Contra-continuous functions defined through $\Lambda_I$-closed sets

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Abstract: - We introduce some variants of contra-continuity in terms of $\Lambda_I$-closed sets, namely contra-$\Lambda_I$-continuous, contra quasi-$\Lambda_I$-continuous and contra $\Lambda_I$-irresolute functions. The relationships between these functions are investigated and their respective characterizations are established. Moreover, we study their behavior of several topological notions under the direct and inverse images of these functions.

Key-Words: - Ideal; local function, $\Lambda_I$-set, $\Lambda_I$-closed set, contra $\Lambda_I$-continuous function.


1 Introduction

In 1986, Maki [8] introduced and studied the notions of $\Lambda$-sets in topological spaces. Later, in 1997, Arenas et al. [9] defined and studied the notions of $\lambda$-closed and $\lambda$-open sets, using $\Lambda$-sets and closed sets. Particularly, these authors used $\lambda$-closed sets to characterize the axiom $T_{1/2}$. On the other hand, in 1933, Kuratowski [7] introduced a generalization of the closure, called the local function, by the ideal theory on topological spaces. In 1992, Jankovic and Hamlett [8], introduced the concept of $I$-open set via the local function, which is independent of the notion of open set and is a generalization of the notion of pre-open set given by Mashhour et al. [8]. Replacing the class of open sets by the class of $I$-open sets, in 2011, Noiri and Keskin [10] introduced and studied modifications of $\Lambda$-sets and $\lambda$-closed sets in the context of topological spaces equipped with an ideal, which they called $\Lambda_I$-sets and $\Lambda_I$-closed sets, and so characterized two separation properties called spaces $I-T_1$ and spaces $I-T_{1/2}$.

In 1996, Dontchev [2] introduced the notion of contra-continuous function in topological spaces and established interesting results which related contra-continuity with compact spaces, $S$-closed spaces and strongly-$S$-closed spaces. The main purpose of this work is to introduce new variants of contra-continuous functions and characterize its by using $\Lambda_I$-closed sets, in contrast with the variants of continuity studied by Sanabria et al. [11]. Also, we study and investigate the preservation of several separation properties, connectedness and compactness, through direct and inverse images of such functions.

2 Preliminaries

Throughout this paper, $(X, \tau)$ and $(Y, \sigma)$ (or simply $X$ and $Y$) always mean topological spaces in which no separation axioms are assumed, unless explicitly stated. In the same form: $P(X)$, Int(A) and Cl(A) denote the power set of $X$, the interior of $A$ and the closure of $A$, respectively. An ideal $I$ on a topological space $(X, \tau)$ is a nonempty collection of subsets of $X$ which satisfies the following two properties:

1. $A \in I$ and $B \subset A$ implies $B \in I$;
2. $A \in I$ and $B \in I$ implies $A \cup B \in I$.

A topological space $(X, \tau)$ with an ideal $I$ on $X$ is called an ideal topological space and is denoted by $(X, \tau, I)$. Given $(X, \tau, I)$, the application $(.)^* : P(X) \to P(X)$ defined as $A^*(I, \tau) = \{x \in X : U \cap A \notin I \text{ for every } U \in \tau(x)\}$, where $\tau(x) = \{U \in \tau : x \in U\}$, is called the local function of $A$ with respect to $\tau$ and $I$. Briefly, we will write $A^*$ for $A^*(I, \tau)$. In general, $X^*$ is a proper subset of $X$. The equality $X = X^*$ is equivalent to the equality $\tau \cap I = \emptyset$, see [3]. Following [10], we call the
ideal topological spaces which satisfy this hypothesis Hayashi-Samuels spaces (briefly H.S.S.). Observe that Cl*(A) = A ∪ A*(I, τ) defines a Kuratowski closure for a topology τ*(I) (also denoted τ* when there is no chance for confusion), finer than τ. The elements of τ* are called τ*-open and the complement of a τ*-open is called τ*-closed. Observe that a subset A of an ideal topological space (X, τ, I) is τ*-closed if and only if A* ⊆ A, see [3]. The subset A of (X, τ, I) is said to be I-open [2] if A ⊆ Int(A*). Note that X is not an I-open set, in general. A is an I-closed set if its complement is an I-open set. The collection of all I-open sets of an ideal topological space (X, τ, I) is denoted by IO(X, τ). Following to [10], for a subset A of (X, τ, I) we define ΛI(A) as ΛI(A) = ∩{U : A ⊆ U, U ∈ IO(X, τ)}. Also, a subset A is said to be a Λ1-set if A = Λ1(A), while A is said to be a Λ1-closed set if A = U ∩ F, where U is a Λ1-set and F is an τ*-closed set. The complement of a Λ1-closed set is called Λ1-open set. In [10] the following implications are shown:

\[ I\text{-open} \implies \Lambda_1\text{-set} \implies \Lambda_1\text{-closed} \]

**Lemma 2.1.** [11] Lemma 2 If (X, τ, I) is a H.S.S., the every τ*+open set is Λ1-open.

**Lemma 2.2.** [11] Lemma 3 Let \{Bα : α ∈ Δ\} be a collection of subsets of the ideal topological space (X, τ, I). If Bα is Λ1-open for each α ∈ Δ, then \bigcup\{Bα : α ∈ Δ\} is Λ1-open.

Next, we present the definitions and characterizations of Λ1-continuous, quasi-Λ1-continuous and Λ1-irresolute functions given in [11].

**Definition 2.3.** Let (X, τ, I) and (Y, σ, J) be two ideal topological spaces. A function \( f : (X, \tau, I) \rightarrow (Y, \sigma, J) \) is said to be:

1. Λ1-continuous, if \( f^{-1}(V) \) is a Λ1-open set in (X, τ, I) for each open set V in (Y, σ, J).
2. Quasi-Λ1-continuous, if \( f^{-1}(V) \) is a Λ1-open set in (X, τ, I) for each σ*-open set V in (Y, σ, J).
3. Λ1-irresolute, if \( f^{-1}(V) \) is a Λ1-open set in (X, τ, I) for each Λ1-open set V in (Y, σ, J).

**Theorem 2.4.** Let (X, τ, I) and (Y, σ, J) be two ideal topological spaces and \( f : (X, \tau, I) \rightarrow (Y, \sigma, J) \) be a function. The following statements are equivalent:

1. \( f \) is Λ1-continuous (resp. quasi-Λ1-continuous, Λ1-irresolute).
2. \( f^{-1}(B) \) is a Λ1-closed set in (X, τ, I) for each closed (resp. σ*-closed, Λ1-closed) set B in (Y, σ, J).

3. For each \( x \in X \) and each open (resp. σ*-open, Λ1-open) set \( V \) in (Y, σ) such that \( f(x) \in V \), there exists a Λ1-open set \( U \) in (X, τ, I) with \( x \in U \) such that \( f(U) \subseteq V \).

**Proof.** See Theorems 4, 5 and 6 of [13].

### 3 Contra Λ1-continuous functions

The concept of contra-continuous function in topological spaces was given by Dontchev in [4]. A function \( f : X \rightarrow Y \) is said to be contra-continuous, if the preimage of each open set in \( Y \) is a closed set in \( X \). In this section we use open sets, τ*-open sets, Λ1-open and Λ1-closed sets to introduce and characterize new variants of contra-continuous function, called contra Λ1-continuous, contra quasi-Λ1-continuous and contra Λ1-irresolute functions.

**Definition 3.1.** Let (X, τ, I) and (Y, σ, J) be two ideal topological spaces and \( f : (X, \tau, I) \rightarrow (Y, \sigma, J) \) be a function. Then, \( f \) is said to be:

1. Contra Λ1-continuous, if \( f^{-1}(V) \) is a Λ1-closed subset of \( X \) for each open subset \( V \) of \( Y \).
2. Contra quasi-Λ1-continuous, if \( f^{-1}(V) \) is a Λ1-closed subset of \( X \) for each σ*-open set \( V \) of \( Y \).
3. Contra Λ1-irresolute, if \( f^{-1}(V) \) is a Λ1-open subset of \( X \) for each Λ1-open set \( V \) of \( Y \).

**Theorem 3.2.** Let (X, τ, I) and (Y, σ, J) be two ideal topological spaces and \( f : (X, \tau, I) \rightarrow (Y, \sigma, J) \) be a function, where \( (Y, \sigma, J) \) is a H.S.S. If \( f \) is contra Λ1-irresolute, then \( f \) is contra quasi-Λ1-continuous.

**Proof.** Let \( V \) be a σ*-open subset of \( Y \), then by Lemma 2, we have \( V \) is a Λ1-open set of \( Y \) and since \( f \) is contra Λ1-irresolute, it follows that \( f^{-1}(V) \) is a Λ1-closed set of \( X \). Therefore, \( f \) is contra quasi-Λ1-continuous.

In the following example, we show a function that is contra quasi-Λ1-continuous but is not contra Λ1-irresolute.

**Example 3.3.** Let \( X = \{a, b, c\} \), \( \tau = \{\emptyset, \{a, c\}, X\} \), \( I = \{\emptyset, \{c\}\} \), \( \sigma = \{\emptyset, X, \{c\}, \{b, c\}\} \), \( J = \{\emptyset, \{a\}\} \). Then, the collection of all Λ1-closed sets of \( X \) is \( \{\emptyset, X, \{b\}, \{c, b\}\} \), the collection of all σ*-open sets of \( X \) is \( \{\emptyset, X, \{c\}, \{b, c\}\} \), the collection of all Λ1-open sets of \( X \) is \( \{\emptyset, X, \{c\}, \{b, c\}\} \{a, c\}\). and, we have the identity function \( f : (X, \tau, I) \rightarrow (Y, \sigma, J) \) is contra quasi-Λ1-continuous, but is not contra Λ1-irresolute, because \( f^{-1}(\{a, c\}) = \{a, c\} \) is not a Λ1-closed set of \( X \).
The following example shows that in Theorem 3.2, the condition that \((Y, \sigma, J)\) is a H.S.S., cannot be omitted.

**Example 3.4.** Let \(X = \{a, b, c\}, \tau = \{\emptyset, X, \{a\}, \{a, b\}\}, \sigma = \{\emptyset, X, \{c\}, \{b, c\}\}, I = \{\emptyset, \{b\}\}\). Note that \((X, \tau, I)\) is not a H.S.S. because \(\tau \cap I = \{\emptyset, \{b\}\}\). In addition, the collection of all \(\Lambda_1\)-open sets of \(X\) is \(\{\emptyset, X\}\), the collection of all \(\Lambda_1\)-closed sets of \(X\) is \(\{\emptyset, X, \{a\}, \{a, b\}\}\) and, we have the identity function \(f : (X, \tau, I) \rightarrow (X, \tau, I)\) is contr quasi-\(\Lambda_1\)-continuous.

**Theorem 3.5.** Let \((X, \tau, I)\) and \((Y, \sigma, J)\) be two ideal topological spaces and \(f : (X, \tau, I) \rightarrow (Y, \sigma, J)\) be a function. If \(f\) is contra quasi-\(\Lambda_1\)-continuous function, then \(f\) is contra \(\Lambda_1\)-continuous.

**Proof.** Let \(V\) be an open set of \(Y\). Then \(V\) is a \(\sigma^*\)-open set of \(Y\) and, as \(f\) is contra quasi-\(\Lambda_1\)-continuous, it follows that \(f^{-1}(V)\) is a \(\Lambda_1\)-closed set of \(X\). Hence, \(f\) is contra \(\Lambda_1\)-continuous.

Now we show an example of a contra \(\Lambda_1\)-continuous function that is not contra quasi-\(\Lambda_1\)-continuous.

**Example 3.6.** Let \(X = \{a, b, c\}, \tau = \{\emptyset, X, \{a\}, \{a, b\}\}, \sigma = \{\emptyset, X, \{c\}, \{b, c\}\}, I = \{\emptyset, \{c\}\}\) and \(J = \{\emptyset, \{c\}, \{b, c\}, \{b\}\}\). Then, the collection of all \(\sigma^*\)-open sets of \(X\) is \(\{\emptyset, X, \{c\}, \{b, c\}, \{a\}, \{a, b\}, \{a, c\}\}\), the collection of all \(\Lambda_1\)-closed sets of \(X\) is \(\{\emptyset, X, \{b, c\}, \{a, c\}, \{b\}\}\) and, we have the identity function \(f : (X, \tau, I) \rightarrow (X, \sigma, J)\) is contra \(\Lambda_1\)-continuous, but is not contra quasi-\(\Lambda_1\)-continuous, because \(f^{-1}(\{a\})\) and \(f^{-1}(\{a, c\})\) are not \(\Lambda_1\)-closed sets of \(X\).

**Corollary 3.7.** Let \((X, \tau, I)\) and \((Y, \sigma, J)\) be two ideal topological spaces and \(f : (X, \tau, I) \rightarrow (Y, \sigma, J)\) be a function, where \((Y, \sigma, J)\) is a H.S.S. If \(f\) is contra \(\Lambda_1\)-irresolute, then \(f\) is contra \(\Lambda_1\)-continuous.

**Proof.** The proof follows from Theorems 3.2 and 3.5.

According to the previous results, given an H.S.S., we obtain the following diagram, where none of the implications is reversible:

\[
\text{Contra } \Lambda_1\text{-irresolute} \quad \longrightarrow \quad \text{Contra quasi } \Lambda_1\text{-continuous} \quad \quad \downarrow \quad \quad \text{Contra } \Lambda_1\text{-continuous}
\]

**Theorem 3.8.** Let \((X, \tau, I), (Y, \sigma, J)\) and \((Z, \theta, K)\) be three ideal topological spaces, \(f : (X, \tau, I) \rightarrow (Y, \sigma, J)\) and \(g : (Y, \sigma, J) \rightarrow (Z, \theta, K)\) be two functions. The following statements hold:

1. \(g \circ f\) is contra \(\Lambda_1\)-irresolute if \(f\) is \(\Lambda_1\)-irresolute and \(g\) is contra \(\Lambda_1\)-irresolute.
2. \(g \circ f\) is contra \(\Lambda_1\)-irresolute if \(f\) is contra \(\Lambda_1\)-irresolute and \(g\) is \(\Lambda_1\)-irresolute.
3. \(g \circ f\) is \(\Lambda_1\)-irresolute, if \(f\) is contra \(\Lambda_1\)-irresolute and \(g\) is contra \(\Lambda_1\)-irresolute.
4. \(g \circ f\) is contra \(\Lambda_1\)-continuous if \(f\) is contra \(\Lambda_1\)-continuous and \(g\) is contra \(\Lambda_1\)-continuous.
5. \(g \circ f\) is contra \(\Lambda_1\)-continuous if \(f\) is contra \(\Lambda_1\)-continuous and \(g\) is \(\Lambda_1\)-continuous.
6. \(g \circ f\) is \(\Lambda_1\)-continuous, if \(f\) is \(\Lambda_1\)-continuous and \(g\) is \(\Lambda_1\)-continuous.
7. \(g \circ f\) is \(\Lambda_1\)-continuous, if \(f\) is \(\Lambda_1\)-continuous and \(g\) is contra \(\Lambda_1\)-continuous.
8. \(g \circ f\) is contra \(\Lambda_1\)-continuous, if \(f\) is contra \(\Lambda_1\)-irresolute and \(g\) is contra \(\Lambda_1\)-continuous.
9. \(g \circ f\) is contra \(\Lambda_1\)-continuous, if \(f\) is contra \(\Lambda_1\)-irresolute and \(g\) is contra \(\Lambda_1\)-continuous.
10. \(g \circ f\) is contra quasi-\(\Lambda_1\)-continuous, if \(f\) is \(\Lambda_1\)-irresolute and \(g\) is contra quasi-\(\Lambda_1\)-continuous.
11. \(g \circ f\) is contra quasi-\(\Lambda_1\)-continuous, if \(f\) is contra \(\Lambda_1\)-irresolute and \(g\) is quasi \(\Lambda_1\)-continuous.
12. \(g \circ f\) is quasi \(\Lambda_1\)-continuous, if \(f\) is contra \(\Lambda_1\)-irresolute and \(g\) is quasi \(\Lambda_1\)-continuous.

**Proof.** (1) Let \(V\) be a \(\Lambda_K\)-open set of \(Z\). Since \(g\) is contra \(\Lambda_1\)-irresolute, then \((g^{-1}(V))\) is a \(\Lambda_1\)-closed set of \(Y\) and as \(f\) is \(\Lambda_1\)-irresolute, then by Theorem 2.4, we have \(f^{-1}(g^{-1}(V))\) is a \(\Lambda_1\)-closed set of \(X\). But \((g \circ f)^{-1}(V) = (f^{-1} \circ g^{-1})(V) = (f^{-1}(g^{-1}(V)))\) and hence, \((g \circ f)^{-1}(V)\) is a \(\Lambda_1\)-closed set of \(X\). This shows that \(g \circ f\) is contra \(\Lambda_1\)-irresolute.

The proofs of (2)-(12) are analogous to the case (1).

In the next three theorems, we characterize contra \(\Lambda_1\)-continuous, contra quasi-\(\Lambda_1\)-continuous and contra \(\Lambda_1\)-irresolute functions, respectively.

**Theorem 3.9.** Let \(f : (X, \tau, I) \rightarrow (Y, \sigma)\) be a function. The following statements are equivalent:

1. \(f\) is contra \(\Lambda_1\)-continuous.
2. \(f^{-1}(F)\) is a \(\Lambda_1\)-open set of \(X\) for each closed set \(F\) of \(Y\).
3. For each $x \in X$ and each closed set $F$ of $Y$ such that $f(x) \in F$, there exists a $\Lambda_I$-open set $U$ of $X$ with $x \in U$ and $f(U) \subset F$.

Proof. (1) $\Rightarrow$ (2) Let $F$ be any closed subset of $Y$, then $V = Y - F$ is an open subset of $Y$ and since $f$ is contra $\Lambda_I$-continuous, $f^{-1}(V)$ is a $\Lambda_I$-closed subset of $X$, but $f^{-1}(V) = f^{-1}(Y - F) = f^{-1}(Y) - f^{-1}(F) = X - f^{-1}(F)$ and hence, $f^{-1}(F)$ is a $\Lambda_I$-open subset of $X$.

(2) $\Rightarrow$ (3) Let $V$ be any open subset of $Y$, then $F = Y - V$ is a closed subset of $Y$ and by hypothesis, we have $f^{-1}(F)$ is a $\Lambda_I$-open subset of $X$, but $f^{-1}(F) = f^{-1}(Y - V) = f^{-1}(Y) - f^{-1}(V) = X - f^{-1}(V)$ and so, $f^{-1}(V)$ is a $\Lambda_I$-closed subset of $X$. This shows that $f$ is contra $\Lambda_I$-continuous.

(1) $\Rightarrow$ (3) Let $x \in X$ and $F$ be a closed subset of $Y$ such that $f(x) \in F$, then $x \in f^{-1}(F)$ and since $f$ is a contra $\Lambda_I$-continuous function, $f^{-1}(F)$ is a $\Lambda_I$-open subset of $X$. If $U = f^{-1}(F)$, then $U$ is a $\Lambda_I$-open subset of $X$ such that $x \in U$ and $f(U) \subset F$. Thus, $x \in U \subset f^{-1}(f(U)) \subset f^{-1}(F)$ and hence, $f^{-1}(F) = \bigcup\{U_x : x \in f^{-1}(F)\}$. By Lemma 2.2, we obtain that $f^{-1}(F)$ is a $\Lambda_I$-open subset of $X$.

Theorem 3.10. Let $f : (X, \tau, I) \rightarrow (Y, \sigma, J)$ be a function, the following statements are equivalent:

1. $f$ is contra quasi-$\Lambda_I$-continuous.
2. $f^{-1}(F)$ is a $\Lambda_I$-open set of $X$ for each $\sigma^*$-closed set $F$ of $Y$.
3. For each $x \in X$ and for each $\sigma^*$-closed set $F$ of $Y$ such that $f(x) \in F$, there exists a $\Lambda_I$-open set $U$ of $X$ with $x \in U$ and $f(U) \subset F$.

Proof. It is proven in a similar way to the Theorema 3.9.

Theorem 3.11. Let $f : (X, \tau, I) \rightarrow (Y, \sigma, J)$ be a function, the following statements are equivalent:

1. $f$ is contra $\Lambda_I$-irresolute.
2. $f^{-1}(F)$ is a $\Lambda_I$-open set of $X$ for each $\Lambda_J$-closed set $F$ of $Y$.
3. For each $x \in X$ and each $\Lambda_J$-closed set $F$ of $Y$ such that $f(x) \in F$, there exists a $\Lambda_I$-open set $U$ of $X$ with $x \in U$ and $f(U) \subset F$.

Proof. It is proven in a similar way to the Theorema 3.9.

4 Preservation of notions under direct or inverse images

In this section we study the behavior of some topological notions under direct or inverse images of the new variants of contra-continuity introduced in the Section 3. Before continuing our study, we must remember the following definitions introduced in [11]. An ideal topological space $(X, \tau, I)$ is said to be $\Lambda_I$-connected (resp. $\tau^*$-connected) if $X$ cannot be written as a disjoint union of two nonempty $\Lambda_I$-open (resp. $\tau^*$-open) sets.

Theorem 4.1. If $f : (X, \tau, I) \rightarrow (Y, \sigma)$ is a contra $\Lambda_I$-continuous and surjective function and $(X, \tau, I)$ is a $\Lambda_I$-connected space having more than one element, then $(Y, \sigma)$ is not a discrete space.

Proof. Suppose that $(Y, \sigma)$ is a discrete space and $A$ be any nonempty proper subset of $Y$. So, $A$ is an open and closed subset of $Y$ and as $f$ is a contra $\Lambda_I$-continuous function, it follows that $f^{-1}(A)$ is a $\Lambda_I$-open and $\Lambda_I$-closed set of $X$. Since $(X, \tau, I)$ is a $\Lambda_I$-connected space, by [11], $\emptyset$ and $X$ are the only subsets of $X$ which are both $\Lambda_I$-open and $\Lambda_I$-closed. Thus, $f^{-1}(A) = \emptyset$ or $f^{-1}(A) = X$. If $f^{-1}(A) = \emptyset$, then this contradicts the fact that $A \neq \emptyset$ and $f$ is surjective. If $f^{-1}(A) = X$, then $f$ is not a function and, hence, $(Y, \sigma)$ is not a discrete space.

Theorem 4.2. If $f : (X, \tau, I) \rightarrow (Y, \sigma, J)$ is a surjective function, then the following properties hold:

1. If $f$ is contra $\Lambda_I$-irresolute and $(X, \tau, I)$ is a $\Lambda_I$-connected space, then $(Y, \sigma, J)$ is a $\Lambda_J$-connected space.
2. If $f$ is a contra quasi-$\Lambda_I$-continuous function and $(X, \tau, I)$ is a $\Lambda_J$-connected space, then $(Y, \sigma, J)$ is a $\sigma^*$-connected space.
3. If $f$ is a contra $\Lambda_I$-continuous function and $(X, \tau, I)$ is a $\Lambda_I$-connected space, then $(Y, \sigma)$ is connected.

Proof. (1) Assume that $(X, \tau, I)$ is a $\Lambda_I$-connected space and $f : (X, \tau, I) \rightarrow (Y, \sigma)$ is surjective contra $\Lambda_I$-irresolute function. Suppose that $(Y, \sigma)$ is not $\Lambda_J$-connected. Then, there exist nonempty $\Lambda_J$-open subsets $A$ and $B$ of $Y$ such that $A \cap B = \emptyset$ and $Y = A \cup B$. Thus, $B = Y - A$ and $A = Y - B$ are nonempty $\Lambda_J$-closed subsets of $Y$, and as $f$ is a contra $\Lambda_I$-irresolute function, we have $f^{-1}(A)$ and $f^{-1}(B)$ are $\Lambda_I$-open subsets of $X$ such that $f^{-1}(A) \cap f^{-1}(B) = \emptyset$ and $f^{-1}(A) \cup f^{-1}(B) = X$. This contradicts the fact that $(X, \tau, I)$ is a $\Lambda_I$-connected space. Therefore, $(Y, \sigma, J)$ is $\Lambda_J$-connected.

The proofs of (2) and (3) are similar to case (1).
Theorem 4.3. An ideal topological space \((X, \tau, I)\) is \(\Lambda_I\)-connected, if each contra \(\Lambda_I\)-continuous function \(f : (X, \tau, I) \to (Y, \sigma)\), where \((Y, \sigma)\) is a \(T_0\)-space, is a constant function.

Proof. Suppose that \((X, \tau, I)\) is not a \(\Lambda_I\)-connected space and each contra \(\Lambda_I\)-continuous function \(f : (X, \tau, I) \to (Y, \sigma)\), where \((Y, \sigma)\) is a \(T_0\)-space, is a constant function. Since \((X, \tau, I)\) is not \(\Lambda_I\)-connected, by [11, Theorem 11], there exists a nonempty proper subset \(A\) of \(X\) which is both \(\Lambda_I\)-open and \(\Lambda_I\)-closed. Let \(Y = \{a, b\}\), \(\sigma = \{Y, \emptyset, \{a\}, \{b\}\}\) be a topology on \(Y\) and \(f : (X, \tau, I) \to (Y, \sigma)\) be a function such that \(f(A) = \{a\}\) and \(f(X - A) = \{b\}\). Then \(f\) is a non-constant contra \(\Lambda_I\)-continuous function such that \((Y, \sigma)\) is a \(T_0\)-space, which is a contradiction. Therefore, \((X, \tau, I)\) is a \(\Lambda_I\)-connected space.

Theorem 4.4. If \(f : (X, \tau, I) \to (Y, \sigma)\) is a contra \(\Lambda_I\)-continuous function and \((Y, \sigma)\) is a regular space, then \(f\) is \(\Lambda_I\)-continuous.

Proof. Let \(x \in X\) and \(V\) be an open set of \(Y\) such that \(f(x) \in V\). Since \((Y, \sigma)\) is a regular space, there exists an open set \(W\) of \(Y\) such that \(f(x) \in W \subset Cl(W) \subset V\). Now, since \(f\) is a contra \(\Lambda_I\)-continuous function, then by Theorem 2.14 there exists a \(\Lambda_I\)-open set \(U\) of \(X\) such that \(x \in U\) and \(f(U) \subset Cl(W) \subset V\). By Theorem 3.9, we conclude that \(f\) is a \(\Lambda_I\)-continuous function.

Definition 4.5. An ideal topological space \((X, \tau, I)\) is said to be \(\Lambda_I\)-normal, if for each pair of disjoint closed subsets \(A, B \subset X\), there exist \(\Lambda_I\)-open subsets \(U, V \subset X\) such that \(A \subset U\), \(B \subset V\) and \(U \cap V = \emptyset\).

Remark 4.6. Let \((X, \tau, I)\) be a H.S.S. If \((X, \tau, I)\) is normal, then \((X, \tau, I)\) is \(\Lambda_I\)-normal.

Recall that a topological space \((X, \tau)\) is said to be ultra normal [12], if for each pair of nonempty disjoint closed subsets \(A, B \subset X\), there exist two clopen subsets \(G, H \subset X\) such that \(A \subset G\), \(B \subset H\) and \(U \cap V = \emptyset\).

Theorem 4.7. If \(f : (X, \tau, I) \to (Y, \sigma)\) is an injective, closed and contra \(\Lambda_I\)-continuous function and \((Y, \sigma)\) is an ultra normal space, then \((X, \tau, I)\) is a \(\Lambda_I\)-normal space.

Proof. Let \(A, B \subset X\) be two disjoint closed subsets of \(X\). Since \(f\) is closed and injective, then \(f(A)\) and \(f(B)\) are disjoint closed subsets of \(Y\) and as \((Y, \sigma)\) is an ultra normal space, there exist two clopen subsets \(G, H\) of \(Y\) such that \(f(A) \subset G\), \(f(B) \subset H\) and \(G \cap H = \emptyset\). Now, since \(f\) is contra \(\Lambda_I\)-continuous, \(f^{-1}(G)\) and \(f^{-1}(H)\) are \(\Lambda_I\)-closed subsets of \(X\) and also, \(A \subset f^{-1}(f(A)) \subset f^{-1}(G), B \subset f^{-1}(f(B)) \subset f^{-1}(H)\) and \(f^{-1}(G) \cap f^{-1}(H) = f^{-1}(\emptyset) = \emptyset\). In consequence, \((X, \tau, I)\) is \(\Lambda_I\)-normal.

Definition 4.8. A space \((X, \tau, I)\) is said to be \(\Lambda_I\)-T2, if for each pair of distinct points \(x, y \in X\), there exist \(\Lambda_I\)-open subsets \(U, V \subset X\) such that \(x \in U\), \(y \in V\) and \(U \cap V = \emptyset\).

Remark 4.9. Let \((X, \tau, I)\) be a H.S.S. If \((X, \tau, I)\) is \(\Lambda_I\)-T2, then \((X, \tau, I)\) is \(\Lambda_I\)-T2.

Recall that a topological space \((X, \tau)\) is said to be Urysohn [13], if for each pair of distinct points \(x, y \in X\), there exist two open subsets \(U, V \subset X\) such that \(x \in U\), \(y \in V\) and \(Cl(U) \cap Cl(V) = \emptyset\). The following result shows that, the inverse image of an Urysohn space under an injective and contra \(\Lambda_I\)-continuous function, is a \(\Lambda_I\)-T2-space.

Theorem 4.10. If \(f : (X, \tau, I) \to (Y, \sigma)\) is an injective and contra \(\Lambda_I\)-continuous function and \((Y, \sigma)\) is an Urysohn space, then \((X, \tau, I)\) is a \(\Lambda_I\)-T2-space.

Proof. Consider \(x, y\) be two points of \(X\) with \(x \neq y\). By the injectivity of \(f\), we have \(f(x) \neq f(y)\) and as \((Y, \sigma)\) is an Urysohn space, there exist two open subsets \(U, V \subset Y\) such that \(f(x) \in U\), \(f(y) \in V\) and \(Cl(U) \cap Cl(V) = \emptyset\). By Theorem 3.9, there exist two \(\Lambda_I\)-open subsets \(A, B \subset X\) such that \(x \in A\), \(y \in B\), \(f(A) \subset Cl(U)\) and \(f(B) \subset Cl(V)\). Thus, \(f(A) \cap f(B) \subset Cl(U) \cap Cl(V) = \emptyset\), which implies that \(f(A) \cap f(B) = f(A) \cap f(B) = \emptyset\) and hence, \(A \cap B = \emptyset\). This shows that \((X, \tau, I)\) is a \(\Lambda_I\)-T2-space.

Recall that a topological space \((X, \tau)\) is locally indiscrete [3], if each open subset of \(X\) is closed. In the following definition some modifications of a locally indiscrete space are introduced in order to investigate related properties with the functions defined in the Section 3.

Definition 4.11. We say that an ideal topological space \((X, \tau, I)\) is:

1. Locally \(\tau^*\)-indiscrete, if each \(\tau^*\)-open subset of \(X\) is closed in \(X\).
2. Locally \(\Lambda_I\)-indiscrete, if each \(\Lambda_I\)-open subset of \(X\) is closed in \(X\).
3. \(\Lambda_I\)-space, if each \(\Lambda_I\)-open subset of \(X\) is open in \(X\).

Proposition 4.12. Let \((X, \tau, I)\) be an ideal topological space. The following statements hold:

1. If \((X, \tau, I)\) is an H.S.S. locally \(\Lambda_I\)-indiscrete, then \((X, \tau, I)\) is locally \(\tau^*\)-indiscrete.
2. If \((X, \tau, I)\) is a locally \(\tau^*\)-indiscrete space, then 
\((X, \tau)\) is locally indiscrete.

3. \((X, \tau, I)\) is locally \(\tau^*\)-indiscrete space if and only if 
each \(\tau^*\)-closed subset of \(X\) is open in \(X\).

4. \((X, \tau, I)\) is locally \(\Lambda_1\)-indiscrete space if and 
only if each \(\Lambda_1\)-closed subset of \(X\) is open in \(X\).

5. \((X, \tau, I)\) is \(\Lambda_1\)-space if and only if each 
\(\Lambda_1\)-closed subset of \(X\) is closed in \(X\).

**Theorem 4.13.** Let \((X, \tau, I)\) and \((Y, \sigma, J)\) be two 
ideal topological spaces and \(f : (X, \tau, I) \rightarrow 
(Y, \sigma, J)\) be a contra \(\Lambda_1\)-continuous function. The 
following statements hold:

1. If \((X, \tau, I)\) is locally \(\Lambda_1\)-indiscrete, then \(f\) is a 
continuous function.

2. If \((X, \tau, I)\) is a \(\Lambda_1\)-space, then \(f\) is a contra-
continuous function.

**Proof.** (1) Let \(B\) be a closed subset of \(Y\). Since \(f\) is a 
contra \(\Lambda_1\)-continuous function, \(f^{-1}(B)\) is a 
\(\Lambda_1\)-open subset of \(X\) and as \((X, \tau, I)\) is locally 
\(\Lambda_1\)-indiscrete, then \(f^{-1}(B)\) is a closed subset of \(X\). Therefore, \(f\) is a 
continuous function.

The proof of (2) is similar to case (1). \(\square\)

The following result shows that, the direct image of 
a \(\Lambda_1\)-space under a surjective, closed and contra 
\(\Lambda_1\)-irresolute (resp. contra quasi-\(\Lambda_1\)-continuous, 
contra \(\Lambda_1\)-continuous) function is a locally \(\Lambda_1\)-indiscrete 
(resp. locally \(\sigma^*\)-indiscrete, locally indiscrete) space.

**Theorem 4.14.** Let \((X, \tau, I)\) and \((Y, \sigma, J)\) be two 
ideal topological spaces and \(f : (X, \tau, I) \rightarrow 
(Y, \sigma, J)\) be a surjective and closed function. The 
following statements hold:

1. If \(f\) is contra \(\Lambda_1\)-irresolute and \((X, \tau, I)\) is a 
\(\Lambda_1\)-space, then \((Y, \sigma, J)\) is a \(\Lambda_1\)-

2. If \(f\) is contra quasi-\(\Lambda_1\)-continuous and \((X, \tau, I)\) is 
a \(\Lambda_1\)-space, then \((Y, \sigma, J)\) is a locally \(\sigma^*\)-
indiscrete.

3. If \(f\) is contra \(\Lambda_1\)-continuous and \((X, \tau, I)\) is a 
\(\Lambda_1\)-space, then \((Y, \sigma)\) is a locally indiscrete.

**Proof.** Straightforward. \(\square\)

Recall that a topological space \((X, \tau)\) is said to be 
strongly \(S\)-closed \(\[2\]\), if each closed cover of \(X\) has a 
finite subcover. Now we introduce a modification of 
a strongly \(S\)-closed space using \(\Lambda_1\)-closed sets.

**Definition 4.15.** We say that an ideal topological 
space \((X, \tau, I)\) is strongly \(S\)-\(\Lambda_1\)-closed, if each cover 
of \(X\) by \(\Lambda_1\)-closed sets has a finite subcover.

**Remark 4.16.** Let \((X, \tau, I)\) be a H.S.S. If \((X, \tau, I)\) is 
strongly \(S\)-\(\Lambda_1\)-closed, then \((X, \tau)\) is strongly \(S\)-
closed.

The notions of \(\Lambda_1\)-compact space and \(\tau^*\)-compact 
space were introduced in \(\[1\]\). We say that an ideal 
topological space \((X, \tau, I)\) is \(\Lambda_1\)-compact (resp. 
\(\tau^*\)-compact), if each cover of \(X\) by \(\Lambda_1\)-open (resp. \(\tau^*\)-
open) sets has a finite subcover. The following result shows 
that, the direct image of a strongly \(S\)-\(\Lambda_1\)-closed space under a surjective and contra \(\Lambda_1\)-
irresolute (resp. contra quasi-\(\Lambda_1\)-continuous, contra 
\(\Lambda_1\)-continuous) function, is a \(\Lambda_1\)-compact (resp. \(\sigma^*\)-
compact, compact) space.

**Theorem 4.17.** Let \((X, \tau, I)\) and \((Y, \sigma, J)\) be two 
ideal topological spaces and \(f : (X, \tau, I) \rightarrow 
(Y, \sigma, J)\) be a surjective function. The following 
statements hold:

1. If \(f\) is contra \(\Lambda_1\)-irresolute and \((X, \tau, I)\) is 
strongly \(S\)-\(\Lambda_1\)-closed, then \((Y, \sigma, J)\) is 
a \(\Lambda_1\)-

2. If \(f\) is contra quasi-\(\Lambda_1\)-continuous and \((X, \tau, I)\) is 
strongly \(S\)-\(\Lambda_1\)-closed, then \((Y, \sigma, J)\) is a \(\sigma^*\)-
compact.

3. If \(f\) is contra \(\Lambda_1\)-continuous and \((X, \tau, I)\) is 
strongly \(S\)-\(\Lambda_1\)-closed, then \((Y, \sigma)\) is compact.

**Proof.** (1) Let \(\{V_\alpha : \alpha \in \Delta\}\) be a cover of \(Y\) by \(\Lambda_1\)-
open sets. Since \(f\) is contra \(\Lambda_1\)-irresolute, \(\{f^{-1}(V_\alpha) : 
\alpha \in \Delta\}\) is a cover of \(X\) by \(\Lambda_1\)-closed and as 
\((X, \tau, I)\) is strongly \(S\)-\(\Lambda_1\)-closed, there exists a finite subcollection 
\(\{f^{-1}(V_\alpha) : i = 1, \ldots, n\}\) of \(\{f^{-1}(V_\alpha) : \alpha \in 
\Delta\}\) such that \(X = \bigcup_{i=1}^{n} f^{-1}(V_\alpha)\). Thus, \(Y = f(X) = 
\bigcup_{i=1}^{n} f^{-1}(V_\alpha)\) and hence, \((Y, \sigma, J)\) is \(\Lambda_1\)-compact.

The proofs of (2) and (3) are similar to case (1). \(\square\)

5 Conclusion

The notion of continuous function is one of the most 
important in the study of general topology and the 
lines of research derived from it by the use of ideals 
on topological spaces. This has made it possible to 
establish some interesting advances in topics related 
to computing and image design, particularly in digital 
topology. The antagonistic concept to the continuity 
of a function, is the contra-continuity of a function, 
which appeared due to the need to analyze the behav-
ior of certain spaces defined in terms of coverings. In 
this research work, new classes of contra-continuous
functions were studied, which were defined using the notions of open set, $\tau^*$-open set, $\Lambda_I$-open set and $\Lambda_I$-closed set. The relationships between these classes of functions were studied and some compositions involving them were analyzed; as well as characterizations of such functions were established. Similarly, the behavior of modifications of connected spaces, normal spaces, $T_2$-spaces, $\lambda$-spaces, locally indiscrete spaces and strongly $S$-closed spaces, under direct or inverse images of the new classes of functions defined in this work. It should be noted that in some results obtained that involved the use of notions described by means of $I$-open sets, the additional condition had to be requested that the topological space $(X, \tau, I)$ used was Hayashi Samuels to guarantee that $X$ would turn out to be an $I$-open set and, therefore, each $\tau^*$-open set would be $\Lambda_I$-open.

References:

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