_{12}D_0^2X_2^2 + \Omega_1^2X_1^2 \\ &= -(p_{13} + \alpha_1^2p_{14} + \alpha_1p_{15})A_1\left[A_1e^{2i\omega_1T_0} + \bar{A}_1\right] \\ &-(p_{13} + \alpha_2^2p_{14} + \alpha_2p_{15})A_2\left[A_2e^{2i\omega_2T_0} + \bar{A}_2\right] \\ &-2\left[p_{13} + \alpha_1\alpha_2p_{14} + (\alpha_1+\alpha_2)p_{15}\right]A_1 \\ &\times\left[A_2e^{i(\omega_1+\omega_2)T_0} + \bar{A}_2e^{i(\omega_1-\omega_2)T_0}\right] + \text{cc}, \end{aligned} \tag{33}$$

$$\begin{aligned} &p_{21}D_0^2X_1^2 + p_{22}D_0^2X_2^2 + \Omega_2^2X_2^2 \\ &= -(p_{23} + \alpha_1^2p_{24} + \alpha_1p_{25})A_1\left[A_1e^{2i\omega_1T_0} + \bar{A}_1\right] \\ &-(p_{23} + \alpha_2^2p_{24} + \alpha_2p_{25})A_2\left[A_2e^{2i\omega_2T_0} + \bar{A}_2\right] \\ &-2\left[p_{23} + \alpha_1\alpha_2p_{24} + (\alpha_1+\alpha_2)p_{25}\right]A_1 \\ &\times\left[A_2e^{i(\omega_1+\omega_2)T_0} + \bar{A}_2e^{i(\omega_1-\omega_2)T_0}\right] + \text{cc}. \end{aligned} \tag{34}$$

Reference to equations (33) and (34) shows that the two-to-one internal resonance $\omega_1 = 2\omega_2$ could occur on this step, which has been studied in detail in [32]. Thus, in further treatment we assume that $\omega_1 \neq 2\omega_2$.

Therefore to solve equations (33) and (34) let us first apply the operators $(p_{22}D_0^2 + \Omega_2^2)$ and $(-p_{12}D_0^2)$ to (33) and (34), respectively, and then add the resulting equations. This procedure will allow us to eliminate X_2^2. If we apply the operators $(-p_{12}D_0^2)$ and $(p_{11}D_0^2 + \Omega_1^2)$ to (33) and (34), respectively, and then add the resulting equations. This procedure will allow us to eliminate X_1^2. Thus, we obtain

$$\begin{aligned} &\left[(p_{11}p_{22} - p_{12}^2)D_0^4 + (p_{11}\Omega_2^2 + p_{22}\Omega_1^2)D_0^2 + \Omega_1^2\Omega_2^2\right] \\ &\times X_1^2 = -\left[(p_{13} + \alpha_1^2p_{14} + \alpha_1p_{15})(p_{22}D_0^2 + \Omega_2^2)\right. \\ &\left.-(p_{23} + \alpha_1^2p_{24} + \alpha_1p_{25})p_{12}D_0^2\right] \\ &\times A_1\left[A_1e^{2i\omega_1T_0} + \bar{A}_1\right] \\ &-\left[(p_{13} + \alpha_2^2p_{14} + \alpha_2p_{15})(p_{22}D_0^2 + \Omega_2^2)\right. \\ &\left.-(p_{23} + \alpha_2^2p_{24} + \alpha_2p_{25})p_{12}D_0^2\right] \\ &\times A_2\left[A_2e^{2i\omega_2T_0} + \bar{A}_2\right] \\ &-2\left\{\left[p_{13} + \alpha_1\alpha_2p_{14} + (\alpha_1+\alpha_2)p_{15}\right]\right. \\ &\times(p_{22}D_0^2 + \Omega_2^2) \\ &\left.-\left[p_{23} + \alpha_1\alpha_2p_{24} + (\alpha_1+\alpha_2)p_{25}\right]p_{12}D_0^2\right\} \\ &\times A_1\left[A_2e^{i(\omega_1+\omega_2)T_0} + \bar{A}_2e^{i(\omega_1-\omega_2)T_0}\right] + \text{cc}, \end{aligned} \tag{35}$$

$$
\begin{aligned}
&\left[(p_{11}p_{22}-p_{12}^2)D_0^4+(p_{11}\Omega_2^2+p_{22}\Omega_1^2)D_0^2+\Omega_1^2\Omega_2^2\right]\\
&\quad\times X_2^2=-\left[(p_{23}+\alpha_1^2p_{24}+\alpha_1p_{25})(p_{11}D_0^2+\Omega_1^2)\right.\\
&\quad\left.-(p_{13}+\alpha_1^2p_{14}+\alpha_1p_{15})p_{12}D_0^2\right]\\
&\quad\times A_1\left[A_1e^{2i\omega_1T_0}+\bar{A}_1\right]\\
&\quad-\left[(p_{23}+\alpha_2^2p_{24}+\alpha_2p_{25})(p_{11}D_0^2+\Omega_1^2)\right.\\
&\quad\left.-(p_{13}+\alpha_2^2p_{14}+\alpha_2p_{15})p_{12}D_0^2\right]\\
&\quad\times A_2\left[A_2e^{2i\omega_2T_0}+\bar{A}_2\right]\\
&\quad-2\left\{[p_{23}+\alpha_1\alpha_2p_{24}+(\alpha_1+\alpha_2)p_{25}]\right.\\
&\quad\times(p_{11}D_0^2+\Omega_1^2)\\
&\quad\left.-[p_{13}+\alpha_1\alpha_2p_{14}+(\alpha_1+\alpha_2)p_{15}]\,p_{12}D_0^2\right\}\\
&\quad\times A_1\left[A_2e^{i(\omega_1+\omega_2)T_0}+\bar{A}_2e^{i(\omega_1-\omega_2)T_0}\right]+\text{cc.}
\end{aligned}
\tag{36}
$$

The solution of (35) and (36) has the form

$$
\begin{aligned}
&X_1^2=F_1(T_2)\,e^{i\omega_1T_0}+F_2(T_2)\,e^{i\omega_2T_0}\\
&+N_1A_1^2e^{2i\omega_1T_0}+N_2A_2^2e^{2i\omega_2T_0}+N_3A_1\bar{A}_1\\
&+N_4A_2\bar{A}_2+N_5A_1A_2e^{i(\omega_1+\omega_2)T_0}\\
&+N_6A_1\bar{A}_2e^{i(\omega_1-\omega_2)T_0}+\text{cc},
\end{aligned}
\tag{37}
$$

$$
\begin{aligned}
&X_2^2=\alpha_1F_1(T_2)\,e^{i\omega_1T_0}+\alpha_2F_2(T_2)\,e^{i\omega_2T_0}\\
&+E_1A_1^2e^{2i\omega_1T_0}+E_2A_2^2e^{2i\omega_2T_0}+E_3A_1\bar{A}_1\\
&+E_4A_2\bar{A}_2+E_5A_1A_2e^{i(\omega_1+\omega_2)T_0}\\
&+E_6A_1\bar{A}_2e^{i(\omega_1-\omega_2)T_0}+\text{cc},
\end{aligned}
\tag{38}
$$

where $F_1(T_2)$ and $F_2(T_2)$ are unknown complex functions, and coefficients N_i and E_i $(i=1,2,...,6)$ are presented in Appendix.

3.3 Solution of Equations at Order of ε^3 at Three-to-one Internal Resonance

Now substituting (23), (24), (37), and (38) in (21) and (22), we obtain [33]

$$
\begin{aligned}
&p_{11}D_0^2X_1^3+p_{12}D_0^2X_2^3+\Omega_1^2X_1^3\\
&\quad=-\left[2i\omega_1(p_{11}+\alpha_1p_{12})D_2A_1\right.\\
&\quad\left.+K_1A_1^2\bar{A}_1+K_2A_1A_2\bar{A}_2\right]e^{i\omega_1T_0}\\
&\quad-\left[2i\omega_2(p_{11}+\alpha_2p_{12})D_2A_2+L_1A_2^2\bar{A}_2\right.\\
&\quad\left.+L_2A_1\bar{A}_1A_2\right]e^{i\omega_2T_0}\\
&\quad-\left\{M_1A_2^3e^{3i\omega_2T_0}+M_2A_1\bar{A}_2^2e^{i(\omega_1-2\omega_2)T_0}\right\}\\
&\quad+\text{Reg}+\text{cc},
\end{aligned}
\tag{39}
$$

$$
\begin{aligned}
&p_{21}D_0^2X_1^3+p_{22}D_0^2X_2^3+\Omega_2^2X_2^3\\
&\quad=-\left[2i\omega_1(p_{21}+\alpha_1p_{22})D_2A_1\right.\\
&\quad\left.+K_3A_1^2\bar{A}_1+K_4A_1A_2\bar{A}_2\right]e^{i\omega_1T_0}\\
&\quad-\left[2i\omega_2(p_{21}+\alpha_2p_{22})D_2A_2+L_3A_2^2\bar{A}_2\right.\\
&\quad\left.+L_4A_1\bar{A}_1A_2\right]e^{i\omega_2T_0}\\
&\quad-\left\{M_3A_2^3e^{3i\omega_2T_0}+M_4A_1\bar{A}_2^2e^{i(\omega_1-2\omega_2)T_0}\right\}\\
&\quad+\text{Reg}+\text{cc},
\end{aligned}
\tag{40}
$$

where all regular terms are designated by Reg, and coefficients K_i, L_i, and M_i $(i=1,2,3,4)$ are given in Appendix.

Reference to equations (39) and (40) shows that the following three-to-one internal resonance could occur:

$$\omega_1=3\omega_2. \tag{41}$$

Internal Resonance $\omega_1=3\omega_2$. Suppose that $\omega_1\approx 3\omega_2$. Then equations (39) and (40) could be rewritten in the following form:

$$
\begin{aligned}
&p_{11}D_0^2X_1^3+p_{12}D_0^2X_2^3+\Omega_1^2X_1^3=B_1\exp(i\omega_1T_0)\\
&\quad+B_2\exp(i\omega_2T_0)+\text{Reg}+\text{cc},
\end{aligned}
\tag{42}
$$

$$
\begin{aligned}
&p_{21}D_0^2X_1^3+p_{22}D_0^2X_2^3+\Omega_2^2X_2^3=B_3\exp(i\omega_1T_0)\\
&\quad+B_4\exp(i\omega_2T_0)+\text{Reg}+\text{cc},
\end{aligned}
\tag{43}
$$

where

$$
\begin{aligned}
B_1=&-2i\omega_1(p_{11}+\alpha_1p_{12})D_2A_1-K_1A_1^2\bar{A}_1\\
&-K_2A_1A_2\bar{A}_2-M_1A_2^3,
\end{aligned}
\tag{44}
$$

$$
\begin{aligned}
B_2=&-2i\omega_2(p_{11}+\alpha_2p_{12})D_2A_2-L_1A_2^2\bar{A}_2\\
&-L_2A_1\bar{A}_1A_2-M_2A_1\bar{A}_2^2,
\end{aligned}
\tag{45}
$$

$$
\begin{aligned}
B_3=&-2i\omega_1(p_{21}+\alpha_1p_{22})D_2A_1-K_3A_1^2\bar{A}_1\\
&-K_4A_1A_2\bar{A}_2-M_3A_2^3,
\end{aligned}
\tag{46}
$$

$$
\begin{aligned}
B_4=&-2i\omega_2(p_{21}+\alpha_2p_{22})D_2A_2-L_3A_2^2\bar{A}_2\\
&-L_4A_1\bar{A}_1A_2-M_4A_1\bar{A}_2^2.
\end{aligned}
\tag{47}
$$

Let us show that the terms with the exponents $\exp(\pm i\omega_iT_0)$ $(i=1,2)$ produce circular terms in equations (42) and (43). For this purpose we choose a particular solution in the form

$$
\begin{aligned}
X_{1\,p}^2&=C_1\exp(i\omega_1T_0)+\text{cc},\\
X_{2\,p}^2&=C_2\exp(i\omega_1T_0)+\text{cc},
\end{aligned}
\tag{48}
$$

or

$$
\begin{aligned}
X_{1\,p}^2&=C_1'\exp(i\omega_2T_0)+\text{cc},\\
X_{2\,p}^2&=C_2'\exp(i\omega_2T_0)+\text{cc},
\end{aligned}
\tag{49}
$$

where C_1, C_2 and C_1', C_2' are arbitrary constants.

Substituting the proposed solution in (42) and (43) we are led to the following sets of equations, respectively:

$$\begin{cases} p_{12}\omega_1^2\left(\alpha_1 C_1 - C_2\right) = B_1, \\ p_{21}\omega_1^2\left(-C_1 + \frac{1}{\alpha_1}\, C_2\right) = B_3, \end{cases} \tag{50}$$

or

$$\begin{cases} p_{12}\omega_2^2\left(\alpha_2 C_1' - C_2'\right) = B_2, \\ p_{21}\omega_2^2\left(-C_1' + \frac{1}{\alpha_2}\, C_2'\right) = B_4. \end{cases} \tag{51}$$

From the sets of equations (50) and (51) it is evident that the determinants comprised from the coefficients standing at C_1, C_2 and C_1', C_2' are equal to zero, therefore, it is impossible to determine the arbitrary constants C_1, C_2 and C_1', C_2' of the particular solutions (48) and (49), what proves the above proposition concerning the circular terms.

In order to eliminate the circular terms, the terms proportional to $e^{i\omega_1 T_0}$ and $e^{i\omega_2 T_0}$ should be vanished to zero putting $B_i = 0$ $(i = 1, 2, 3, 4)$. So we obtain four equations for defining two unknown amplitudes $A_1(t)$ and $A_2(t)$. However, it is possible to show that not all of these four equations are linear independent from each other.

For this purpose, let us first apply the operators $(p_{22}D_0^2 + \Omega_2^2)$ and $(-p_{12}D_0^2)$ to (42) and (43), respectively, and then add the resulting equations. This procedure will allow us to eliminate X_2^3. If we apply the operators $(-p_{12}D_0^2)$ and $(p_{11}D_0^2 + \Omega_1^2)$ to (42) and (43), respectively, and then add the resulting equations. This procedure will allow us to eliminate X_1^3. Thus, we obtain

$$\begin{aligned} &\left[(p_{11}p_{22} - p_{12}^2)D_0^4 + (p_{11}\Omega_2^2 + p_{22}\Omega_1^2)D_0^2 + \Omega_1^2\Omega_2^2\right] X_1^3 \\ &\quad = \left[(p_{22}D_0^2 + \Omega_2^2)B_1 - p_{12}D_0^2 B_3\right]\exp(i\omega_1 T_0) \\ &\quad + \left[(p_{22}D_0^2 + \Omega_2^2)B_2 - p_{12}D_0^2 B_4\right]\exp(i\omega_2 T_0) \\ &\quad + \mathrm{Reg} + \mathrm{cc}, \end{aligned} \tag{52}$$

$$\begin{aligned} &\left[(p_{11}p_{22} - p_{12}^2)D_0^4 + (p_{11}\Omega_2^2 + p_{22}\Omega_1^2)D_0^2 + \Omega_1^2\Omega_2^2\right] X_2^3 \\ &\quad = \left[-p_{12}D_0^2 B_1 + (p_{11}D_0^2 + \Omega_1^2)B_3\right]\exp(i\omega_1 T_0) \\ &\quad + \left[-p_{12}D_0^2 B_2 + (p_{11}D_0^2 + \Omega_1^2)B_4\right]\exp(i\omega_2 T_0) \\ &\quad + \mathrm{Reg} + \mathrm{cc}. \end{aligned} \tag{53}$$

To eliminate the circular terms from equations (52) and (53), it is necessary to vanish to zero the terms in each square bracket. As a result we obtain

$$\begin{cases} (\Omega_2^2 - p_{22}\omega_1^2)B_1 + p_{12}\omega_1^2 B_3 = 0 \\ p_{12}\omega_1^2 B_1 + (\Omega_1^2 - p_{11}\omega_1^2)B_3 = 0 \end{cases} \tag{54}$$

and

$$\begin{cases} (\Omega_2^2 - p_{22}\omega_2^2)B_2 + p_{12}\omega_2^2 B_4 = 0 \\ p_{12}\omega_2^2 B_2 + (\Omega_1^2 - p_{11}\omega_2^2)B_4 = 0 \end{cases} \tag{55}$$

From equations (54) and (55) it is evident that the determinant of each set of equations is reduced to the characteristic equation (31), whence it follows that each pair of equations is linear dependent, therefore for further treatment we should take only one equation from each pair in order that these two chosen equations are to be linear independent. Thus, for example, taking the first equations from each pair and considering relationships (28) and (30), we have

$$B_1 + \alpha_1 B_3 = 0, \tag{56}$$

$$B_3 + \alpha_2 B_4 = 0. \tag{57}$$

Substituting (44)-(47) in (56) and (57), we obtain the following solvability equations:

$$2i\omega_1 D_2 A_1 + p_1 A_1^2 \bar{A}_1 + p_2 A_1 A_2 \bar{A}_2 + p_3 A_2^3 = 0, \tag{58}$$

$$2i\omega_2 D_2 A_2 + p_4 A_2^2 \bar{A}_2 + p_5 A_1 \bar{A}_1 A_2 + p_6 A_1 \bar{A}_2^2 = 0, \tag{59}$$

where

$$p_1 = \frac{K_1 + \alpha_1 K_3}{k_1}, \quad p_2 = \frac{K_2 + \alpha_1 K_4}{k_1},$$

$$p_3 = \frac{M_1 + \alpha_1 M_3}{k_1}, \quad p_4 = \frac{L_1 + \alpha_2 L_3}{k_2},$$

$$p_5 = \frac{L_2 + \alpha_2 L_4}{k_2}, \quad p_6 = \frac{M_2 + \alpha_2 M_4}{k_2},$$

$$k_1 = \frac{\Omega_1^2 + \alpha_1 \Omega_2^2}{\omega_1^2}, \quad k_2 = \frac{\Omega_1^2 + \alpha_2 \Omega_2^2}{\omega_2^2}.$$

Let us multiply equations (58) and (59) by $\bar{A}_1$ and $\bar{A}_2$, respectively, and find their complex conjugates. After adding every pair of the mutually adjoint equations with each other and subtracting one from another, as a result we obtain

$$\begin{aligned} &2i\omega_1\left(\bar{A}_1 D_2 A_1 - A_1 D_2 \bar{A}_1\right) + 2p_1 A_1^2 \bar{A}_1^2 \\ &\quad + 2p_2 A_1 \bar{A}_1 A_2 \bar{A}_2 + p_3\left(\bar{A}_1 A_2^3 + A_1 \bar{A}_2^3\right) = 0, \end{aligned} \tag{60}$$

$$2i\omega_1\left(\bar{A}_1 D_2 A_1 + A_1 D_2 \bar{A}_1\right) + p_3\left(\bar{A}_1 A_2^3 - A_1 \bar{A}_2^3\right) = 0, \tag{61}$$

$$\begin{aligned} &2i\omega_2\left(\bar{A}_2 D_2 A_2 - A_2 D_2 \bar{A}_2\right) + 2p_4 A_2^2 \bar{A}_2^2 \\ &\quad + 2p_5 A_1 \bar{A}_1 A_2 \bar{A}_2 + p_6\left(A_1 \bar{A}_2^3 + \bar{A}_1 A_2^3\right) = 0, \end{aligned} \tag{62}$$

$$2i\omega_2\left(\bar{A}_2 D_2 A_2 + A_2 D_2 \bar{A}_2\right) + p_6\left(A_1 \bar{A}_2^3 - \bar{A}_1 A_2^3\right) = 0. \tag{63}$$

Representing $A_1(T_2)$ and $A_2(T_2)$ in equations (60)–(63) in the polar form

$$A_i(T_2) = a_i(T_2)e^{i\varphi_i(T_2)} \quad (i = 1, 2), \tag{64}$$

we are led to the system of four nonlinear differential equations in $a_1(T_2)$, $a_2(T_2)$, $\varphi_1(T_2)$, and $\varphi_2(T_2)$

$$(a_1^2)^{\cdot} = -\frac{p_3}{\omega_1}\, a_1 a_2^3 \sin\delta, \tag{65}$$

$$2\dot\varphi_1 - \frac{p_1}{\omega_1}\, a_1^2 - \frac{p_2}{\omega_1}\, a_2^2 - \frac{p_3}{\omega_1}\, a_1^{-1} a_2^3 \cos\delta = 0, \tag{66}$$

$$(a_2^2)^{\cdot} = \frac{p_6}{\omega_2}\, a_1 a_2^3 \sin\delta, \tag{67}$$

$$2\dot\varphi_2 - \frac{p_5}{\omega_2}\, a_1^2 - \frac{p_4}{\omega_2}\, a_2^2 - \frac{p_6}{\omega_2}\, a_1 a_2 \cos\delta = 0, \tag{68}$$

where $\delta = 3\varphi_2 - \varphi_1$, and a dot denotes differentiation with respect to T_2.

From equations (65) and (67) we could find that

$$\frac{p_6}{\omega_2}(a_1^2)^{\cdot} + \frac{p_3}{\omega_1}(a_2^2)^{\cdot} = 0 \tag{69}$$

Multiplying equation (69) by MV_0 and integrating over T_2, we obtain the first integral of the set of equations (65)–(68), which is the law of conservation of energy,

$$MV_0\left(\frac{p_6}{\omega_2}\, a_1^2 + \frac{p_3}{\omega_1}\, a_2^2\right) = T_0, \tag{70}$$

where T_0 is the initial energy.

Considering that $T_0 = \frac{1}{2}\, MV_0^2$, equation (70) is reduced to the following form:

$$\frac{p_6}{\omega_2}\, a_1^2 + \frac{p_3}{\omega_1}\, a_2^2 = \frac{V_0}{2}. \tag{71}$$

Let us introduce into consideration a new function $\xi(T_2)$ in the following form:

$$a_1^2 = \frac{\omega_2}{p_6}\,\frac{V_0}{2}\,\xi(T_2), \quad a_2^2 = \frac{\omega_1}{p_3}\,\frac{V_0}{2}\left[1 - \xi(T_2)\right]. \tag{72}$$

It is easy to verify by the direct substitution that formulas (72) satisfy equation (71), while the value $\xi(0)$ $(0 \le \xi(0) \le 1)$ governs the energy distribution between two subsystems, X_1^1 and X_2^1, at the moment of impact.

Substituting (72) in (65) yields

$$\dot\xi = -b\,\frac{V_0}{2}(1-\xi)\sqrt{\xi(1-\xi)}\,\sin\delta, \tag{73}$$

where

$$b = \sqrt{\frac{\omega_1 p_6}{\omega_2 p_3}}.$$

Subtracting equation (66) from the triple equation (68), we have

$$\dot\delta = b\,\frac{V_0}{2}\left(\frac{3}{2}\,\xi - \frac{1}{2}(1-\xi)\right)\sqrt{\frac{1-\xi}{\xi}}\cos\delta + \frac{V_0}{2}\left(\frac{3p_5}{2\omega_2} - \frac{p_1}{2\omega_1}\right)\frac{\omega_2}{p_6}\,\xi + \frac{V_0}{2}\left(\frac{3p_4}{2\omega_2} - \frac{p_2}{2\omega_1}\right)\frac{\omega_1}{p_3}(1-\xi). \tag{74}$$

Equation (74) could be rewritten in another form considering that

$$\dot\delta = \frac{d\delta}{d\xi}\,\dot\xi,$$

or with due account for (73)

$$\dot\delta = -b\,\frac{V_0}{2}(1-\xi)\sqrt{\xi(1-\xi)}\,\frac{d\delta}{d\xi}\,\sin\delta. \tag{75}$$

Substituting (75) in equation (74) yields

$$\frac{d\cos\delta}{d\xi} + \frac{1-4\xi}{2\xi(1-\xi)}\,\cos\delta - \frac{\Gamma_1}{\sqrt{\xi(1-\xi)}} - \frac{\Gamma_2}{1-\xi}\sqrt{\frac{\xi}{1-\xi}} = 0, \tag{76}$$

where

$$\Gamma_1 = \frac{1}{b}\left(\frac{3p_4}{2\omega_2} - \frac{p_2}{2\omega_1}\right)\frac{\omega_1}{p_3},$$

$$\Gamma_2 = \frac{1}{b}\left(\frac{3p_5}{2\omega_2} - \frac{p_1}{2\omega_1}\right)\frac{\omega_2}{p_6}.$$

Integrating (76), we have

$$\cos\delta = \frac{G_0}{(1-\xi)\sqrt{\xi(1-\xi)}} - \frac{\Gamma_1}{2}\sqrt{\frac{1-\xi}{\xi}} + \frac{\Gamma_2}{2}\,\frac{\xi}{1-\xi}\sqrt{\frac{\xi}{1-\xi}}, \tag{77}$$

where G_0 is a constant of integration to be determined from the initial conditions.

Based on relationship (77), it is possible to introduce into consideration the stream function $G(\delta, \xi)$ of the phase fluid on the plane $\delta\xi$ such that

$$G(\delta,\xi) = (1-\xi)\sqrt{\xi(1-\xi)}\cos\delta + \frac{\Gamma_1}{2}(1-\xi)^2 - \frac{\Gamma_2}{2}\,\xi^2 = G_0, \tag{78}$$

which is one more first integral of the set of equations (65)–(68).

It is easy to verify that the function (78) is really a stream function, since

$$v_\delta = \dot{\delta} = -b\,\frac{V_0}{2}\,\frac{\partial G}{\partial \xi}, \quad v_\xi = \dot{\xi} = b\,\frac{V_0}{2}\,\frac{\partial G}{\partial \delta}. \quad (79)$$

In order to find the T_2-dependence of ξ, it is necessary to express $\sin\delta$ in terms of ξ in equation (73) with a help of relationship (77). As a result we obtain

$$\dot{\xi} = -b\,\frac{V_0}{2}\sqrt{\xi(1-\xi)^3 - \left[G_0 - \frac{\Gamma_1}{2}(1-\xi)^2 + \frac{\Gamma_2}{2}\,\xi^2\right]^2}$$

or

$$\int_{\xi_0}^{\xi} \frac{d\xi}{\sqrt{\xi(1-\xi)^3 - \left[G_0 - \frac{\Gamma_1}{2}(1-\xi)^2 + \frac{\Gamma_2}{2}\,\xi^2\right]^2}} = -b\,\frac{V_0}{2}\,T_2, \quad (80)$$

where ξ_0 is the initial magnitude of the function $\xi = \xi(T_2)$.

In other words, the calculation of the T_2-dependence of ξ is reduced to the calculation of the incomplete elliptic integral in the left hand-side of (80).

3.4 Phase portraits

The qualitative analysis of the case of the three-to-one internal resonance (41) could be carried out with the help of the stream-function $G(\xi, \delta)$ defined by relationship (78). The phase portrait to be constructed according to (78) depends essentially on the magnitudes of the coefficients Γ_1 and Γ_2. Let us carry out the phenomenological analysis of the phase portraits constructing them at different magnitudes of the system parameters.

3.4.1 The case when $\Gamma_1 = \Gamma_2 = 0$

Let us first consider the case when $\Gamma_1 = \Gamma_2 = 0$. Then (78) is reduced to

$$G(\delta, \xi) = (1-\xi)\sqrt{\xi(1-\xi)}\cos\delta = G_0, \quad (81)$$

and the stream-lines of the phase fluid in the phase plane $\xi - \delta$ for this particular case are presented in Figure 2. Magnitudes of G are indicated by digits near the curves which correspond to the stream-lines; the flow direction of the phase fluid elements are shown by arrows on the stream-lines.

Reference to Figure 2 shows that the phase fluid flows within the circulation zones, which tend to be located around the perimeter of the rectangles bounded by the lines $\xi = 0$, $\xi = 1$, and $\delta = \pm(\pi/2) \pm 2\pi n$ $(n = 0, 1, 2, ...)$. As this takes place, the flow in each such rectangle becomes isolated. On all four rectangle sides $G = 0$ and inside it the value G preserves its sign. The function G attains its extreme magnitudes at the points with the coordinates $\xi = \frac{1}{4}$, $\delta = \pm\pi n$ $(n = 0, 1, 2, ...)$.

Along the lines $\delta = \pm(\pi/2) \pm 2\pi n$ $(n = 0, 1, 2, ...)$ the solution could be written as

$$\xi = \left[1 + \frac{1}{[c_0 + f(T_2)]^2}\right]^{-1},$$

$$\delta(T_2) = \delta_0 = \frac{\pi}{2} \pm \pi n, \;\; n = 0, 1, 2, ...$$

where

$$f(T_2) = -\frac{bV_0}{2}\,T_2, \quad c_0 = \sqrt{\frac{\xi_0}{1-\xi_0}}.$$

Along the line $\xi = 1$ the stationary boundary regime is realized, because when $\xi = \xi_0 = 1$ the amplitudes $a_1 = \text{const}$ and $a_2 = 0$, and from (73) and (75) it follows that $\dot{\xi} = \dot{\delta} = 0$.

The transition of fluid elements from the points $\xi = 0$, $\delta = \pi/2 \pm 2\pi n$ to the points $\xi = 0$, $\delta = -\pi/2 \pm 2\pi n$ $(n = 0, 1, 2...)$ proceeds instantly, because according to the distribution of the phase velocity along the section $\delta = 0$ (see Figure 2) the magnitude of $\mathbf{v}$ tends to infinity as $\xi \to 0$. The distribution of the velocity along the vertical lines $\delta = \pm\pi n$ has the aperiodic character, while in the vicinity of the line $\xi = 1/4$ it possesses the periodic character.

3.4.2 The case when $\Gamma_1 = 0$ and $\Gamma_2/2 = 1$

In this case, the stream-function is defined as

$$G(\xi, \delta) = \xi^{1/2}(1-\xi)^{3/2}\cos\delta + (1-\xi)^2 = G(\xi_0, \delta_0),$$

and Figure 3 shows the streamlines of the phase fluid in the phase plane.

As in the previous case, the phase fluid flows in an infinitely long channel, the boundaries of which are the straight lines $\xi = 0$ and $\xi = 1$, corresponding to the phase modulated motions. In one part the streamlines are non-closed, what corresponds to the periodic change of amplitudes and the aperiodic change of phases; in another part they are closed, what corresponds to the periodic change of both amplitudes and phases. The aperiodic regime lines are the boundaries of the closed and unclosed streamline areas. From the phase portrait in Figure 3 it is seen that the circulation zones are located in a staggered arrangement by the right and left channel sides (this configuration resembles that of von Kármán staggered vortex tracks).

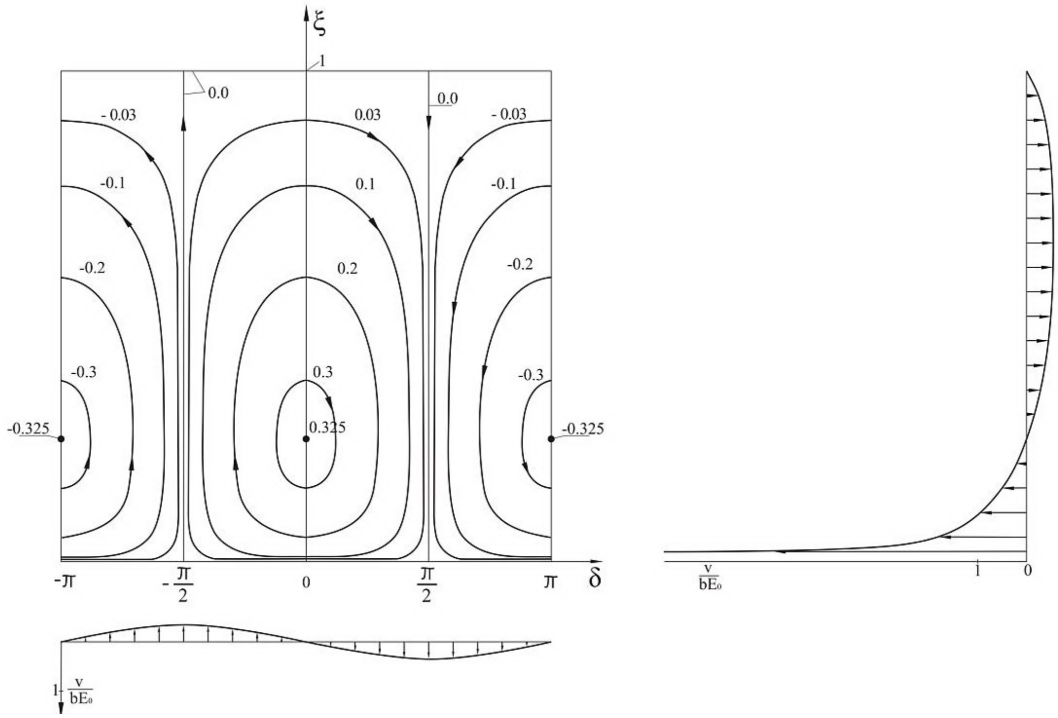

Figure 2: Phase portrait for the case of 1:3 internal resonance at $\Gamma_1 = \Gamma_2 = 0$

Each zone by the side $\xi = 1$ is surrounded by a line with the value $G = 0$. This line consists of two parts connected with each other at the branch points with the coordinates $\xi = 1$, $\delta = \pi/2 \pm \pi n$ $(n = 0, 1, 2...)$. One branch of this line corresponds to the phase-modulated regime $\xi = 1$, and the other to the aperiodic regime, wherein ξ varies from $\xi_{\min} = 0.5$ to $\xi_{\max} = 1$. At the branch point itself, the phase fluid flow velocity is equal to zero. Along the separatrix, the analytic solution can be constructed in the following form:

$$\frac{2\sqrt{2}}{1-\xi}\sqrt{(1-\xi)(2-\xi)}\Big|_{\xi_0}^{\xi} = -\frac{bV_0}{2}\,T_2,$$

$$\cos\delta = -\sqrt{\frac{1-\xi}{\xi}}.$$

The circulation zones by the side $\xi = 0$ are surrounded by the line with the value $G = 1$. However, only those parts of the line $G = 1$ which bound these zones from above and come closer to the side $\xi = 0$ at the points $\xi = 0$, $\delta = \pi/2 \pm \pi n$ belong to the domain of the fluid flow. The transition of fluid elements from the points $\xi = 0$, $\delta = (\pi/2) \pm \pi n$ to the points $\xi = 0$, $\delta = (3\pi/2) \pm \pi n$ proceeds instantly. The line $G = 1$ conforms to the periodic change of the amplitudes and the aperiodic change of the phase. The separatrix $G = 1$ is defined by the following equations:

$$\int_{\xi_0}^{\xi} \frac{d\xi}{\sqrt{\xi(1 - 7\xi + 7\xi^2 - 2\xi^3)}}$$

$$= \int_{\xi_0}^{\xi} \frac{d\xi}{\sqrt{\xi(0.170515 - \xi)(2\xi^2 - 6.659\xi + 5.865)}}$$

$$= -\frac{bV_0}{2}\,T_2,$$

$$\cos\delta = \frac{2-\xi}{1-\xi}\sqrt{\frac{\xi}{1-\xi}},$$

wherein ξ varies from $\xi_{\min} = 0$ to $\xi_{\max} = 0.170515$.

Inside the both circulation zones there are points with the extreme values of the stream-function: maximal $G_{\max} = 1.11$ and minimal $G_{\min} = -0.0475$, respectively. These points are the centers corresponding to the stable stationary regimes $\xi = \xi_0 = 0.0443$, $\delta = \delta_0 = \pm 2\pi n$ and $\xi = \xi_0 = 0.7057$, $\delta = \delta_0 = \pi \pm 2\pi n$, respectively.

Between the lines corresponding to $G = 0$ and $G = 1$, unclosed streamlines are located which are in accordance with the periodic change of the amplitudes and the aperiodic change of the phase difference.

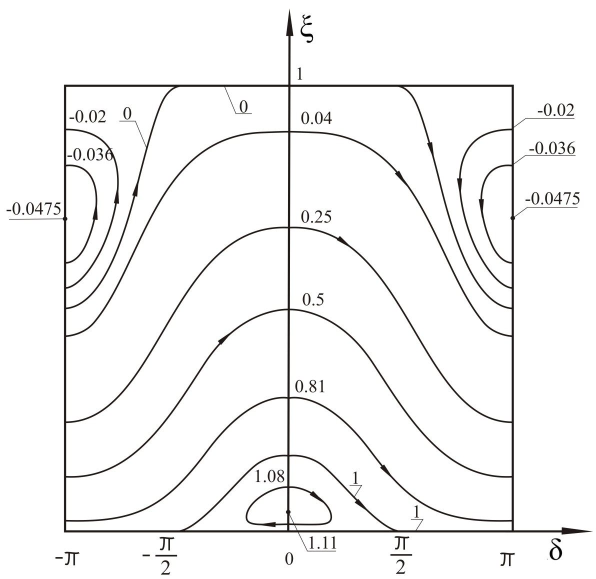

Figure 3: Phase portrait for the case of 1:3 internal resonance at $\Gamma_1 = 0$, and $\Gamma_2/2 = 1$

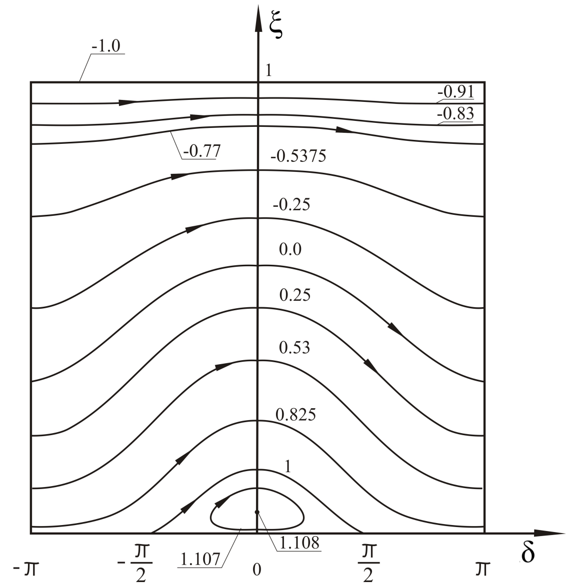

Figure 4: Phase portrait for the case of 1:3 internal resonance at $\Gamma_1/2 = \Gamma_2/2 = 1$

3.4.3 The case when $\Gamma_1/2 = \Gamma_2/2 = 1$

In this case, the stream-function is defined as

$$G(\xi,\delta)=\xi^{1/2}(1-\xi)^{3/2}\cos\delta-\xi^2+(1-\xi)^2=G(\xi_0,\delta_0),$$

and Figure 4 shows the streamlines of the phase fluid in the phase plane.

From Figure 4 it is seen that, unlike the previous case presented in Figure 4, the circulation zones by the side $\xi = 1$ and the aperiodic regime disappear. If $\xi \to 1$, then the streamlines level off and tend to the line $\xi = 1$ where $G = -1$. If $\xi \to 0$, then the streamlines tend to the piecewise continuous line $G = 1$ determined on the segments $[-(\pi/2)\pm 2\pi n,\ (\pi/2)\pm 2\pi n]$. The transition of fluid elements from the points $\xi = 0$, $\delta = (\pi/2) \pm 2\pi n$ to the points $\xi = 0$, $\delta = (3\pi/2) \pm 2\pi n$ proceeds instantly. The line $G = 1$ conforms to the periodic change of the amplitudes and the aperiodic change of the phase difference. The separatrix $G = 1$ is defined by the following equations:

$$\int_{\xi_0}^{\xi} \frac{d\xi}{\sqrt{\xi(1-7\xi+3\xi^2-\xi^3)}}$$

$$= \int_{\xi_0}^{\xi} \frac{d\xi}{\sqrt{\xi(0.1523-\xi)(\xi^2-2.8477\xi+6.5663)}}$$

$$= -\frac{bV_0}{2}\,T_2,$$

$$\cos\delta = \frac{2}{1-\xi}\sqrt{\frac{\xi}{1-\xi}},$$

wherein ξ varies from $\xi_{\min} = 0$ to $\xi_{\max} = 0.1523$.

Inside each circulation zone there is a point with the maximal value of the stream-function $G_{\max} = 1.108$. These points are the centers corresponding to the stable stationary regimes $\xi = \xi_0 = 0.04$, $\delta = \delta_0 = \pm 2\pi n$.

Between the lines corresponding to $G = -1$ and $G = 1$, unclosed streamlines are located which are in accordance with the periodic change of the amplitudes and the aperiodic change of the phase difference.

3.4.4 The case when $\Gamma_1 = -21.84$ and $= \Gamma_2 = 0.01$

In this case, the stream-function is defined as

$$G(\xi,\delta) = \xi^{1/2}(1-\xi)^{3/2}\cos\delta - 0.005\xi^2$$
$$-10.92(1-\xi)^2 = G(\xi_0,\delta_0),$$

and Figure 5 shows the streamlines of the phase fluid in the phase plane.

Figure 5 illustrates the phase portrait with only unclosed phase fluid streamlines along which the fluid flows in the direction of an increase in δ. With $\xi \to 0$ and $\xi \to 1$, the streamlines level off and tend, respectively, to the lines $\xi = 0$ with $G = G_{\min} = \Gamma_1/2 = -10.92$ and $\xi = 1$ with $G = G_{\max} = -\Gamma_2/2 = -0.005$.

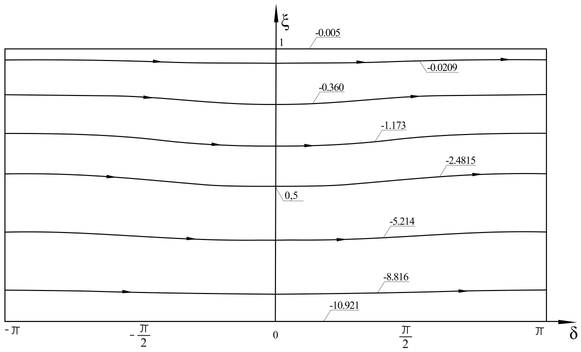

Figure 5: Phase portrait for the case of 1:3 internal resonance at $\Gamma_1 = -21.84$ and $\Gamma_2 = 0.01$

3.5 Initial conditions

In order to construct the final solution of the problem under consideration, i.e. to solve the set of Eqs. (65)-(68) involving the functions $a_1(T_2)$, $a_2(T_2)$, or $\xi(T_2)$, as well as $\varphi_1(T_2)$, and $\varphi_2(T_2)$, or $\delta(T_2)$, it is necessary to use the initial conditions

$$w(x, y, 0) = 0, \tag{82}$$

$$\dot{w}(x_0, y_0, 0) = \varepsilon V_0, \tag{83}$$

$$\frac{p_6}{\omega_2}\, a_1^2(0) + \frac{p_3}{\omega_1}\, a_2^2(0) = \frac{V_0}{2}. \tag{84}$$

The two-term relationship for the displacement w (9) within an accuracy of ε according to (16) has the form

$$w(x, y, t) = \epsilon \left[X^1_{\alpha\beta}(T_0, T_2) \sin\left(\tfrac{\alpha\pi x}{a}\right) \sin\left(\tfrac{\beta\pi y}{b}\right) + X^1_{\gamma\delta}(T_0, T_2) \sin\left(\tfrac{\gamma\pi x}{a}\right) \sin\left(\tfrac{\delta\pi y}{b}\right) \right] + O(\varepsilon^3). \tag{85}$$

Substituting (23) and (24) in (85) with due account for (64) yields

$$\begin{aligned} w(x, y, t) &= 2\epsilon \Big\{ a_1(\varepsilon^2 t) \cos\left[\omega_1 t + \varphi_1(\varepsilon^2 t)\right] \\ &+ a_2(\varepsilon^2 t) \cos\left[\omega_2 t + \varphi_2(\varepsilon^2 t)\right] \Big\} \sin\left(\tfrac{\alpha\pi x}{a}\right) \sin\left(\tfrac{\beta\pi y}{b}\right) \\ &+ 2\epsilon \Big\{ \alpha_1 a_1(\varepsilon^2 t) \cos\left[\omega_1 t + \varphi_1(\varepsilon^2 t)\right] + \alpha_2 a_2(\varepsilon^2 t) \\ &\times \cos\left[\omega_2 t + \varphi_2(\varepsilon^2 t)\right] \Big\} \sin\left(\tfrac{\gamma\pi x}{a}\right) \sin\left(\tfrac{\delta\pi y}{b}\right) + O(\varepsilon^3). \end{aligned} \tag{86}$$

Differentiating (86) with respect to time t and limiting ourselves by the terms of the order of ε, we could find the velocity of the shell at the point of impact as follows

$$\begin{aligned} \dot{w}(x_0, y_0, t) &= -2\epsilon \Big\{ \omega_1 (s_1 + \alpha_1 s_2) a_1(\varepsilon^2 t) \\ &\times \sin\left[\omega_1 t + \varphi_1(\varepsilon^2 t)\right] + \omega_2 (s_1 + \alpha_2 s_2) \\ &\times a_2(\varepsilon^2 t) \sin\left[\omega_2 t + \varphi_2(\varepsilon^2 t)\right] \Big\} + O(\varepsilon^3). \end{aligned} \tag{87}$$

Substituting (86) in the first initial condition (82) and assuming that $a_1(0) > 0$ and $a_2(0) > 0$, we have

$$\cos\varphi_1(0) = 0, \quad \cos\varphi_2(0) = 0, \tag{88}$$

whence it follows that

$$\varphi_1(0) = \pm\frac{\pi}{2}, \quad \varphi_2(0) = \pm\frac{\pi}{2}, \tag{89}$$

and

$$\cos\delta_0 = \cos\left[3\varphi_2(0) - \varphi_1(0)\right] = \mp 1, \tag{90}$$

i.e.,

$$\delta_0 = \pm\pi(n+1) \quad (n = 0, 1, 2, ...). \tag{91}$$

The signs in (89) should be chosen considering the fact that the initial amplitudes are positive values, i.e. $a_1(0) > 0$ and $a_2(0) > 0$. Assume for definiteness that

$$\varphi_1(0) = -\frac{\pi}{2}, \quad \varphi_2(0) = -\frac{\pi}{2}. \tag{92}$$

Substituting now (87) in the second initial condition (83) with due account for (92), we obtain

$$\omega_1(s_1 + \alpha_1 s_2) a_1(0) + \omega_2(s_1 + \alpha_2 s_2) a_2(0) = \frac{V_0}{2}. \tag{93}$$

From equations (84) and (93) we could determine the initial amplitudes

$$a_2(0) = \frac{V_0}{2\omega_2(s_1 + \alpha_2 s_2)} - \frac{\omega_1(s_1 + \alpha_1 s_2)}{\omega_2(s_1 + \alpha_2 s_2)}\, a_1(0), \tag{94}$$

$$d_1 a_1^2(0) + d_2 a_1(0) + d_3 = 0, \tag{95}$$

where

$$d_1 = 1 + \frac{\omega_1^2 (s_1 + \alpha_1 s_2)^2}{b^2 \omega_2^2 (s_1 + \alpha_2 s_2)^2},$$

$$d_2 = -\frac{V_0 \omega_1 (s_1 + \alpha_1 s_2)}{b^2 k_1 \omega_2^2 (s_1 + \alpha_2 s_2)^2},$$

$$d_3 = \frac{V_0^2}{4 b^2 \omega_2^2 (s_1 + \alpha_2 s_2)^2} - \frac{V_0 \omega_2}{2 p_6}.$$

It should be noted that the initial amplitudes depend not only on the initial velocity of the impactor, but according to (94) and (95) they are defined also by the parameters of two impact-induced modes coupled by the three-to-one internal resonance (41).

Considering (90), from (78) we find the value of constant G_0, which defines the trajectory of a point on the phase plane

$$\begin{aligned} G_0 &= \frac{4}{V_0^2}\left[\pm\frac{p_3}{\omega_1}\sqrt{\frac{p_3p_6}{\omega_1\omega_2}}a_1(0)a_2^3(0)\right.\\ &+ \left.\frac{\Gamma_1 p_3^2}{2\omega_1^2}\,a_2^4(0)-\frac{\Gamma_2 p_6^2}{2\omega_2^2}\,a_1^4(0)\right]. \qquad (96)\end{aligned}$$

Thus, we have determined all necessary constants from the initial conditions, therefore we could proceed to the construction of the solution for the contact force.

3.6 The contact force

Now knowing $a_1(0)$, $a_2(0)$, $\varphi_1(0)$, and $\varphi_2(0)$, it is possible to calculate the value $P(t)$, which within an accuracy of ε has the form:

$$P(t) = -\varepsilon M\left[\ddot{X}_1^1(t)s_1+\ddot{X}_2^1(t)s_2\right]+O(\varepsilon^3), \quad (97)$$

or with due account for (87)

$$\begin{aligned} P(t) &= 2\epsilon M\Big\{\omega_1^2 a_1(\varepsilon^2 t)\cos\left[\omega_1 t+\varphi_1(\varepsilon^2 t)\right]\\ &+\omega_2^2 a_2(\varepsilon^2 t)\cos\left[\omega_2 t+\varphi_2(\varepsilon^2 t)\right]\Big\}s_1\\ &+2\epsilon M\Big\{\alpha_1\omega_1^2 a_1(\varepsilon^2 t)\cos\left[\omega_1 t+\varphi_1(\varepsilon^2 t)\right]\\ &+\alpha_2\omega_2^2 a_2(\varepsilon^2 t)\cos\left[\omega_2 t+\varphi_2(\varepsilon^2 t)\right]\Big\}s_2+O(\varepsilon^3). \end{aligned} \quad (98)$$

Considering (92) and (41), Eq. (98) is reduced to

$$\begin{aligned} P(t) &= 2\varepsilon M\omega_2^2[9a_1(0)(s_1+\alpha_1 s_2)\sin 3\omega_2 t\\ &+ a_2(0)(s_1+\alpha_2 s_2)\sin\omega_2 t]\\ &= 18M\varepsilon(s_1+\alpha_1 s_2)\omega_2^2 a_1(0)\sin\omega_2 t\\ &\times\left(3-4\sin^2\omega_2 t+\frac{1}{9}\,\text{æ}\right)+O(\varepsilon^3), \end{aligned} \quad (99)$$

where the dimensionless coefficient æ

$$\text{æ} = \frac{a_2(0)(s_1+\alpha_2 s_2)}{a_1(0)(s_1+\alpha_1 s_2)}$$

is defined by the parameters of two impact-induced modes coupled by the three-to-one internal resonance (41), as well as by the coordinates of the point of impact and the initial velocity of impact.

The contact force in the dimensionless form could be written as

$$P^*(t) = \left(3-4\sin^2\tau+\frac{1}{9}\,\text{æ}\right)\sin\tau, \qquad (100)$$

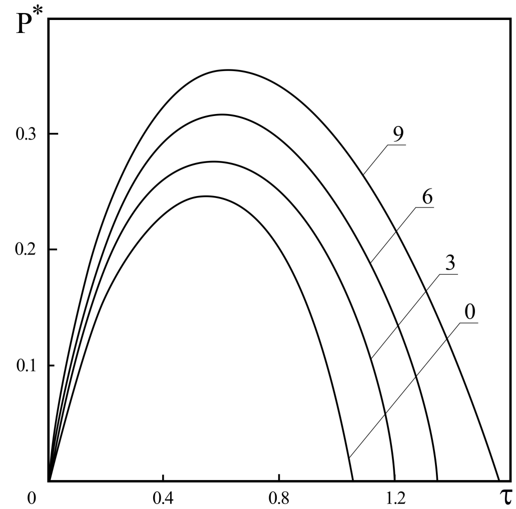

Figure 6: Dimensionless time dependence of the dimensionless contact force

where

$$P^*(t) = \frac{P(t)}{18\varepsilon M\omega_2^2(s_1+\alpha_1 s_2)a_1(0)}.$$

The dimensionless time $\tau = \omega_2 t$ dependence of the dimensionless contact force P^* defined by (100) is shown in Figure 6 for the different magnitudes of the parameter æ: 0, 3, 6, and 9. Reference to Figure 6 shows that the increase in the parameter æ results in the increase of both the maximal contact force and the duration of contact. In other words, from Figure 6 it is evident that the peak contact force and the duration of contact depend essentially upon the parameters of two impact-induced modes coupled by the three-to-one internal resonance (41).

4 Conclusion

In the present paper, a new approach has been proposed for the analysis of the impact interactions of nonlinear doubly curved shallow shells with rectangular base under the low-velocity impact by an elastic sphere. It has been assumed that the shell is simply supported and partial differential equations have been obtained in terms of shell's transverse displacement and Airy's stress function. The equations of motion have been reduced to a set of infinite nonlinear ordinary differential equations of the second order in time and with cubic and quadratic nonlinearities in terms of the generalized displacements.

The approach utilized in the present paper is based on the fact that during impact only two modes strongly coupled by the three-to-one internal resonance condition are initiated. Such an approach differs from the Galerkin method, wherein resonance phenomena are not involved. Since is it assumed that shell's displacements are finite, then the local bearing of the shell and impactor's materials is neglected with respect to the shell deflection in the contact region. In other words, the Hertz's theory, which is traditionally in hand for solving impact problems, is not used in the present study; instead, the method of multiple time scales is adopted, which is used with much success for investigating vibrations of nonlinear systems subjected to the conditions of the internal resonance, as well as to find the time dependence of the contact force.

It has been shown that the time dependence of the contact force depends essentially on the position of the point of impact and the parameters of two impact-induced modes coupled by the three-to-one internal resonance. Besides, the contact force depends essentially on the magnitude of the initial energy of the impactor. This value governs the place on the phase plane, where a mechanical system locates at the moment of impact, and the phase trajectory, along which it moves during the process of impact.

The procedure suggested in the present paper could be generalized for the analysis of impact response of plates and shells when their motions are described by three or five nonlinear differential equations.

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Appendix

$$\begin{aligned} N_1 = &- \left[(p_{13} + \alpha_1^2 p_{14} + \alpha_1 p_{15})(\Omega_2^2 - 4\omega_1^2 p_{22}) \right. \\ &\left. + (p_{23} + \alpha_1^2 p_{24} + \alpha_1 p_{25}) 4\omega_1^2 p_{12}\right] \\ &\times \left[16\omega_1^4 (p_{11}p_{22} - p_{12}^2) - 4\omega_1^2 (p_{11}\Omega_2^2 + p_{22}\Omega_1^2) \right. \\ &\left. + \Omega_1^2 \Omega_2^2\right]^{-1}, \end{aligned} \tag{101}$$

$$N_2 = -\left[(p_{13}+\alpha_2^2 p_{14}+\alpha_2 p_{15})(\Omega_2^2-4\omega_2^2 p_{22}) + (p_{23}+\alpha_2^2 p_{24}+\alpha_2 p_{25})4\omega_2^2 p_{12}\right] \times \left[16\omega_2^4(p_{11}p_{22}-p_{12}^2)-4\omega_2^2(p_{11}\Omega_2^2+p_{22}\Omega_1^2) + \Omega_1^2\Omega_2^2\right]^{-1}, \tag{102}$$

$$N_3 = -\frac{p_{13}+\alpha_1^2 p_{14}+\alpha_1 p_{15}}{\Omega_1^2}, \tag{103}$$

$$N_4 = -\frac{p_{13}+\alpha_2^2 p_{14}+\alpha_2 p_{15}}{\Omega_1^2}, \tag{104}$$

$$N_5 = -2\left\{[p_{13}+\alpha_1\alpha_2 p_{14}+(\alpha_1+\alpha_2)p_{15}] \times \left[\Omega_2^2-p_{22}(\omega_1+\omega_2)^2\right]+p_{12}(\omega_1+\omega_2)^2 \times [p_{23}+\alpha_1\alpha_2 p_{24}+(\alpha_1+\alpha_2)p_{25}]\right\} \times \left[(\omega_1+\omega_2)^4(p_{11}p_{22}-p_{12}^2)-(\omega_1+\omega_2)^2 \times (p_{11}\Omega_2^2+p_{22}\Omega_1^2)+\Omega_1^2\Omega_2^2\right]^{-1}, \tag{105}$$

$$N_6 = -2\left\{[p_{13}+\alpha_1\alpha_2 p_{14}+(\alpha_1+\alpha_2)p_{15}] \times \left[\Omega_2^2-p_{22}(\omega_1-\omega_2)^2\right]+p_{12}(\omega_1-\omega_2)^2 \times [p_{23}+\alpha_1\alpha_2 p_{24}+(\alpha_1+\alpha_2)p_{25}]\right\} \times \left[(\omega_1-\omega_2)^4(p_{11}p_{22}-p_{12}^2)-(\omega_1-\omega_2)^2 \times (p_{11}\Omega_2^2+p_{22}\Omega_1^2)+\Omega_1^2\Omega_2^2\right]^{-1}, \tag{106}$$

$$E_1 = -\left[(p_{23}+\alpha_1^2 p_{24}+\alpha_1 p_{25})(\Omega_1^2-4\omega_1^2 p_{11}) + (p_{13}+\alpha_2^2 p_{14}+\alpha_2 p_{15})4\omega_1^2 p_{12}\right] \times \left[16\omega_1^4(p_{11}p_{22}-p_{12}^2)-4\omega_1^2(p_{11}\Omega_2^2+p_{22}\Omega_1^2) + \Omega_1^2\Omega_2^2\right]^{-1}, \tag{107}$$

$$E_2 = -\left[(p_{23}+\alpha_1^2 p_{24}+\alpha_1 p_{25})(\Omega_1^2-4\omega_2^2 p_{11}) + (p_{13}+\alpha_2^2 p_{14}+\alpha_2 p_{15})4\omega_2^2 p_{12}\right] \times \left[16\omega_2^4(p_{11}p_{22}-p_{12}^2)-4\omega_2^2(p_{11}\Omega_2^2+p_{22}\Omega_1^2) + \Omega_1^2\Omega_2^2\right]^{-1}, \tag{108}$$

$$E_3 = -\frac{p_{23}+\alpha_1^2 p_{24}+\alpha_1 p_{25}}{\Omega_2^2}, \tag{109}$$

$$E_4 = -\frac{p_{23}+\alpha_2^2 p_{24}+\alpha_2 p_{25}}{\Omega_2^2}, \tag{110}$$

$$E_5 = -2\left\{[p_{23}+\alpha_1\alpha_2 p_{24}+(\alpha_1+\alpha_2)p_{25}] \times \left[\Omega_1^2-p_{11}(\omega_1+\omega_2)^2\right]+p_{12}(\omega_1+\omega_2)^2 \times [p_{13}+\alpha_1\alpha_2 p_{14}+(\alpha_1+\alpha_2)p_{15}]\right\} \times \left[(\omega_1+\omega_2)^4(p_{11}p_{22}-p_{12}^2)-(\omega_1+\omega_2)^2 \times (p_{11}\Omega_2^2+p_{22}\Omega_1^2)+\Omega_1^2\Omega_2^2\right]^{-1}, \tag{111}$$

$$E_6 = -2\left\{[p_{23}+\alpha_1\alpha_2 p_{24}+(\alpha_1+\alpha_2)p_{25}] \times \left[\Omega_1^2-p_{11}(\omega_1-\omega_2)^2\right]+p_{12}(\omega_1-\omega_2)^2 \times [p_{13}+\alpha_1\alpha_2 p_{14}+(\alpha_1+\alpha_2)p_{15}]\right\} \times \left[(\omega_1-\omega_2)^4(p_{11}p_{22}-p_{12}^2)-(\omega_1-\omega_2)^2 \times (p_{11}\Omega_2^2+p_{22}\Omega_1^2)+\Omega_1^2\Omega_2^2\right]^{-1}. \tag{112}$$

$$K_1 = 3\left(p_{16}+\alpha_1^2 p_{17}\right) +(2p_{13}+\alpha_1 p_{15})(D_1+2D_3) +(2\alpha_1 p_{14}+p_{15})(E_1+2E_3), \tag{113}$$

$$K_2 = 6p_{16}+2\alpha_2\left(2\alpha_1+\alpha_2\right)p_{17} +(2p_{13}+\alpha_1 p_{15})2D_4 +(2p_{13}+\alpha_2 p_{15})(D_5+D_6) +(2\alpha_1 p_{14}+p_{15})2E_4 +(2\alpha_2 p_{14}+p_{15})(E_5+E_6), \tag{114}$$

$$K_3 = 3\alpha_1\left(\alpha_1^2 p_{26}+p_{27}\right) +(2\alpha_1 p_{23}+p_{25})(E_1+2E_3) +(2p_{24}+\alpha_1 p_{25})(D_1+2D_3), \tag{115}$$

$$K_4 = 6\alpha_1\alpha_2^2 p_{26}+2p_{27}(\alpha_1+2\alpha_2) +(2p_{24}+\alpha_1 p_{25})2D_4 +(2p_{24}+\alpha_2 p_{25})(D_5+D_6) +(2\alpha_1 p_{23}+p_{25})2E_4 +(2\alpha_2 p_{23}+p_{25})(E_5+E_6), \tag{116}$$

$$L_1 = 3\left(p_{16}+\alpha_2^2 p_{17}\right) +(2p_{13}+\alpha_2 p_{15})(D_2+2D_4) +(2\alpha_2 p_{14}+p_{15})(E_2+2E_4), \tag{117}$$

$$L_2 = 6p_{16}+2\alpha_1\left(\alpha_1+2\alpha_2\right)p_{17} +(2p_{13}+\alpha_2 p_{15})2D_3 +(2p_{13}+\alpha_1 p_{15})(D_5+D_6) +(2\alpha_2 p_{14}+p_{15})2E_3 +(2\alpha_1 p_{14}+p_{15})(E_5+E_6), \tag{118}$$

$$L_3 = 3\alpha_2\left(\alpha_2^2 p_{26}+p_{27}\right) +(2p_{24}+\alpha_2 p_{25})(D_2+2D_4) +(2\alpha_2 p_{23}+p_{25})(E_2+2E_4), \tag{119}$$

$$L_4 = 6\alpha_1^2\alpha_2 p_{26}+2p_{27}(2\alpha_1+\alpha_2) +(2p_{24}+\alpha_2 p_{25})2D_3 +(2p_{24}+\alpha_1 p_{25})(D_5+D_6) +(2\alpha_2 p_{23}+p_{25})2E_3 +(2\alpha_1 p_{23}+p_{25})(E_5+E_6), \tag{120}$$

$$M_1 = p_{16}+\alpha_2^2 p_{17}+(2p_{13}+\alpha_2 p_{15})D_2 +(2\alpha_2 p_{14}+p_{15})E_2, \tag{121}$$

$$M_2 = 3p_{16} + \alpha_2\left(2\alpha_1 + \alpha_2\right)p_{17}$$
$$+(2p_{13} + \alpha_1 p_{15})D_2 + (2\alpha_1 p_{14} + p_{15})E_2$$
$$+(2p_{13} + \alpha_2 p_{15})D_6 + (2\alpha_2 p_{14} + p_{15})E_6, \quad (122)$$

$$M_3 = \alpha_2\left(\alpha_2^2 p_{26} + p_{27}\right) + (2p_{24} + \alpha_2 p_{25})D_2$$
$$+(2\alpha_2 p_{23} + p_{25})E_2, \quad (123)$$

$$M_4 = 3\alpha_1\alpha_2^2 p_{26} + p_{27}(\alpha_1 + 2\alpha_2)$$
$$+(2p_{24} + \alpha_1 p_{25})D_2 + (2\alpha_1 p_{23} + p_{25})E_2$$
$$+(2p_{24} + \alpha_2 p_{25})D_6 + (2\alpha_2 p_{23} + p_{25})E_6. \quad (124)$$