Abstract: - The present paper is devoted to the study of $k, t, d$-proximity on rough sets from the relation point of view. The operator $O_0(\varphi, A)$ and $O(\varphi, A)$ is defined using the upper approximation and closure operators derived from the relation. The properties of induced operator and their connections are henceforth obtained. Moreover, our approach represents a new generalization of the operator using upper approximation only.

Key-Words: - Proximity spaces; Grill operator; Closure operators; Rough sets.


1 Introduction

A classical rough theory set theory based on an equivalence relation was proposed by Pawlak in 1982 [17]. Rough set theory is a powerful mathematical tool to deal with vagueness. Recently, rough set has been combined with some mathematical theories such as algebra and topology. Rough set theory has wide application in varied fields. Chen, et.al. [2] used decision theoretic rough sets in data mining. Wang [15] used rough set theory for scene image classification. Landowski and Landowska [4] presented the utilization of rough set hypothesis to get information from experimental data obtained from the examination of traffic intensity in chosen areas. In the present study, the concept rough sets are used to study $k, t$-proximities. The notion of basic proximity is due to Cech and the notion of Grill is due to Choquet [2]. The comprehensive study on the theory of proximity spaces was done by Naimpally and Warrack [8, 14]. The concept of proximities spaces has been generalized both in fuzzy setting [7]. Hosny [13] studied a type of proximity space based on Ideal $I$ and three types of proximity $\delta$. Peters, Tiwari and Singh [9] studied a associated $\varepsilon$-approach merotopies. A proximity can be obtained from merotopy by restricting the cardinality.

2 Preliminaries

In this section, we collect the basic definitions of rough set, proximity spaces and other fundamental concepts which are used throughout this paper.

2.1 Definition

[17] Let $X$ be a nonempty set and $R$ be an equivalence relation on $X$. Then the pair $(X, R)$ is called an approximation space.

2.2 Definition

[17] Let $(X, R)$ be an approximation space and $[x]_R$ be the equivalence class of $x$ under $R$. Then lower approximation and upper approximation of $A \subseteq X$ are, respectively, defined to be the sets:

\[
\tilde{A} = \{x \in X | [x]_R \cap A \neq \emptyset\}, \quad A = \{x \in X | [x]_R \subseteq A\}.
\]

For an approximation space $(X, R)$, $A \subseteq X$ is called a Definable set if it is a union of equivalence classes under $R$ and a pair $(L, U)$ of definable sets is called a rough set in $(X, R)$ if $L \subseteq U$, also if equivalence class of $x$ is a singleton set $\{x\}$ such that $\{x\} \in U$, then $\{x\} \in L$.

2.3 Definition

[17] Let $\emptyset$ be the empty set and $A^c$ is the compliment of $A$ in $X$, then we can get the following properties of the Pawalak’s rough sets:

(i) $\tilde{A} \subseteq A \subseteq \overline{A}$

(ii) $\overline{X} \subseteq X \subseteq X$

(iii) $\emptyset = \emptyset = \emptyset$
(iv) If $A \subseteq B$ then $A \subseteq B$ and $\overline{A} \subseteq \overline{B}$.
(v) $A = B$ if $A \subseteq B$ and $\overline{A} = \overline{B}$.
(vi) $(A \cap B) = (A) \cap (B)$ and $(A \cup \overline{B}) = (\overline{A}) \cup (\overline{B})$.
(vii) $(\overline{A} \cap B) \subseteq (\overline{A}) \cap (\overline{B})$ and $(\overline{A} \cup B) \supseteq (A) \cup (B)$.

3 Symmetric Relation of Rough Set

This section is devoted to study the symmetric relation on rough set determined by equivalence relation. Let $X$ be a set and $R$ be the equivalence relation on $X$. Let $U_X^R$ denotes the approximation space.

3.1 Definition

A *rough-grill* $\mathcal{G}$ on $U_X^R$ is a collection of upper approximations of rough sets defined on $U_X^R$, satisfying: $\emptyset \notin \mathcal{G}$; if $\overline{A} \in \mathcal{G}$ and $B \supseteq \overline{A}$, then $\overline{B} \in \mathcal{G}$; $\overline{A \cup B} \in \mathcal{G}$ implies that $\overline{A} \in \mathcal{G}$ or $\overline{B} \in \mathcal{G}$. The family of all rough grills is denoted by $\Gamma(U_X^R)$.

3.2 Definition

Let $U_X^R$ be an approximation space. Let $A$ be the rough set in the form of $(\overline{A}, \overline{A})$. The family of all symmetric relation $\varphi$ on $U_X^R$ with the condition: $\varphi(A) = \{B \in U_X^R : (A,B) \in \varphi, A \neq \emptyset\} \in \Gamma(U_X^R)$ denoted by $\Gamma(U_X^R)$.

3.2.1 Remark

Every $\varphi \in U_X^R$ is the rough proximity on $X$ if $\overline{A} \cap \overline{B} \neq \emptyset$ implies that $A \in \varphi(B)$.

3.3 Lemma

(i) For every $\varphi \in U_X^R$ and $A \in U_X^R$, the operator given by $cl_R(A) = A \cup \{x \in X : [x]_R, A \in \varphi\}$ is a (Rough) $\overline{C}$ech closure operator on $X$.
(ii) For every $A_1, A_2 \in U_X^R$, $cl_R(X-\{A_1 \cup A_2\}) = X-(A_1 \cup A_2)$ if $cl_R(X-A_1) = X-A_1$ and $cl_R(X-A_2) = X-A_2$.

3.4 Definition

Let $A, E, F \in U_X^R$ and $\varphi \in U_X^R$.
Define $\overline{O}_0(\varphi, A), O_0(\varphi, A), \overline{O}(\varphi, A)$, and $\underline{O}(\varphi, A)$ as follows:
(i) $\overline{O}_0(\varphi, A) = \{E : \overline{E} \supseteq \overline{A} \text{ and } cl_R(X-E) = X-E\}$
(ii) $O_0(\varphi, A) = \{E : E \supseteq A \text{ and } cl_R(X-E) = X-E\}$
(iii) $\overline{O}(\varphi, A) = \{F : F \in \overline{O}_0(\varphi, A) \text{ and } (A, X-cl_R(F)) \notin \varphi\}$
(iv) $\underline{O}(\varphi, A) = \{F : F \in O_0(\varphi, A) \text{ and } (A, X-cl_R(F)) \notin \varphi\}$

3.4.1 Remark

(i) For $A, B \in U_X^R$ with $B \subseteq A$,
\[\overline{O}_0(\varphi, A) \subseteq \overline{O}_0(\varphi, B)\]
\[O_0(\varphi, A) \subseteq O_0(\varphi, B)\]
\[\overline{O}(\varphi, A) \subseteq \overline{O}(\varphi, B)\]
\[\underline{O}(\varphi, A) \subseteq \underline{O}(\varphi, B)\]

(ii) Universal rough set contained in $\overline{O}(\varphi, A)$ for all $A \in U_X^R$.
(iii) For $A= \emptyset$, the operators defined in 3.4 are nonempty as it contains null set.

3.5 Lemma

For the sets $A_1, A_2 \in U_X^R$ and $\varphi \in U_X^R$,
(i) $\overline{O}_0(\varphi, A_1 \cup A_2) = \overline{O}_0(\varphi, A_1) \cap \overline{O}_0(\varphi, A_2)$.
(ii) $O_0(\varphi, A_1 \cup A_2) = O_0(\varphi, A_1) \cap O_0(\varphi, A_2)$.
(iii) $\overline{O}(\varphi, A_1 \cup A_2) = \overline{O}(\varphi, A_1) \cap \overline{O}(\varphi, A_2)$.
(iv) $\underline{O}(\varphi, A_1 \cup A_2) = \underline{O}(\varphi, A_1) \cap \underline{O}(\varphi, A_2)$.
(v) $O_0(\varphi, A_1 \cup A_2)$ is contained in $\mathcal{G}$ iff $O_0(\varphi, A_1)$ is contained in $\mathcal{G}$ or $O_0(\varphi, A_2)$ is contained in $\mathcal{G}$.
(vi) $\overline{O}(\varphi, A_1 \cup A_2)$ is contained in $\mathcal{G}$ iff $\overline{O}(\varphi, A_1)$ is contained in $\mathcal{G}$ or $\overline{O}(\varphi, A_2)$ is contained in $\mathcal{G}$.
(vii) $\overline{O}(\varphi, A_1) \cup \overline{O}(\varphi, A_2)$ is contained in $\mathcal{G}$ iff $\overline{O}(\varphi, A_1)$ is contained in $\mathcal{G}$ or $\overline{O}(\varphi, A_2)$ is contained in $\mathcal{G}$.
(viii) $\overline{O}(\varphi, A_1) \cup \overline{O}(\varphi, A_2)$ is contained in $\mathcal{G}$ iff $\overline{O}(\varphi, A_1)$ or $\overline{O}(\varphi, A_2)$ is contained in $\mathcal{G}$.

**Proof.** (i) Using remark 3.4.1, $\overline{O}_0(\varphi, A_1 \cup A_2) \subseteq \overline{O}_0(\varphi, A_1) \cap \overline{O}_0(\varphi, A_2)$.
For converse, it is sufficient to note that if $\overline{D} \in \overline{O}_0(\varphi, A_1) \cap \overline{O}_0(\varphi, A_2)$, then $\overline{D} \supseteq \overline{A_1} \cup \overline{A_2}$.
(ii) Analogous to (i).
(iii) Since $A_1 \subseteq A_1 \cup A_2$, we get $\overline{O}_0(\varphi, A_1 \cup A_2) \subseteq \overline{O}_0(\varphi, A_1) \cap \overline{O}_0(\varphi, A_2)$.
Let $D \in \overline{O}(\varphi, A_1) \cap \overline{O}(\varphi, A_2)$. Then using (i) $D \in \overline{O}(\varphi, A_1 \cup A_2)$. Since $\varphi(X-cl_R(D))$ is a grill and
\[ A_1, A_2 \in \phi(X - cl_R(D)) \text{, we get } (A_1 \cup A_2, X - cl_R(D)) \notin \phi. \]

(iv) Analogous to (iii).

(v) Let \( \overline{G}(\phi, A_1) \subseteq G \). Then \( \overline{G}(\phi, A_1 \cup A_2) \subseteq G \), follows from remark 3.4.1. Conversely, if \( D \in \overline{G}(\phi, A_1 \cup A_2) \), then \( \overline{D} \supseteq \overline{A_1} \cup \overline{A_2} \). Let \( \overline{G}(\phi, A_1) \not\subseteq G \) and \( \overline{G}(\phi, A_2) \not\subseteq G \). Then \( D_1 \in \overline{G}(\phi, A_1) \) and \( D_2 \in \overline{G}(\phi, A_2) \) such that \( D_1, D_2 \notin G \). This implies that \( D_1 \cup D_2 \supseteq A_1 \cup A_2 \) but \( D_1 \cup D_2 \notin G \). Hence, \( \overline{G}(\phi, A_1 \cup A_2) \not\subseteq G \).

(vi) Analogous to (v).

(vii) Let \( \overline{G}(\phi, A_1) \not\subseteq G \) and \( \overline{G}(\phi, A_2) \not\subseteq G \). Then \( D_1 \in \overline{G}(\phi, A_1) \) and \( D_2 \in \overline{G}(\phi, A_2) \) such that \( D_1, D_2 \notin G \). Therefore \( D_1 \cup D_2 \in \overline{G}(\phi, A_1 \cup A_2) \) and \( D_1 \cup D_2 \not\subseteq G \). Let \( D_1 \cup D_2 \in \phi(X - cl_R(A_1 \cup A_2)) \). Then \( A_1 \in \phi(X - cl_R(D_1 \cup D_2)) \) or \( A_2 \in \phi(X - cl_R(D_1 \cup D_2)) \).

3.6 Definition

(i) The set of all functions \( v: U^R_X \times \Gamma(U^R_X) \rightarrow P(U^R_X) \), \( v(\Pi, G) \in U^R_X \) is denoted by \( \Phi(X) \) where \( G \) and \( \Pi \) are rough grills and rough proximities on \( X \).

(ii) For \( G \in \Gamma(U^R_X) \) and \( \phi \in U^R_X \), we define:

\[
\begin{align*}
& t(\phi, G) = \{ A \in U^R_X : \overline{G}(\phi, A) \not\subseteq G \} \\
& k(\phi, G) = \{ A \in U^R_X : \overline{G}(\phi, A) \subseteq G \} \\
& d(\phi, G) = \{ A \in U^R_X : \exists D \ni \phi(D) = G \text{ and } E_A \cap E_D \neq \emptyset \} \\
& \in \overline{G}(\phi, A) \text{ and all } E_D \subseteq \overline{G}(\phi, D) \}.
\end{align*}
\]

3.7 Theorem

The functions \( t, k, d \in \Phi(X) \).

3.8 Definition

A rough proximity \( \Pi \) on \( U^R_X \) is said to be \( \lambda \)-proximity on \( U^R_X \) iff for all \( A \in U^R_X \), there exists a function \( \lambda \in \Phi(X) \) satisfying \( \lambda(\Pi, \Pi(A)) \subseteq \Pi(A) \). Further we denote the set of all \( \lambda \)-proximities on \( U^R_X \) by \( R^\lambda \). A rough grill operator will be in class \( A_0 \) if \( \lambda(\phi, G_1) \subseteq \lambda(\phi, G_2) \) where \( G_1 \) and \( G_2 \) are rough grills with \( G_1 \subseteq G_2 \); for all \( \phi \in (U^R_X) \); \( \lambda \) will be in class \( A_1 \) if \( \lambda(\phi, G_1 \cup G_2) \subseteq \lambda(\phi, G_1) \cup \lambda(\phi, G_2) \), where \( G_1 \) and \( G_2 \) are rough grills.

3.9 Proposition

(i) \( t, k, d \in A_0 \cap A_1 \).

(ii) \( d \in A_1 \).

3.10 Proposition

Let \( \Pi \) be a rough proximity on \( U^R_X \). Then,

(i) \( \Pi \in R^\lambda_k \) iff it satisfies: \( G \subseteq \Pi(F) \iff \exists H \in \overline{G}(\Pi(G), F) \subseteq \Pi(F) \).

(ii) \( \Pi \in R^\lambda_t \) iff it satisfies: \( G \subseteq \Pi(F) \iff \exists H \in \overline{G}(\Pi(G), F) \subseteq \Pi(F) \).

(iii) \( \Pi \in R^\lambda_d \) iff it satisfies \( B \subseteq \Pi(C) \iff \exists H \in \overline{G}(\phi, C) \subseteq \Pi \).

3.11 Theorem

\( \overline{R^\lambda_k} \subseteq \overline{R^\lambda_t} \subseteq \overline{R^\lambda_d} \).

Proof. Let \( \Pi \in \overline{R^\lambda_k} \). Then \( G \subseteq \Pi(F) \iff \exists H \subseteq \overline{G}(\Pi(G), F) \subseteq \Pi(F) \).

3.12 Definition

For a rough closure space \( (U^R_X, cl_R) \) and the operator \( \overline{G}(cl_R, F) \), the rough sets \( H_1 \) and \( H_2 \) are said to be separated with respect to \( cl_R \) iff \( \exists G \subseteq \overline{G}(cl_R, H_1), i = 1, 2, 3,... \) such that \( G_1 \) and \( G_2 \) are disjoint rough sets.

3.13 Proposition

For a rough closure space \( (U^R_X, cl_R) \),

\( \Pi_{cl_R^*} = \{ (F, G) : F \) and \( G \) are not separated with respect to \( cl_R \} \)

is a rough proximity on \( U^R_X \).

Proof. Let \( (F, G) \in \Pi_{cl_R^*} \). Then there exist \( H_1 \subseteq \overline{G}(cl_R, F) \) and \( H_2 \subseteq \overline{G}(cl_R, G) \) such that \( H_1 \cap H_2 = \emptyset \). Since \( H_1 \supseteq F \) and \( H_2 \supseteq G \), we get \( F \cap G = \emptyset \). Since \( \emptyset \subseteq \overline{G}(cl_R, \phi) \), \( \emptyset \) and \( F \) are separated for all \( \emptyset \).

Hence \( \emptyset \notin \Pi_{cl_R^*} \). Then \( G \notin \Pi_{cl_R^*} \) and \( G \subseteq \overline{G}(cl_R, F) \).

Then for all \( H_1 \subseteq \overline{G}(cl_R, F) \) and all \( H_2 \subseteq \overline{G}(cl_R, G) \), \( H_1 \cap H_2 \neq \emptyset \). Since for \( H \in \Pi_{cl_R^*} \),

\( \overline{G}(cl_R, F) \subseteq \overline{G}(cl_R, G) \). Let \( F, G \notin \Pi_{cl_R^*} \).

Then there exists \( S_1 \subseteq \overline{G}(cl_R, F) \) and \( H_1 \subseteq \overline{G}(cl_R, H) \).
such that $S_1 \cap H_1 = \emptyset$. Also there exist $S_2 \in \overline{O}_d(c_l g, G)$ and $H_2 \in \overline{O}_d(c_l r, H)$ such that $S_2 \cap H_2 = \emptyset$. Since $S_1 \cup S_2 \in \overline{O}_d(c_l p, F \cup G), H_1 \cap H_2 \in \overline{O}_d(c_l r, H)$ and $(S_1 \cup S_2) \cap (H_1 \cap H_2) = (S_1 \cap (H_1 \cap H_2)) \cup (S_2 \cap (H_1 \cap H_2)) = \emptyset$. We have $F \cup G \notin \Pi^+_{cl_R}(H)$.

3.14 Theorem
If $\Pi \in R_I$, where $i = k, t$ or $d$, then $(X, C_H)$ is a regular rough topological space satisfying property RT, where

Proof. It is sufficient to show for $d$. Let $x \notin C_H(B)$, then $H_1 \cap \overline{O}(\varphi, (x))$ and $H_2 \in \overline{O}(\varphi, B)$ such that $H_1 \cap H_2 = \emptyset$.

$\Rightarrow B = H_1 \cap H_2 \subset X - H_1 \cap C_H(X - H_1) = X - H_x$.

$\Rightarrow C_H(B) \subset X - H_x \Rightarrow \exists C_H(C_H(B)) \subset X - H_x$.

Hence $(X, C_H)$ is a topological space. Now to show regularity:

Let $x \notin F$. Then $\{x\} \notin \Pi(F) \Rightarrow E_x \supseteq \{x\}$ and $E_F \supseteq F$ such that $E_x \cap E_F = \emptyset$.

This proves that $(X, C_H)$ is regular.

3.15 Theorem
Let $(X, C)$ be a rough topological space satisfying property RT. Then $\Pi^+ = M_1(X, C_H), i = k, t, d$.

Proof. Since $(X, C)$ is a regular topological space satisfying property RT, and by [16] a rough topological space induced by similarity relation is regular, the result follows for $d$—proximity. Hence the result follows for $k$ and $t$ proximities as well.

3.16 Proposition
Let $i = k, t$ and $d$. Then $M_i(X, C_H) \neq \emptyset$ iff $(X, C_H)$ is a regular topological space satisfying RT. Moreover $\Pi^+_i$ is the smallest $i$—proximity in each case.

3.17 Lemma
Let $(X, C)$ be a regular topological space. Then $\Pi^+_i \in M_i(X, C_H), i = k, t, d$.

Proof. First, we have to show that $C = C_{\Pi^+_i}$ (for $k$—proximity). Let $x \notin C_{\Pi^+_i}(A)$.

$\Rightarrow \{x\}, A \notin \Pi^+_i \Rightarrow N_x \cap N_A = \emptyset$ for some $N_x \in O_0(C_H, \{x\})$ and $N_A \in (C_H, A)$.

$\Rightarrow x \in N_x$ and $A \subseteq A$ such that $N_x \cap N_A = \emptyset \Rightarrow x \notin (A)$. [Because this is the property of regular topological space and $C$ is regular topological space]. Since $\overline{R}_k \subseteq \overline{R}_t \subseteq \overline{R}_d$. We have to show only that $\Pi^+_i \in \Pi^+_k(X, C_H)$.

Let $G \notin \Pi^+_k(F)$. Then $F$ and $G$ are separated. This implies that $H_F \in \overline{O}_d(c_l r, F)$ and $H_G \in \overline{O}_d(c_l r, G)$ such that $H_F \cap H_G = \emptyset$. This implies $F \subseteq H_F$ and $G \subseteq H_G$ and $H_F \cap H_G = \emptyset$.

This implies that $X \subseteq \overline{cl}_R H_G$ and $G$ are disjoint sets.

$\Rightarrow G \notin \Pi^+_k\{X - \overline{cl}_R H_G\} \Rightarrow H_G \notin \Pi^+_k(F)$. [Because $F \subseteq H_F$ and $H_F \cap H_G = \emptyset$. So $F \cap H_G = \emptyset$].

4 Conclusion
This paper investigates $k, t$ and $d$—proximities on rough sets based on approximation operations. The method basically deals with general symmetric relation and approximation operation. The present study deals with proximities taking upper approximations only; however, the results hold good for lower approximations also. The non-trivial semi-proximity are in correspondence with the digital image used in computer graphics and can be seen in Latecki and Prokop [3]. The significance of this generalization is that we can generate new method to get rough proximities spaces with respect to each similarity relation and corresponding closure operator induced from relation. In view of this the present work will help to investigate or generalized the concept of proximities on rough set theory and generalized rough set theory.

References:

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