Almost weak continuity for multifunctions in ideal topological spaces

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Abstract: The main goal of this paper is to introduce the concepts of upper and lower almost weakly $*$-continuous multifunctions. Some characterizations of upper and lower almost weakly $*$-continuous multifunctions are investigated.

Key-Words: $*$-open set, upper almost weakly $*$-continuous multifunction, lower almost weakly $*$-continuous multifunction


1 Introduction

Continuity is an important concept for the study and investigation in topological spaces. The concept of weak continuity due to Levine [16] is one of the most important weak forms of continuity in topological spaces. Ekici et al. [8] established a new class of functions called weakly $\lambda$-continuous functions which is weaker than $\lambda$-continuous functions and investigated some fundamental properties of weakly $\lambda$-continuous functions. Popa and Noiri [19] introduced a new notion of weakly $(\tau, m)$-continuous functions as functions from a topological space into a set satisfying some minimal conditions and obtained some characterizations and several properties of such functions. Janković [14] defined almost weakly continuous functions as a generalization of both weakly continuous functions and almost continuous functions in the sense of Husain [12]. Smithson [22] and Popa [20, 21] extended independently these concepts to multifunctions and introduced upper (resp. lower) weakly continuous multifunctions and upper (resp. lower) almost continuous multifunctions. The present authors introduced and studied other weak forms of continuous functions: weakly quasicontinuous multifunctions, strong $\lambda$-continuous multifunctions, weakly $\alpha$-continuous multifunctions and weakly $\beta$-continuous multifunctions. These multifunctions have similar properties.

The purpose of the present paper is to introduce the concepts of upper and lower almost weakly $*$-continuous multifunctions as a multifunction from an ideal topological space into an ideal topological space and investigate several characterizations of such multifunctions.

The concept of ideals in topological spaces has been introduced and studied by Kuratowski [15] and Vaidyanathswamy [23]. Every topological space is an ideal topological space and all the results of ideal topological spaces are generalizations of the results established in topological spaces. In 1990, Janković and Hamlett [13] introduced the notion of $\mathcal{I}$-open sets in ideal topological spaces. Abd El-Monsef et al. [1] further investigated $\mathcal{I}$-open sets and $\mathcal{I}$-continuous functions. Later, several authors studied ideal topological spaces giving several convenient definitions. Some authors obtained decompositions of continuity. For instance, Açikgöz et al. [2] introduced and investigated the notions of weakly-$\mathcal{I}$-continuous and weak$^{*}$-$\mathcal{I}$-continuous functions in ideal topological spaces. Donthev [6] introduced the notion of per-$\mathcal{I}$-open sets and obtained a decomposition of $\mathcal{I}$-continuity. In [1], the present authors introduced the notions of semi-$\mathcal{I}$-open sets, $\alpha$-$\mathcal{I}$-open sets and $\beta$-$\mathcal{I}$-open sets via idealization and using these sets obtained new decompositions of continuity. Hatir and Noiri [10] investigated further properties of semi-$\mathcal{I}$-open sets and semi-$\mathcal{I}$-continuity. Moreover, the present authors [8] introduced and investigated the notions of strong $\beta$-$\mathcal{I}$-open sets and strongly $\beta$-$\mathcal{I}$-continuous functions.

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1 Introduction

Continuity is an important concept for the study and investigation in topological spaces. The concept of weak continuity due to Levine [16] is one of the most important weak forms of continuity in topological spaces. Ekici et al. [8] established a new class of functions called weakly $\lambda$-continuous functions which is weaker than $\lambda$-continuous functions and investigated some fundamental properties of weakly $\lambda$-continuous functions. Popa and Noiri [19] introduced a new notion of weakly $(\tau, m)$-continuous functions as functions from a topological space into a set satisfying some minimal conditions and obtained some characterizations and several properties of such functions. Janković [14] defined almost weakly continuous functions as a generalization of both weakly continuous functions and almost continuous functions in the sense of Husain [12]. Smithson [22] and Popa [20, 21] extended independently these concepts to multifunctions and introduced upper (resp. lower) weakly continuous multifunctions and upper (resp. lower) almost continuous multifunctions. The present authors introduced and studied other weak forms of continuous functions: weakly quasicontinuous multifunctions, strong $\lambda$-continuous multifunctions, weakly $\alpha$-continuous multifunctions and weakly $\beta$-continuous multifunctions. These multifunctions have similar properties. Noiri and Popa [18] have defined and investigated the notion of almost weakly continuous multifunctions. Ekici et al. [7] introduced the concept of almost contra-continuous multifunctions and investigated several characterizations of almost contra-continuous multifunctions. In [17], the present authors introduced the concept of upper (resp. lower) almost $m$-continuous multifunctions as multifunctions from a set satisfying some minimal conditions into a topological space and obtained some characterizations of such multifunctions.

The purpose of the present paper is to introduce the concepts of upper and lower almost weakly $*$-continuous multifunctions as a multifunction from an ideal topological space into an ideal topological space and investigate several characterizations of such multifunctions.
2 Preliminaries

Throughout the present paper, spaces \((X, \tau)\) and \((Y, \sigma)\) (or simply \(X\) and \(Y\)) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. In a topological space \((X, \tau)\), the closure and the interior of any subset \(A\) of \(X\) will denoted by \(\text{Cl}(A)\) and \(\text{Int}(A)\), respectively. An ideal \(\mathcal{I}\) on a topological space \((X, \tau)\) is a nonempty collection of subsets of \(X\) satisfying the following properties: (1) \(A \in \mathcal{I}\) and \(B \subseteq A\) implies \(B \in \mathcal{I}\); (2) \(A \in \mathcal{I}\) and \(B \in \mathcal{I}\) implies \(A \cup B \in \mathcal{I}\). A topological space \((X, \tau)\) with an ideal \(\mathcal{I}\) on \(X\) is called an ideal topological space and is denoted by \((X, \tau, \mathcal{I})\). Let \(A\) be a subset of \((X, \tau, \mathcal{I})\) and is denoted by \(\mathcal{I}\) for confusion, \(\mathcal{I}(A)\) the closure and the interior of any subset \(A\) is called an ideal topological space and is denoted by \((X, \tau, \mathcal{I})\).

Definition 1. \([2] \) A subset \(A\) of an ideal topological space \((X, \tau, \mathcal{I})\) and \(x \in X\). Then, the following properties hold:

\begin{enumerate}
  \item \(x \in \text{pCl}(A)\) if and only if \(U \cap A \neq \emptyset\) for every \(\mathcal{I}\)-open set \(U\) of \(X\) containing \(x\).
  \item \(A\) is \(\mathcal{I}\)-closed if and only if \(A = \text{pCl}(A)\).
  \item \(\text{pCl}(X - A) = X - \text{pInt}(A)\).
  \item \(\text{pInt}(X - A) = X - \text{pCl}(A)\).
\end{enumerate}

Lemma 2. \([2] \) Let \(A\) be a subset of an ideal topological space \((X, \tau, \mathcal{I})\) and \(x \in X\). Then, the following properties hold:

\begin{enumerate}
  \item \(x \in \text{Int}(\mathcal{I}\text{-cl}(A))\) if and only if \(U \cap A \neq \emptyset\) for every \(\mathcal{I}\)-open set \(U\) of \(X\) containing \(x\).
  \item \(A\) is \(\mathcal{I}\)-closed if and only if \(A = \text{pCl}(A)\).
  \item \(\text{pCl}(X - A) = X - \text{pInt}(A)\).
  \item \(\text{pInt}(X - A) = X - \text{pCl}(A)\).
\end{enumerate}

Lemma 3. For a subset \(A\) of an ideal topological space \((X, \tau, \mathcal{I})\), the following properties hold:

\begin{enumerate}
  \item \(\text{pCl}(A) = A \cup \text{Cl}(\mathcal{I}\text{-cl}(A))\) \([2]\).
  \item \(\text{pInt}(A) = A \cap \text{Int}(\mathcal{I}\text{-cl}(A))\).
\end{enumerate}

3 Some characterizations

In this section, we introduce the notions of upper and lower almost weakly \(\ast\)-continuous multifunctions. Moreover, several interesting characterizations of upper and lower almost weakly \(\ast\)-continuous multifunctions are investigated.

Definition 4. A multifunction \(F : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma, \mathcal{J})\) is said to be:

\begin{enumerate}
  \item upper almost weakly \(\ast\)-continuous if for each \(x \in X\) and each \(\ast\)-open set \(V\) of \(Y\) such that \(F(x) \subseteq V\), \(x \in \text{Int}(\mathcal{I}\text{-cl}(F^+(\text{Cl}(\mathcal{I}\text{-cl}(V))))))\);
  \item lower almost weakly \(\ast\)-continuous if for each \(x \in X\) and each \(\ast\)-open set \(V\) of \(Y\) such that \(F(x) \cap V \neq \emptyset\), \(x \in \text{Int}(\mathcal{I}\text{-cl}(F^-(\text{Cl}(\mathcal{I}\text{-cl}(V))))))\).
\end{enumerate}

The following results give some characterizations of upper and lower almost weakly \(\ast\)-continuous multifunctions.

Theorem 5. For a multifunction \(F : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma, \mathcal{J})\), the following properties are equivalent:

\begin{enumerate}
  \item \(F\) is upper almost weakly \(\ast\)-continuous;
  \item \(F^+(\mathcal{I}\text{-cl}(V)) \subseteq \text{Int}(\mathcal{I}\text{-cl}(F^+(\text{Cl}(\mathcal{I}\text{-cl}(V))))))\) for every \(\ast\)-open set \(V\) of \(Y\);
  \item \(\text{Cl}(\text{Int}(F^+(\mathcal{I}\text{-cl}(V)))) \subseteq F^-(\text{Cl}(\mathcal{J}\text{-cl}(V)))\) for every \(\ast\)-open set \(V\) of \(Y\);
  \item \(\text{pCl}(F^-(\mathcal{I}\text{-cl}(V))) \subseteq F^-(\text{Cl}(\mathcal{J}\text{-cl}(V)))\) for every \(\ast\)-open set \(V\) of \(Y\);
  \item \(F^+(\mathcal{I}\text{-cl}(V)) \subseteq \text{pInt}(F^+(\text{Cl}(\mathcal{J}\text{-cl}(V))))\) for every \(\ast\)-open set \(V\) of \(Y\);
\end{enumerate}
(6) for each \( x \in X \) and each \(*\)-open set \( V \) containing \( F(x) \), there exists a pre-\( \mathcal{J} \)-open set \( U \) of \( X \) containing \( x \) such that \( F(U) \subseteq \text{Cl}^*(V) \).

**Proof.** \((1) \Rightarrow (2)\): Let \( V \) be any \(*\)-open set of \( Y \) and \( x \in F^+(V) \). Then \( F(x) \subseteq V \). Since \( F \) is upper almost weakly \(*\)-continuous,

\[
x \in \text{Int}((\text{Cl}^*(F^+(\text{Cl}^*(V))))).
\]

Consequently, we obtain

\[
F^+(V) \subseteq \text{Int}((\text{Cl}^*(F^+(\text{Cl}^*(V))))).
\]

\((2) \Rightarrow (3)\): Let \( V \) be any \(*\)-open set of \( Y \). Since \( Y - \text{Cl}^*(V) \) is \(*\)-open and \((2)\), we have

\[
X - F^-(\text{Cl}^*(V)) = F^+(Y - \text{Cl}^*(V))
\]

\[
\subseteq \text{Int}((\text{Cl}^*(F^+(\text{Cl}^*(Y - \text{Cl}^*(V))))))
\]

\[
= \text{Int}((\text{Cl}^*(F^+(Y - \text{Int}^*(\text{Cl}^*(V))))))
\]

\[
\subseteq \text{Int}((\text{Cl}^*(F^+(Y - V))))
\]

\[
= \text{Int}((\text{Cl}^*(X - F^-(V))))
\]

\[
= X - \text{Int}((\text{Cl}^*(F^-(V))))
\]

and hence \( \text{Cl}(\text{Int}^*(F^-(V))) \subseteq F^-(\text{Cl}^*(V)) \).

\((3) \Rightarrow (4)\): Let \( V \) be any \(*\)-open set of \( Y \). By \((3)\) and Lemma 3\((1)\),

\[
p\text{Cl}(F^-(V)) = F^-(V) \cup \text{Cl}(\text{Int}^*(F^-(V)))
\]

\[
\subseteq F^-(\text{Cl}^*(V)) \cup \text{Cl}(\text{Int}^*(F^-(V)))
\]

\[
= F^-(\text{Cl}^*(V)).
\]

\((4) \Rightarrow (5)\): Let \( V \) be any \(*\)-open set of \( Y \). Since \( Y - \text{Cl}^*(V) \) is \(*\)-open and \((4)\), we have

\[
X - p\text{Int}(F^+(\text{Cl}^*(V))) = p\text{Cl}(X - F^+(\text{Cl}^*(V)))
\]

\[
= p\text{Cl}(F^-(Y - \text{Cl}^*(V)))
\]

\[
\subseteq F^-(\text{Cl}^*(Y - \text{Cl}^*(V)))
\]

\[
\subseteq F^-(\text{Cl}^*(Y - V))
\]

\[
= F^-(Y - V)
\]

\[
= X - F^+(V)
\]

and hence \( F^+(V) \subseteq p\text{Int}(F^+(\text{Cl}^*(V))) \).

\((5) \Rightarrow (6)\): Let \( x \in X \) and \( V \) be any \(*\)-open set of \( Y \) containing \( F(x) \). By \((5)\), we have

\[
x \in F^+(V) \subseteq p\text{Int}(F^+(\text{Cl}^*(V)))
\]

and so there exists a pre-\(*\)-open set \( U \) of \( X \) containing \( x \) such that \( U \subseteq F^+(\text{Cl}^*(V)) \); hence

\[
F(U) \subseteq \text{Cl}^*(V).
\]

Thus, \( U \subseteq F^+(\text{Cl}^*(V)) \) and hence

\[
x \in U \subseteq \text{Int}(\text{Cl}^*(U)) \subseteq \text{Int}(\text{Cl}^*(F^+(\text{Cl}^*(V)))).
\]

This shows that \( F \) is upper almost weakly \(*\)-continuous. \(\square\)

**Theorem 6.** For a multifunction \( F : (X, \tau, \mathcal{J}) \rightarrow (Y, \sigma, \mathcal{J}) \), the following properties are equivalent:

\((1)\) \( F \) is lower almost weakly \(*\)-continuous;

\((2)\) \( F^-(V) \subseteq \text{Int}(\text{Cl}^*(F^-(\text{Cl}^*(V)))) \) for every \(*\)-open set \( V \) of \( Y \);

\((3)\) \( \text{Cl}(\text{Int}^*(F^+(V))) \subseteq F^+(\text{Cl}^*(V)) \) for every \(*\)-open set \( V \) of \( Y \);

\((4)\) \( p\text{Cl}(F^+(V)) \subseteq F^+(\text{Cl}^*(V)) \) for every \(*\)-open set \( V \) of \( Y \);

\((5)\) \( F^-(V) \subseteq p\text{Int}(F^-(\text{Cl}^*(V))) \) for every \(*\)-open set \( V \) of \( Y \);

\((6)\) for each \( x \in X \) and each \(*\)-open set \( V \) containing \( F(x) \), there exists a pre-\( \mathcal{J} \)-open set \( U \) of \( X \) containing \( x \) such that \( F(U) \subseteq \text{Cl}^*(V) \).

**Proof.** The proof is similar to that of Theorem 5 \(\square\)

**Theorem 7.** For a multifunction \( F : (X, \tau, \mathcal{J}) \rightarrow (Y, \sigma, \mathcal{J}) \), the following properties are equivalent:

\((1)\) \( F \) is upper almost weakly \(*\)-continuous;

\((2)\) \( \text{Cl}(\text{Int}^*(F^-(\text{Int}^*(K)))) \subseteq F^- K \) for every \(*\)-closed set \( K \) of \( Y \);

\((3)\) \( p\text{Cl}(F^-(\text{Int}^*(K))) \subseteq F^- K \) for every \(*\)-closed set \( K \) of \( Y \);

\((4)\) \( p\text{Cl}(F^+(\text{Int}^*(\text{Cl}^*(B)))) \subseteq F^- \text{Cl}^*(B) \) for every subset \( B \) of \( Y \);

\((5)\) \( F^+(\text{Int}^*(B)) \subseteq p\text{Int}(F^+(\text{Cl}^*(\text{Int}^*(B)))) \) for every subset \( B \) of \( Y \).
Proof. (1) \(\Rightarrow\) (2): Let \(K\) be any \(*\)-closed set of \(Y\). Then \(Y - K\) is \(*\)-open in \(Y\) and by Theorem 5, we have

\[
\begin{align*}
X - F^-(K) &= F^+(Y - K) \\
&\subseteq \text{Int}(\text{Cl}^*(F^+(\text{Cl}^*(Y - K)))) \\
&= \text{Int}(\text{Cl}^*(F^+(Y - \text{Int}^*(K)))) \\
&= \text{Int}^*(X - F^-((\text{Int}^*)(K))) \\
&= X - \text{Cl}(\text{Int}^*(F^-(\text{Int}^*(K)))) \\
&= X - \text{Cl}(\text{Int}^*(F^-(\text{Int}^*(K)))) \\
\end{align*}
\]

and hence \(\text{Cl}(\text{Int}^*((F^-((\text{Int}^*(K)))))) \subseteq F^-(K).

(2) \(\Rightarrow\) (3): Let \(K\) be any \(*\)-closed set of \(Y\). By (2) and Lemma 5, (1),

\[
\begin{align*}
\text{pCl}(F^-(\text{Int}^*(K))) &= F^-((\text{Int}^*(K)) \cup \text{Cl}(\text{Int}^*(F^-((\text{Int}^*(K)))))) \\
&\subseteq F^-((\text{Int}^*(K)) \cup F^-(K)) = F^-(K).
\end{align*}
\]

(3) \(\Rightarrow\) (4): The proof is obvious.

(4) \(\Rightarrow\) (5): Let \(B\) be any subset of \(Y\). Then by (4), we have

\[
\begin{align*}
X - \text{pInt}(F^+(\text{Cl}^*(\text{Int}^*(B)))) &= \text{pCl}(X - F^+(\text{Cl}^*(\text{Int}^*(B)))) \\
&= \text{pCl}(F^-(Y - \text{Cl}^*(\text{Int}^*(B)))) \\
&= \text{pCl}(F^-(\text{Int}^*(\text{Cl}^*(Y - B)))) \\
&\subseteq F^-(\text{Cl}^*(Y - B)) \\
&= X - F^+(\text{Int}^*(B)) \\
\end{align*}
\]

and hence \(F^+(\text{Int}^*(B)) \subseteq \text{pInt}(F^+(\text{Cl}^*(\text{Int}^*(B))))\).

(5) \(\Rightarrow\) (1): Let \(V\) be any \(*\)-open set of \(Y\). Then by (5), we have \(F^+(V) \subseteq \text{pInt}(F^+(\text{Cl}^*(V)))\) and hence \(F\) is upper almost weakly \(*\)-continuous by Theorem 5.

\(\square\)

Theorem 8. For a multifunction \(F : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma, \mathcal{J})\), the following properties are equivalent:

(1) \(F\) is lower almost weakly \(*\)-continuous;

(2) \(\text{Cl}(\text{Int}^*((F^+(\text{Int}^*(K)))))) \subseteq F^+(K)\) for every \(*\)-closed set \(K\) of \(Y\);

(3) \(\text{pCl}(F^+(\text{Int}^*(K))) \subseteq F^+(K)\) for every \(*\)-closed set \(K\) of \(Y\);

(4) \(\text{pCl}(F^+(\text{Int}^*(\text{Cl}^*(B)))) \subseteq F^+(\text{Cl}^*(B))\) for every subset \(B\) of \(Y\);

(5) \(F^-(\text{Int}^*(B)) \subseteq \text{pInt}(F^-(\text{Cl}^*(\text{Int}^*(B))))\) for every subset \(B\) of \(Y\).

Proof. The proof is similar to that of Theorem 9.

\(\square\)

Definition 9. A function \(f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma, \mathcal{J})\) is said to be almost weakly \(*\)-continuous if

\[
f^{-1}(V) \subseteq \text{Int}(\text{Cl}^*(f^{-1}(\text{Cl}^*(V))))
\]

for each \(*\)-open set \(V\) of \(Y\).

Corollary 10. For a function \(f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma, \mathcal{J})\), the following properties are equivalent:

(1) \(f\) is almost weakly \(*\)-continuous;

(2) \(\text{Cl}(\text{Int}^*(f^{-1}(V))) \subseteq f^{-1}(\text{Cl}^*(V))\) for every \(*\)-open set \(V\) of \(Y);\)

(3) \(\text{pCl}(f^{-1}(V)) \subseteq f^{-1}(\text{Cl}^*(V))\) for every \(*\)-open set \(V\) of \(Y);\)

(4) \(f^{-1}(V) \subseteq \text{pInt}(f^{-1}(\text{Cl}^*(V)))\) for every \(*\)-open set \(V\) of \(Y);\)

(5) for each \(x \in X\) and each \(*\)-open set \(V\) of \(Y\) containing \(f(x)\), there exists a \(-\)-preopen set \(U\) of \(X\) containing \(x\) such that \(f(U) \subseteq \text{Cl}^*(V)\).

Definition 11. A point \(x\) in an ideal topological space \((X, \tau, \mathcal{I})\) is called a \(*\)-cluster point of \(A\) if

\[
\text{Cl}^*(U) \cap A \neq \emptyset
\]

for every \(*\)-open set \(U\) of \(X\) containing \(x\). The set of all \(*\)-cluster points of \(A\) is called the \(*\)-closure of \(A\) and is denoted by \(*\)-Cl\((A)\).

Definition 12. A subset \(A\) of an ideal topological space \((X, \tau, \mathcal{I})\) is called

(1) \(*\)-closed if \(*\)-Cl\((A) = A\).

(2) \(*\)-open if its complement is \(*\)-closed.

Lemma 13. For a subset \(A\) of an ideal topological space \((X, \tau, \mathcal{I})\), the following properties hold:

(1) If \(A\) is \(*\)-open in \(X\), then \(\text{Cl}^*(A) = \ast\text{Cl}(A)\).

(2) \(*\)-Cl\((A)\) is \(*\)-closed in \(X\).

Definition 14. A subset \(A\) of an ideal topological space \((X, \tau, \mathcal{I})\) is said to be:

(1) \(-\text{I}^*\)-open if \(A = \text{Int}^*(\text{Cl}^*(A))\);

(2) \(-\text{I}^*\)-closed if its complement is \(-\text{I}^*\)-open;

(3) \(\text{I}^*\)-preopen if \(A \subseteq \text{Int}^*(\text{Cl}^*(A))\);

(4) \(\text{I}^*\)-preclosed if its complement is \(\text{I}^*\)-preopen.

Theorem 15. For a multifunction \(F : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma, \mathcal{J})\), the following properties are equivalent:

(1) \(F\) is upper almost weakly \(*\)-continuous;
It follows from Theorem 5 that

\[ p \text{Cl}(F^{-}(\text{Int}^{*}(\ast_{0}\text{Cl}(B)))) \subseteq F^{-}(\ast_{0}\text{Cl}(B)) \]

for every subset \( B \) of \( Y \);

(3) \( p \text{Cl}(F^{-}(\text{Int}^{*}(\text{Cl}^{*}(V)))) \subseteq F^{-}(\text{Cl}^{*}(V)) \) for every \( \ast \)-open set \( V \) of \( Y \);

(4) \( p \text{Cl}(F^{-}(\text{Int}^{*}(\text{Cl}^{*}(V)))) \subseteq F^{-}(\text{Cl}^{*}(V)) \) for every \( \ast \)-preopen set \( V \) of \( Y \);

(5) \( p \text{Cl}(F^{-}(\text{Int}^{*}(K))) \subseteq F^{-}(K) \) for every \( R-\mathcal{J} \)-closed set \( K \) of \( Y \).

Proof. (1) \( \Rightarrow \) (2): Let \( B \) be any subset of \( Y \). Suppose that \( x \in X - F^{-}(\ast_{0}\text{Cl}(B)) \). Then, we have

\[ F(x) \subseteq Y - \ast_{0}\text{Cl}(B). \]

Since \( Y - \ast_{0}\text{Cl}(B) \) is \( \ast \)-open in \( Y \) and by Theorem 5, there exists a \( \ast \)-open set \( U \) of \( X \) containing \( x \) such that \( F(U) \subseteq \text{Cl}^{*}(Y - \ast_{0}\text{Cl}(B)) \). Therefore,

\[ U \subseteq F^{+}(\text{Cl}^{*}(Y - \ast_{0}\text{Cl}(B))) \]

\[ = F^{+}(Y - \text{Int}^{*}(\ast_{0}\text{Cl}(B))) \]

\[ = X - F^{-}(\text{Int}^{*}(\ast_{0}\text{Cl}(B))). \]

This implies that \( U \cap F^{-}(\text{Int}^{*}(\ast_{0}\text{Cl}(B))) = \emptyset \) and hence \( x \in X - p \text{Cl}(F^{-}(\text{Int}^{*}(\ast_{0}\text{Cl}(B)))) \). Consequently, we obtain

\[ p \text{Cl}(F^{-}(\text{Int}^{*}(\ast_{0}\text{Cl}(B)))) \subseteq F^{-}(\ast_{0}\text{Cl}(B)). \]

(2) \( \Rightarrow \) (3): This follows from Lemma 13(1).

(3) \( \Rightarrow \) (4): Let \( V \) be any \( \mathcal{J} \)-preopen set of \( Y \). Then \( V \subseteq \text{Int}^{*}(\text{Cl}^{*}(V)) \) and by (3),

\[ p \text{Cl}(F^{-}(\text{Int}^{*}(\text{Cl}^{*}(V)))) \]

\[ = p \text{Cl}(F^{-}(\text{Int}^{*}(\text{Cl}^{*}(\text{Int}^{*}(\text{Cl}^{*}(V)))))) \]

\[ \subseteq F^{-}(\text{Cl}^{*}(\text{Int}^{*}(\text{Cl}^{*}(V)))) \]

\[ = F^{-}(\text{Cl}^{*}(V)). \]

(4) \( \Rightarrow \) (5): Let \( K \) be any \( R-\mathcal{J} \)-closed set of \( Y \). Since \( \text{Int}^{*}(K) \) is \( \mathcal{J} \)-preopen in \( Y \) and by (4),

\[ p \text{Cl}(F^{-}(\text{Int}^{*}(K))) = p \text{Cl}(F^{-}(\text{Int}^{*}(\text{Cl}^{*}(\text{Int}^{*}(K)))))) \]

\[ \subseteq F^{-}(\text{Cl}^{*}(\text{Int}^{*}(K))) = F^{-}(K). \]

(5) \( \Rightarrow \) (1): Let \( V \) be any \( \ast \)-open set of \( Y \). Then \( \text{Cl}^{*}(V) \) is \( R-\mathcal{J} \)-closed in \( Y \) and by (5), we have

\[ p \text{Cl}(F^{-}(V)) \subseteq p \text{Cl}(F^{-}(\text{Int}^{*}(\text{Cl}^{*}(V)))) \]

\[ \subseteq F^{-}(\text{Cl}^{*}(V)). \]

It follows from Theorem 5 that \( F \) is upper almost weakly \( \ast \)-continuous.

\[ \square \]

Theorem 16. For a multifunction \( F : (X, \tau, \mathcal{J}) \rightarrow (Y, \sigma, \mathcal{J}) \), the following properties are equivalent:

(1) \( F \) is lower almost weakly \( \ast \)-continuous;

(2) \( p \text{Cl}(F^{+}(\text{Int}^{*}(\ast_{0}\text{Cl}(B)))) \subseteq F^{+}(\ast_{0}\text{Cl}(B)) \) for every subset \( B \) of \( Y \);

(3) \( p \text{Cl}(F^{+}(\text{Int}^{*}(\text{Cl}^{*}(V)))) \subseteq F^{+}(\text{Cl}^{*}(V)) \) for every \( \ast \)-open set \( V \) of \( Y \);

(4) \( p \text{Cl}(F^{+}(\text{Int}^{*}(\text{Cl}^{*}(V)))) \subseteq F^{+}(\text{Cl}^{*}(V)) \) for every \( \mathcal{J} \)-open set \( V \) of \( Y \);

(5) \( p \text{Cl}(F^{+}(\text{Int}^{*}(K))) \subseteq F^{+}(K) \) for every \( R-\mathcal{J} \)-closed set \( K \) of \( Y \).

Proof. The proof is similar to that of Theorem 13.

\[ \square \]

Corollary 17. For a function \( f : (X, \tau, \mathcal{J}) \rightarrow (Y, \sigma, \mathcal{J}) \), the following properties are equivalent:

(1) \( f \) is almost weakly \( \ast \)-continuous;

(2) \( p \text{Cl}(f^{-1}(\text{Int}^{*}(\ast_{0}\text{Cl}(B)))) \subseteq f^{-1}(\ast_{0}\text{Cl}(B)) \) for every subset \( B \) of \( Y \);

(3) \( p \text{Cl}(f^{-1}(\text{Int}^{*}(\text{Cl}^{*}(V)))) \subseteq f^{-1}(\text{Cl}^{*}(V)) \) for every \( \ast \)-open set \( V \) of \( Y \);

(4) \( p \text{Cl}(f^{-1}(\text{Int}^{*}(\text{Cl}^{*}(V)))) \subseteq f^{-1}(\text{Cl}^{*}(V)) \) for every \( \mathcal{J} \)-open set \( V \) of \( Y \);

(5) \( p \text{Cl}(f^{-1}(\text{Int}^{*}(K))) \subseteq f^{-1}(K) \) for every \( R-\mathcal{J} \)-closed set \( K \) of \( Y \).

4 Conclusion

The branch of mathematics called topology is concerned with all questions directly or indirectly related to continuity. Especially, the study of continuity has been found to be very useful in computer science and digital topology. Continuity of multifunctions in topological spaces has been researched by many mathematicians. This paper deals with the concepts of upper and lower almost weakly \( \ast \)-continuous multifunctions in ideal topological spaces. Several characterizations of upper and lower almost weakly \( \ast \)-continuous multifunctions are obtained. The ideas and results of this paper may motivate further research.

References:


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