Abstract: - The purpose of the present paper is to introduce the new class of $\omega b$–topological vector spaces. We study several basic and fundamental properties of $\omega b$–topological and investigate their relationships with certain existing spaces. Along with other results, we prove that transformation of an open (resp. closed) set in a $\omega b$–topological vector space is $\omega b$–open (resp. closed). In addition, some important and useful characterizations of $\omega b$–topological vector spaces are established. We also introduce the notion of almost $\omega b$–topological vector spaces and present several general properties of almost $\omega b$–topological vector spaces.

Key-Words: - $\omega b$–open set, $\omega b$–closed set, topological vector space $\omega b$–topological vector space, almost $\omega b$–topological vector space


1 Introduction

Functional analysis in its traditional sense deals primarily with Banach spaces and, in particular, Hilbert spaces. However, many classical vector spaces have natural topologies not given by the norm. Such, for example, are many spaces of smooth, holomorphic and generalized function. The theory of topological vector spaces is the science of spaces of precisely this kind. The concept of topological vector spaces was first introduced and studied by Kolmogroff [19] in 1934. Since then, different researchers have explored many interesting and useful properties of topological vector spaces. Due to nice properties, topological vector spaces earn a great importance and remain a fundamental notion in fixed point theory, operator theory, and various advanced branches of Mathematics. Now a days the researchers not only make use of topological vector spaces in other fields for developing new notions but also strength and extend the concept of topological vector spaces with every possible way, making the field of study a more convenient and understandable. Recently, Khan et al [17] defined $s$–topological vector spaces as a generalization of topological vector spaces. Khan and Iqbal [18], in 2016, put forth the concept of irresolute topological vector spaces which is independent of topological vector spaces but is included in $s$–topological vector spaces. Ibrahim [15] introduced the study of $\alpha$–topological vector spaces. In 2018, Sharma et al [31] introduced and studied another class of spaces, namely almost pre topological vector spaces. In [28, 31, 32, 33] the Ram et al introduced the $\beta$–topological vector spaces and pre topological vector spaces, almost $\beta$–topological vector spaces and almost $s$–topological vector spaces and studied properties and characterizations of these spaces. In 2018, Noiri et al [26] introduced and investigated a new class of sets called $\omega b$–open sets which is weaker than $\omega$–open sets and $b$–open sets. In the present paper, we introduce and investigate several properties and characterizations of $\omega b$–topological vector spaces. The relationship of $\omega b$–topological vector spaces with certain types of spaces is investigated as well. We will also introduce and study almost $\omega b$–topological vector spaces.

2 Preliminaries

Throughout this paper, $(X, \tau)$ and $(Y, \sigma)$ (or simply, $X$ and $Y$) denote topological spaces on which no separation axioms are assumed unless explicitly stated.
For a subset $A$ of a space $(X, \tau)$, $Cl(A)$, $Int(A)$ and $X - A$ denote the closure of $A$, the interior of $A$ and the complement of $A$ in $X$, respectively. Recently, as generalization of closed sets, the notion of $\omega$-closed sets were introduced and studied by Hdeib [13]. Let $(X, \tau)$ be a topological space and let $A$ be a subset of $X$. A point $x$ of $X$ is called a condensation point of $A$ if for each open set $U$ with $x \in U$, the set $U - A$ is uncountable. A subset $A$ is said to be $\omega$-closed [7] if it contains all its condensation points. The complement of an $\omega$-closed set is said to be an $\omega$-open set. It is well known that a subset $W$ of a space $(X, \tau)$ is $\omega$-open if and only if for each $x \in W$, there exists an open set $U$ such that $x \in U$ and $U - W$ is countable. The family of all $\omega$-open subsets of a topological space $(X, \tau)$, is denoted by $\omega O(X, \tau)$, forms a topology on $X$ which is finer than $\tau$. The set of all $\omega$-open sets of $(X, \tau)$ containing a point $x \in X$ is denoted by $\omega O(X, x)$. The complement of an $\omega$-open set is said to be an $\omega$-closed set. The intersection of all $\omega$-closed sets of $X$ containing $A$ is called the $\omega$-closure of $A$ and is denoted by $\omega Cl(A)$. The union of all $\omega$-open sets of $X$ contained in $A$ is called $\omega$-interior of $A$ and is denoted by $\omega Int(A)$. A subset $A$ of a topological space $X$ is said to be $b$-open if $A \subseteq Int[Cl(A)] \cup Cl[Int(A)]$. The complement of a $b$-open set is called a $b$-closed set. The intersection of all $b$-closed sets containing $A$ is called the $b$-closure of $A$ and is denoted by $b Cl(A)$. The union of all $b$-open sets of $X$ contained in $A$ is called the $b$-interior of $A$ and is denoted by $b Int(A)$.

**Definition 2.1.** A subset $A$ of a space $X$ is said to be $\omega b$-open if for every $x \in A$, there exists a $b$-open subset $U_x \subseteq X$ containing $x$ such that $U_x - A$ is countable. The complement of an $\omega b$-open subset of $X$ is called an $\omega b$-closed subset of $X$.

The family of all $\omega b$-open sets in a topological space $(X, \tau)$ is denoted by $\omega b - O(X, \tau)$ or $\omega b - O(X)$. The family of all $\omega b$-closed sets in a topological space $(X, \tau)$ is denoted by $\omega b - C(X, \tau)$ or $\omega b - C(X)$.

For any $x \in X$, we present $\omega b - O(X, x) = \{U \subseteq X : x \in U \text{ and } U \text{ is } \omega b\text{-open in } X\}$.

**Lemma 2.2.** For a subset of a topological space, both $\omega$-openness and $b$-openness imply $\omega b$-openness.

**Proof.** (1) Assume $A$ is $\omega$-open, then for each $x \in A$, there is an open set $U_x$ containing $x$ such that $U_x - A$ is a countable set. Since every open set is $b$-open, $A$ is $\omega b$-open.

(2) Let $A$ be $b$-open. For each $x \in A$, there exists a $b$-open set open set $U_x = A$ such that $x \in U_x$ and $U_x - A = \emptyset$. Therefore, $A$ is $\omega b$-open.

The following diagram shows the implications for properties of subsets

$$
\begin{align*}
\text{open set} & \Rightarrow \text{b-open set} \\
\downarrow & \\
\omega\text{-open set} & \Rightarrow \omega b\text{-open set}
\end{align*}
$$

The converses need not be true as shown by the examples by Noiri et al. [26] in 2008.

**Lemma 2.3.** A subset $A$ of a space $X$ is $\omega b$-open if and only if for every $x \in A$, there exists a $b$-open subset $U$ containing $x$ and a countable subset $C$ such that $U - C \subseteq A$.

**Lemma 2.4.** Let $(X, \tau)$ be a topological space.

(1) The intersection of an open set and a $b$-open set is $b$-open set.

(2) The union of any family of $b$-open sets is a $b$-open set.

**Proposition 2.5.** The intersection of an $\omega b$-open set and a $b$-open set is $\omega b$-open.

**Proposition 2.6.** The intersection of an $\omega b$-open set with an open set is $\omega b$-open.

The intersection of two $\omega b$-open sets is not always $\omega b$-open.

**Example 2.7.** Let $X = \mathbb{R}$ with the usual topology $\tau$. Let $A = Q$ be the set of rational numbers and $B = [0, 1)$. Then $A$ and $B$ are $\omega b$-open, but $A \cap B$ is not $\omega b$-open, since each $b$-open set containing $0$ is uncountable set.

**Proposition 2.8.** The union of any family of $\omega b$-open set is $\omega b$-open.

**Proof.** Let $\{A_i : i \in I\}$ be a collection of $\omega b$-open subsets of a space $X$. Then for any $x \in \bigcup_{i \in I} A_i$, there exists $\mu \in I$ such that $x \in A_\mu$. Hence there exists a $b$-open subset $U$ of $X$ such that $U - A_\mu$ is countable. Now because $(\bigcup_{i \in I} A_i) \subseteq U - A_\mu$, and hence $(\bigcup_{i \in I} A_i)$ is countable. Therefore, $x \in \bigcup_{i \in I} A_i$ is $\omega b$-open.
The intersection of all $\omega b$–closed sets of $X$ containing $A$ is called the $\omega b$–closure of $A$ and is denoted by $\omega b Cl (A)$. The union of all $\omega b$–open sets of $X$ contained in $A$ is called the $\omega b$–interior of $A$ and is denoted by $\omega blnt (A)$.

**Definition 2.9.** A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be $\omega b$–continuous if for each open set $W$ in $Y$, the inverse image $f^{-1}(W) \in \omega b – O(X)$.

Equivalently a mapping $f : (X, \tau) \rightarrow (Y, \sigma)$ is called $\omega b$–continuous if for each open set $V$ in $Y$ containing $f (x)$, there exists an $\omega b$–open set $U$ in $X$ containing $x$ such that $f (U) \subseteq V$.

**Definition 2.10.** Let $X$ be a vector space over the field $K$, where $K = \mathbb{R}$ or $\mathbb{C}$ with standard topology. Let $\tau$ be a topology on $X$ such that the following conditions are satisfied:

1. For each $x, y \in X$ and each open set $W \subseteq X$ containing $x + y$, there exist open sets $U$ and $V$ in $X$ containing $x$ and $y$, respectively, such that $U + V \subseteq W$.

2. For each $\lambda \in K$, $x \in X$ and each open set $W \subseteq X$ containing $\lambda x$, there exist open sets $U$ in $K$ containing $\lambda$ and $V$ in $X$ containing $x$, such that $U \cdot V \subseteq W$.

Then the pair $(X, \tau)$ is called a topological vector space.

### 3 $\omega b$–Topological Vector Spaces

**Definition 3.1.** Let $X$ be a vector space over the field $K$, where $K = \mathbb{R}$ or $\mathbb{C}$ with standard topology. Let $\tau$ be a topology on $X$ such that the following conditions are satisfied:

1. For each $x, y \in X$ and each open set $W \subseteq X$ containing $x + y$, there exist $\omega b$–open sets $U$ and $V$ in $X$ containing $x$ and $y$, respectively, such that $U + V \subseteq W$.

2. For each $\lambda \in K$, $x \in X$ and each open set $W \subseteq X$ containing $\lambda x$, there exist $\omega b$–open sets $U$ in $K$ containing $\lambda$ and $V$ in $X$ containing $x$, such that $U \cdot V \subseteq W$.

Let $X = \mathbb{R}$ be the vector space of real numbers over the field $K = \mathbb{R}$, where $X = K = \mathbb{R}$ is endowed with standard topology. Then $(X, \tau)$ is a $\omega b$–topological vector space.

Obviously, every topological vector space is an $\omega b$–topological vector space but the converse is not always true. The following are examples of $\omega b$–topological vector spaces which are not topological vector space.

**Example 3.2.** Let $X = \mathbb{R}$ be the vector space of real numbers over the field $K = \mathbb{R}$, where $X = K = \mathbb{R}$ is endowed with standard topology. Then $(X, \tau)$ is a $\omega b$–topological vector space.

**Example 3.3.** Consider the field $K = \mathbb{R}$ with standard topology. Let $X = \mathbb{R}$ be endowed with the topology $\tau = \{ \emptyset, D, \mathbb{R} \}$, where $D$ denotes the set of irrational numbers. We show that $(X, \tau)$ is an $\omega b$–TVS. For this purpose, we have to prove the following:

1. Let $x, y \in X$. If $x + y$ is rational, then the only open set in $X$ containing $x + y$ is $\mathbb{R}$. So, there is nothing to prove.

If $x + y$ is irrational, then for open neighbourhood $W = D$ of $x + y$ in $X$, we have following cases:

Case (i). If both $x$ and $y$ are irrational, we can choose $\omega b$–open sets $U = \{ x \}$ and $V = \{ y \}$ in $X$ such that $U + V \subseteq W$.

Case (ii). If one of $x$ or $y$ is rational, say $y$. Then, for the selection of $\omega b$–open sets $U = \{ x \}$ and $V = \{ p, y \}$ in $X$, where $p \in D$ such that $p + x \in D$, we have $U + V \subseteq W$. 


This proves the first condition of the definition of \( \omega b \)-topological vector space.

(2) Let \( \lambda \in \mathbb{R} \) and \( x \in X \). If \( \lambda x \) is rational, then verification is straightforward.

Suppose \( \lambda x \) is irrational. Let \( W = D \) be an open neighbourhood of \( \lambda x \). Then the following cases arise:

Case (i). If \( \lambda \) and \( x \) are irrational, then, choose \( \omega b \)-open sets \( U = \left( (\lambda - \varepsilon, \lambda + \varepsilon) I Q \right) \) and \( V = \{ x \} \) in \( X \), we have \( U.V \subseteq W \).

Case (ii). If \( \lambda \) is rational and \( x \) is irrational, then for the selection of \( \omega b \)-open sets \( U = \left( (\lambda - \varepsilon, \lambda + \varepsilon) I Q \right) \) and \( V = \{ x \} \) in \( X \), we have \( U.V \subseteq W \).

Case (iii). Finally, suppose \( \lambda \) is irrational and \( x \) is rational. Then for the choice of \( \omega b \)-open sets, take \( U = \left( (\lambda - \varepsilon, \lambda + \varepsilon) I D \right) \) in \( X \) and \( V = \{ x, p \} \) in \( X \), where the selection of \( p \in D \) is such that \( p.q \) is irrational for each \( q \in U \). Then we obtain \( U.V \subseteq W \).

Hence \( (X, \tau) \) is a \( \omega b \)-topological vector space.

**Example 3.4.** Let \( X = \mathbb{R} \) be a vector space of real numbers over the field \( K = \mathbb{R} \) and let \( \tau \) be a topology on \( X \) induced by open intervals \( (a, b) \) and \( [a, b] \) where \( a, b \in \mathbb{R} \). We observe that \( (\mathbb{R}, \tau) \) is an \( \omega b \)-topological vector space over the field \( \mathbb{R} \) with the topology \( \tau \) defined on \( \mathbb{R} \). We note that for all \( x, y \in \mathbb{R} \) and each open neighbourhood \( W \) of \( x+y \) in \( X \), there exist \( \omega b \)-open neighbourhoods \( U \) and \( V \) of \( x \) and \( y \) respectively in \( X \) such that \( U+V \subseteq W \).

Also for each \( \lambda \in \mathbb{R} \), \( x \in \mathbb{R} \) and for each open neighbourhood \( W \) of \( \lambda x \) in \( X \), there exist \( \omega b \)-open neighbourhoods \( U \) of \( \lambda \) in \( \mathbb{R} \) and \( V \) of \( x \) in \( X \) such that \( U\lambda V \subseteq W \). However, \( (\mathbb{R}, \tau) \) is not a topological vector space because, for instance if, we choose \( x = -3 \), \( y = 4 \) and an open neighbourhood \( W = [1, 2] \) of \( x+y \) in \( X \), we can not find open neighbourhoods \( U \) and \( V \) containing \( x \) and \( y \) respectively in \( X \) which satisfy the condition \( U.V \subseteq W \).

**Theorem 3.5.** Let \( (X, \tau) \) be an \( \omega b \)-topological vector space. Suppose \( T_\lambda : X \to X \) is a translation mapping defined by \( T_\lambda (y) = y + x \) for each \( y \in X \) (where \( x \) is any fixed element of \( X \)), and \( M_\lambda : X \to X \) is a multiplication mapping defined by \( M_\lambda (x) = \lambda x \), for each \( x \in X \) (\( \lambda \neq 0 \) is a fixed scalar). Then prove that \( T_\lambda \) and \( M_\lambda \) both are \( \omega b \)-continuous.

**Proof.** Let \( y \) be an arbitrary element in \( X \) and \( W \) be an open neighbourhood of \( T_\lambda (y) = y + x \). By definition of \( \omega b \)-topological vector spaces, there exist \( \omega b \)-open neighbourhoods \( U \) and \( V \) containing \( y \) and \( x \) respectively, such that \( U + V \subseteq W \). In particular, we have \( U + x \subseteq W \) which means \( T_\lambda (U) \subseteq W \). The inclusion shows that \( T_\lambda \) is \( \omega b \)-continuous at \( y \). Hence \( T_\lambda \) is \( \omega b \)-continuous on \( X \). Now we prove the statement for multiplication mapping. Let \( \lambda \in K \) and \( x \in X \). Let \( W \) be an open neighbourhood of \( M_\lambda (x) = \lambda x \). By definition of \( \omega b \)-topological vector spaces, there exist \( \omega b \)-open neighbourhoods \( U \) and \( V \) containing \( \lambda \) and \( x \) respectively, such that \( U \lambda V \subseteq W \). In particular, we have \( \lambda \lambda V \subseteq W \), which means \( M_\lambda (V) \subseteq W \). The inclusion shows that \( M_\lambda \) is \( \omega b \)-continuous at \( x \). Hence \( M_\lambda \) is \( \omega b \)-continuous on \( X \).

**Theorem 3.6.** Let \( A \) be an open subset of an \( \omega b \)-topological vector space \( (X, \tau) \). Then the following statements are true:

1. \( x + A \in \omega b - O(X) \) for each \( x \in X \).

2. \( \lambda A \in \omega b - O(X) \) for each non-zero scalar \( \lambda \).

**Proof.** (1). Let \( y \in x + A \). Then there exists an \( a \in A \) such that \( y = x + a \). Hence, \( a = (x + y) + y \in A \). Since \( A \) is open set in \( X \), there exist \( \omega b \)-open sets \( U, V \in \omega b - O(X) \) containing \( x \) and \( y \), respectively such that \( U + V \subseteq A \). Therefore, \( \neg x + V \subseteq A \Rightarrow \).
\[ V \subseteq x + A \Rightarrow y \in \omega b - \text{Int}(x + A) \Rightarrow x + A \subseteq \omega b - \text{Int}(x + A). \] Hence it follows that \[ x + A = \omega b - \text{Int}(x + A). \] This proves that \( x + A \) is an \( \omega b \)-open set in \( X \).

(2). Let \( x \in A \). Then \( x = \lambda a \) for some \( a \in A \). Hence \[ a = \left( \frac{1}{\lambda} \right) x \in A \] and \( A \) is an open set in \( X \). So by the definition of \( \omega b \)-topological vector spaces, there exist \( \omega b \)-open sets \( U \) in \( K \) containing \( \frac{1}{\lambda} \) and \( V \) in \( X \) containing \( x \) such that \( U V \subseteq A \). Whence we find that \( x \in V \subseteq \lambda A \). This shows that \( x \in \omega b \text{Int}(\lambda A) \). Since \( x \in \lambda A \) was arbitrary, it follows that \( \lambda A = \omega b \text{Int}(\lambda A) \). Thus, \( \lambda A \in \omega b - O(X) \).

**Corollary 3.7.** Let \( A \) be any non-empty open subset of a \( \omega b \)-topological vector space \( X \). Then the following statements are true:

(i). \( x + A \subseteq \text{Int}[\text{Cl}(\text{Int}(x + A))] \) for each \( x \in X \).

(ii). \( \lambda A \subseteq \text{Int}[\text{Cl}(\text{Int}(\lambda A))] \) for each non-zero scalar \( \lambda \).

**Corollary 3.3.** If \( U \) is any non-empty open set and \( B \) is any set in an \( \omega b \)-topological vector space \( X \), then \( B + U \in \omega b - O(X) \).

**Theorem 3.4.** Let \( (X, \tau) \) be an \( \omega b \)-topological vector space and let \( F \) be a closed subset of \( X \). Then the following statements are true:

(i). \( x + F \in \omega b - C(X) \) for each \( x \in X \),

(ii). \( \lambda F \in \omega b - C(X) \) for each non-zero scalar \( \lambda \).

**Proof.** (1). Suppose that \( y \in \omega b - \text{Cl}(x + F) \). Consider \( z = -x + y \) and let \( W \) be any open set in \( X \) containing \( z \). Then there exists \( \omega b \)-open sets \( U \) and \( V \) in \( X \) such that \( -x \in U \), \( y \in V \) and \( U + V \subseteq W \). Since \( y \in \omega b - \text{Cl}(x + F) \), \( (x + F) \cap V \neq \emptyset \). So, there is an \( a \in (x + F) \cap V \).

\[ -x + a \in F I (U + V) \subseteq F I W \Rightarrow F I W \neq \emptyset \Rightarrow z \in \text{Cl}(F) = F \Rightarrow y \in x + F \] It follows that \( x + F = \omega b - \text{Cl}(x + F) \). This proves that \( x + F \) is \( \omega b \)-closed set in \( X \).

(2). Assume that \( x \in \omega b - \text{Cl}(\lambda F) \) and let \( W \) be any open set in \( X \) containing \( \frac{1}{\lambda} x \). Since \( X \) is \( \omega b \)-topological vector space, there exist \( \omega b \)-open sets \( U \) in \( K \) containing \( \frac{1}{\lambda} \) and \( V \) in \( X \) containing \( x \) such that \( U V \subseteq A \). Whence we find that \( x \in V \subseteq \lambda A \). This shows that \( x \in \omega b \text{Int}(\lambda A) \). Since \( x \in \lambda A \) was arbitrary, it follows that \( \lambda A = \omega b \text{Int}(\lambda A) \). Thus, \( \lambda A \in \omega b - O(X) \).

**Theorem 3.5.** Let \( A \) and \( B \) be any subsets of a \( \omega b \)-topological vector space \( X \). Then \( \omega b - \text{Cl}(A) + \omega b - \text{Cl}(B) \subseteq \text{Cl}(A + B) \).

**Proof.** Let \( z \in \omega b - \text{Cl}(A) + \omega b - \text{Cl}(B) \). Then \( z = x + y \) for some \( x \in \omega b - \text{Cl}(A) \) and \( y \in \omega b - \text{Cl}(B) \). Let \( W \) be any open neighbourhood of \( z \) in \( X \). By definition of \( \omega b \)-topological vector spaces, there exist \( \omega b \)-open sets \( U \) and \( V \) in \( X \) containing \( x \) and \( y \) respectively, such that \( U + V \subseteq W \). Hence \( \omega b - \text{Cl}(A) + \omega b - \text{Cl}(B) \subseteq \text{Cl}(A + B) \).

**Theorem 3.6.** Let \( A \) and \( B \) be any subsets of an \( \omega b \)-topological vector space \( X \). Then \( A + \text{Int}(B) \subseteq \omega b - \text{Int}(A + B) \).

**Proof.** Let \( z \in A + \text{Int}(B) \) be arbitrary. Then \( z = x + y \) for some \( x \in A \) and \( y \in \text{Int}(B) \). This results in \( -x + z \in \text{Int}(B) \) and consequently by definition of
\( \omega b \text{-TVS} \), there exist \( \omega b \text{-} \)open sets \( U \) and \( V \) in \( X \) containing \( -x \) and \( z \) respectively, such that \( U + V \subseteq \text{Int}(B) \). In particular, \( -x + V \subseteq \text{Int}(B) \) \( \Rightarrow \) \( V \subseteq x + \text{Int}(B) \subseteq A + B \). Since \( V \in \omega b-O(X) \), \( z \in V \subseteq \omega b-\text{Int}(A + B) \) \( \Rightarrow \) \( z \in \omega b-\text{Int}(A + B) \). Thus \( A + \text{Int}(B) \subseteq \omega b-\text{Int}(A + B) \).

4 Characterizations of \( \omega b \text{-TVS} \).

**Theorem 4.1.** Let \( A \) be any subset of an \( \omega b \text{-} \)topological vector space \( X \). Then the following assertions are true:

1. \( \omega b-\text{Cl}(x + A) \subseteq x + \text{Cl}(A) \) for each \( x \in X \).
2. \( x + \omega b-\text{Cl}(A) \subseteq \text{Cl}(x + A) \) for each \( x \in X \).
3. \( x + \text{Int}(A) \subseteq \omega b-\text{Int}(x + A) \) for each \( x \in X \).
4. \( \text{Int}(x + A) \subseteq x + \omega b-\text{Int}(A) \) for each \( x \in X \).

**Proof.** (1). Let \( y \in \omega b-\text{Cl}(x + A) \) and consider \( z = -x + y \) in \( X \). Let \( W \) be any open set in \( X \) containing \( z \). Since \( X \) is an \( \omega b \text{-} \)topological vector space, there exist \( \omega b \text{-} \)open sets \( U \) and \( V \) in \( X \) containing \( -x \) and \( y \) respectively, such that \( U + V \subseteq W \). Since \( y \in \omega b-\text{Cl}(x + A) \), there is \( a \in (x + A) I V \). Then it follows that \( -x + a \in (-x + x + A) I (U + V) = A1 (U + V) \subseteq A1 W \). showing that \( A1 W \neq \phi \) and hence \( z \in \text{Cl}(A) \). That is, \( y \in x + \text{Cl}(A) \). Therefore, \( \omega b-\text{Cl}(x + A) \subseteq x + \text{Cl}(A) \).

(2). Let \( z \in x + \omega b-\text{Cl}(A) \). Then \( z = x + y \) for some \( y \in \omega b-\text{Cl}(A) \). Let \( W \) be an open neighbourhood of \( x + y \) in \( X \). Since \( X \) is an \( \omega b \text{-} \)topological vector space, there exist \( \omega b \text{-} \)open sets \( U \) and \( V \) in \( X \) containing \( x \) and \( y \) respectively, such that \( U + V \subseteq W \). Since \( y \in \omega b-\text{Cl}(A) \), \( A1 V \neq \phi \). So, there is \( a \in A1 V \) and thus \( x + a \in (x + A) I (U + V) \subseteq (x + A) I W \). Hence we have \( (x + A) I W \neq \phi \). Therefore it implies \( z \in \text{Cl}(x + A) \). This proves the assertion.

3. Let \( y \in x + \text{Int}(A) \). Then \( y = x + a \) for some \( a \in \text{Int}(A) \) and hence there exist \( \omega b \text{-} \)open sets \( U \) and \( V \) in \( X \) containing \( -x \) and \( y \) respectively, such that \( U + V \subseteq \text{Int}(A) \). Now, \( -x + V \subseteq U + V \subseteq \text{Int}(x + A) \subseteq x + A \) implies that \( V \subseteq x + A \). Since \( V \) is \( \omega b \text{-} \)open set in \( X \) containing \( y \), we must have \( y \in \omega b-\text{Int}(x + A) \). Thus \( x + \text{Int}(A) \subseteq x + \omega b-\text{Int}(x + A) \).

(4). Let \( z \in \text{Int}(x + A) \). Then \( z = x + y \) for some \( y \in A \). Since \( X \) is an \( \omega b \text{-} \)topological vector space, there exist \( \omega b \text{-} \)open sets \( U \) and \( V \) in \( X \) containing \( x \) and \( y \) respectively, such that \( x + V \subseteq U + V \subseteq \text{Int}(x + A) \subseteq x + A \) implies that \( x + V \subseteq x + \omega b-\text{Int}(A) \). Since \( z = x + y = x + V \), so \( z \in x + \omega b-\text{Int}(A) \). Therefore it follows that \( \text{Int}(x + A) \subseteq x + \omega b-\text{Int}(A) \).

**Theorem 4.2.** Let \( A \) be any subset of a \( \omega b \text{-} \)topological vector space \( X \). Then the following assertions are true:

(a). \( \omega b-\text{Cl}(\lambda A) \subseteq \lambda \text{Cl}(A) \) for each non-zero scalar \( \lambda \),
(b). \( \lambda [\omega b-\text{Cl}(A)] \subseteq \text{Cl}(\lambda A) \) for each non-zero scalar \( \lambda \),
(c). \( \lambda \text{Int}(A) \subseteq \omega b-\text{In}(\lambda A) \) for each non-zero scalar \( \lambda \),
(d). \( \text{Int}(\lambda A) \subseteq \lambda [\omega b-\text{Int}(A)] \) for each non-zero scalar \( \lambda \).

**Proof.** (a). Assume that \( x \in \omega b-\text{Cl}(\lambda A) \) and let \( W \) be any open set in \( X \) containing \( \frac{1}{\lambda} x \). Since \( X \) is a \( \omega b \text{-} \)topological vector space, there exist \( \omega b \text{-} \)open sets \( U \) and \( V \) in \( X \) containing \( x \) such that \( U V \subseteq W \). Since \( x \in \omega b-\text{Cl}(\lambda A) \), there
exists $a \in V I (\lambda A)$. This implies that
\[
\frac{1}{\lambda} a \in \frac{1}{\lambda} A (U V) \subseteq A I W \Rightarrow A I W \neq \phi \Rightarrow \frac{1}{\lambda} x \in Cl (A) \Rightarrow x \in \lambda . Cl (A).
\]
It proves that $\omega b - Cl (\lambda A) \subseteq \lambda Cl (A)$.

(b) Suppose that $B = \lambda A$ and $\lambda^* = \frac{1}{\lambda}$. Then, by (a), it follows that $\omega b - Cl (\lambda^* B) \subseteq \lambda^* Cl (B) \Rightarrow \omega b - Cl \left( \frac{1}{\lambda} \lambda^* A \right) \subseteq \frac{1}{\lambda} Cl (\lambda A) \Rightarrow \lambda [\omega b - Cl (A)] \subseteq Cl (\lambda A)$.

(c) Let $y = \lambda x$ for any $x \in Int (A)$. Then $x = \frac{1}{\lambda} y \in Int (A)$. Since $X$ is $\omega b$-topological vector space. So there exist $\omega b$-open sets $U$ and $V$ in $X$ containing $\frac{1}{\lambda}$ and $V$ in $X$ containing $y$ such that $U \cdot V \subseteq \epsilon (A)$. Then it implies that $x = \frac{1}{\lambda} y \in \epsilon - V \subseteq U \cdot \epsilon \subseteq \epsilon (A) \subseteq A \Rightarrow y \in V \subseteq \epsilon A$. Since $V$ is $\omega b$-open set. Hence $y \in \omega b - Int (\lambda A)$. Therefore it follows that $\lambda . Int (A) \subseteq \omega b - Int (\lambda A)$.

(d) Suppose that $B = \lambda A$ and $\lambda^* = \frac{1}{\lambda}$. Then by (c), it follows that $\lambda^* [Int (B)] \subseteq \omega b - Int (\lambda^* B) \Rightarrow \frac{1}{\lambda} [\epsilon (\lambda A)] \subseteq \omega b - Int (A) \Rightarrow \epsilon (\lambda A) \subseteq \lambda [\omega b - Int (A)]$.

Theorem 4.3. Let $A$ be any subset of an $\omega b$-topological vector space $X$. Then $\omega b - Cl [x + Cl (A)] \subseteq x + Cl (A)$ for each $x \in X$.

Proof. Assume $y \in \omega b - Cl [x + Cl (A)]$. Consider $z = -x + y$ and let $W$ be an open neighbourhood of $z \in X$. Then there exist $\omega b$-open neighbourhoods $U$ and $V$ of $-x$ and $y$ in $X$ respectively, such that $U + V \subseteq W$. Then $-x + V \subseteq U + V \subseteq W \Rightarrow V \subseteq x + W$. Since $y \in \omega b - Cl [x + Cl (A)]$, we have $V [x + Cl (A)] \neq \phi$, which implies that $(x + W) [x + Cl (A)] \neq \phi \Rightarrow W \subseteq Cl (A) \neq \phi$. Since $W$ is open, $W \subseteq A \neq \phi$. Hence $z \in Cl (A)$, that is $-x + y \in Cl (A) \Rightarrow y \in x + Cl (A)$. Consequently, $\omega b - Cl [x + Cl (A)] \subseteq x + Cl (A)$.

Theorem 4.4. Let $A$ be any subset of an $\omega b$-topological vector space $X$. Then prove that $x + Int (A) \subseteq \omega b - Int [x + \omega b - Int (A)]$ for each $x \in X$.

Proof. Let $y \in x + Int (A)$. Then $y = x + a$ for some $a \in Int (A)$. As a result of this, we get $\omega b$-open sets $U$ and $V$ in $X$ containing $-x$ and $y$ respectively, such that $U + V \subseteq \epsilon (A)$. Now $-x + V \subseteq U \subseteq \epsilon (A) \subseteq \omega b - Int (A)$. Then it implies $V \subseteq x + \omega b - Int (A)$. Since $V$ is $\omega b$-open set in $X$ such that $y \in V$, we have that $y \in \omega b - Int [x + \omega b - Int (A)]$ proving that $x + Int (A) \subseteq \omega b - Int [x + \omega b - Int (A)]$.

Definition 4.5. Let $B$ be a subset of a space $X$. A collection $\{U_i : i \in I\}$ of $\omega b$-open sets of $X$ is called a $\omega b$-open cover of $B$ if $B \subseteq \bigcup_{i \in I} U_i$. A topological space $X$ is said to be $\omega b$-compact if every $\omega b$-open cover of $X$ has a finite sub cover. A subset $B$ of $X$ is said to be $\omega b$-compact relative to $X$ if every cover of $X$ by $\omega b$-open sets of $X$ has a finite sub cover.

Theorem 4.6. Let $A$ be any $\omega b$-compact set in a $\omega b$-topological vector space $X$. Then $x + A$ is compact, for each $x \in X$.

Proof. Let $\mathcal{P} = \{U_i : i \in I\}$ be an open cover of $x + A$. Then $A \subseteq \bigcup_{i \in I} (-x + U_i)$. By hypothesis and Theorem 3.6, $A \subseteq \bigcup_{i \in I} (-x + U_i)$ for some finite subset $I_0$ of $I$. Hence we find that $x + A \subseteq \bigcup_{i \in I_0} U_i$. Therefore, it follows that $x + A$ is compact.
Theorem 4.7. Let $X$ be an $\omega b$–topological vector space. Then scalar multiple of any $\omega b$–compact subset of $X$ is compact.

**Proof.** Suppose that $A$ is a $\omega b$–compact set in $X$. Let $\lambda$ be any scalar. If $\lambda = 0$, we are done. Assume that $\lambda \neq 0$. We have to show that $\lambda A$ is compact. Let $\mathcal{U} = \{U_i : i \in I\}$ be an open cover of $\lambda A$. Then $\lambda A \subseteq \bigcup U_i \implies A \subseteq \frac{1}{\lambda} \bigcup \{U_i : i \in I\}$.

Consequently, $\left\{\frac{1}{\lambda} U_i : i \in I\right\}$ is an $\omega b$–open cover of $A$. But $A$ is $\omega b$–compact, there exists a finite subset $I_o$ of $I$ such that $A \subseteq \bigcup \left\{\frac{1}{\lambda} U_i : i \in I_o\right\}$. It implies that $\lambda A \subseteq \bigcup \left\{U_i : i \in I_o\right\}$. Hence $\lambda A$ is $\omega b$–compact.

Theorem 4.8. Let $X$ be an $\omega b$–topological vector space and $Y$ be a topological vector space over the same field $K$. Let $f : X \to Y$ be a linear map such that $f$ is continuous at $0$. Then $f$ is $\omega b$–continuous everywhere.

**Proof.** Let $x$ be any non-zero element of $X$ and $V$ be any open set in $Y$ containing $f(x)$. Since the translation of an open set in a topological vector space is open, $V - f(x)$ is open set in $Y$ containing $0$. Since $f$ is continuous at $0$, there exists an open set $U$ in $X$ containing $0$ such that $f(U) \subseteq V - f(x)$.

Furthermore, linearity of $f$ implies that $f(x + U) \subseteq V$. By theorem 3.6, $x + U$ is $\omega b$–open and hence $f$ is $\omega b$–continuous at $x$. By hypothesis, $f$ is $\omega b$–continuous at $0$. This reflects that $f$ is $\omega b$–continuous.

Corollary 4.9. Let $X$ be an $\omega b$–topological vector space over the field $K$. Let $f : X \to K$ be a linear functional which is continuous at $0$. Then the set $F = \{x \in X : f(x) = 0\}$ is $\omega b$–closed.

5 Almost $\omega b$ – Topological Vector Spaces

In this section, we define almost $\omega b$–topological vector spaces and investigate their relationships with certain other types of spaces. Some general properties of almost $\omega b$–topological vector spaces are also discussed.

Definition 5.1. Let $X$ be a vector space over the field $K$, where $K = \mathbb{R}$ or $\mathbb{C}$ with standard topology. Let $\tau$ be a topology on $X$ such that the following conditions are satisfied:

1. For each $x, y \in X$ and each regular open set $W \subseteq X$ containing $x + y$, there exist $\omega b$–open sets $U$ and $V$ in $X$ containing $x$ and $y$ respectively such that $U + V \subseteq W$,

2. For each $x \in X$ and every $\lambda \in K$ and each regular open set $W \subseteq X$ containing $\lambda x$, there exist $\omega b$–open sets $U$ in $K$ containing $\lambda$ and $V$ in $X$ containing $x$ such that $U + V \subseteq W$.

Then the pair $(X, \tau)$ is called an almost $\omega b$–topological vector space.

Theorem 5.2. Let $A$ be any $\delta$–open subset of an almost $\omega b$–topological vector space $X$. Then the following statements are true:

1. $x + A \in \omega b - O(X)$ for each $x \in X$

2. $\lambda A \in \omega b - O(X)$ for each non-zero scalar $\lambda$.

**Proof.** (1) Let $y \in x + A$. Then $y = x + a$ for some $a \in A$. Since $A$ is $\delta$–open set in $X$, there exists a regular open set $W$ in $X$ such that $a \in W \subseteq A$. This implies that $-x + y \in W$. Since $X$ is an almost $\omega b$–topological vector space, there exist $\omega b$–open sets $U$ and $V$ in $X$ such that $-x \in U$, $y \in V$ and $U + V \subseteq W$ implies $-x + V \subseteq A$.

Since $V$ is $\omega b$–open. Therefore
In an almost \( \omega b \)-topological vector space, there exist \( \omega b \)-open sets \( U \) in the topological field \( K \) containing \( \frac{1}{\lambda} \) and \( V \) in \( X \) containing \( x \) such that \( U V \subseteq W \). Then

\[
\frac{1}{\lambda} V \subseteq U V \subseteq W \subseteq A \quad \text{implies that} \quad x \in V \subseteq \lambda A. \quad \text{Since} \quad V \quad \text{is} \quad \omega b \quad \text{-open. Therefore} \quad x \in \omega b - \text{int}(\lambda A) \quad \text{and hence} \quad \lambda A = \omega b - \text{int}(\lambda A). \quad \text{Thus} \quad \lambda A \quad \text{is} \quad \omega b \quad \text{-open in} \quad X; \quad \text{that is,} \quad \lambda A \in \omega b - O(X).
\]

**Corollary 5.3.** For any \( \delta \)-open set \( A \) in an almost \( \omega b \)-topological vector space \( X \), the following statements are valid:

1. \( x + A \subseteq \text{Cl} \left( \text{Int}(x + A) \right) \) for each \( x \in X \),
2. \( \lambda A \subseteq \text{Cl} \left( \text{Int}(\lambda A) \right) \) for each non-zero scalar \( \lambda \).

**Theorem 5.4.** Let \( A \) be any \( \delta \)-open set in an almost \( \omega b \)-topological vector space \( X \). Then

\[
A + B \in \omega b - O(X) \quad \text{for any subset} \quad B \subseteq X.
\]

**Proof.** Follows directly from Theorem 5.2.

**Theorem 5.5.** Let \( B \) be any \( \delta \)-closed closed set in an almost \( \omega b \)-topological vector space \( X \). Then the following statements are true:

1. \( x + B \in \omega b - C(X) \) for each \( x \in X \),
2. \( \lambda B \in \omega b - C(X) \) for each non-zero scalar \( \lambda \).

**Proof.** (1). We need to show that \( x + B = \omega b - \text{Cl}(x + B) \). For, let \( y \in \omega b - \text{Cl}(x + B) \) be an arbitrary element. Fix \( z = -x + y \). Let \( W \) be any \( \delta \)-open set in \( X \) containing \( z \). By definition of \( \delta \)-open sets, there is a regular open set \( G \) in \( X \) such that \( z \in G \subseteq W \). Then there exist \( U, V \in \omega b - O(X) \) such that \( -x \in U, \ y \in V \) and \( U + V \subseteq G \).

Since \( y \in \omega b - \text{Cl}(x + B) \), then by definition,

\[
(x + B) \cap V \neq \emptyset. \quad \text{Let} \quad a \in (x + B) \cap V. \quad \text{Then} \quad x + a \in B, \quad (-x + V) \subseteq B, \quad U + V \subseteq B. \quad \text{This shows that} \quad B \cap W \neq \emptyset. \quad \text{Thus shows that} \quad y \in x + B. \quad \text{Thus} \quad x + B = \omega b - \text{Cl}(x + B). \quad \text{Hence} \quad x + B = \omega b - C(X).
\]

(2). We have to prove that \( \lambda B = \omega b - \text{Cl}(\lambda B) \). Therefore \( \lambda B = \omega b - C(X) \).

**Theorem 5.6.** Let \( A \) be a subset of an almost \( \omega b \)-topological vector space \( X \). Then the following assertions hold:

1. \( x + \omega b - \text{Cl}(A) \subseteq \delta \text{Cl}(x + A) \) for each \( x \in X \),
2. \( \lambda [\omega b - \text{Cl}(A)] \subseteq \delta \text{Cl}(\lambda A) \) for each non-zero scalar \( \lambda \).

**Proof.** (1). Let \( z \in x + \omega b - \text{Cl}(A) \). Then \( z = x + y \) for some \( y \in \omega b - \text{Cl}(A) \). Let \( W \) be an open set in \( X \) containing \( z \), then \( z \in W \subseteq \text{Int} \left( \text{Cl}(W) \right) \). Since \( X \) is an almost \( \omega b \)-topological vector space, then there exist \( U, V \in \omega b - O(X) \) containing \( x \) and \( y \) respectively such that \( U + V \subseteq \text{Int} \left( \text{Cl}(W) \right) \). Since \( y \in \omega b - \text{Cl}(A) \), then there exists some \( a \in A \) such that \( U + V \subseteq (x + A) \) and \( \text{Int} \left( \text{Cl}(W) \right) \).
This implies \((x + A)\) is an open set in \(X\). Therefore it follows that \(x + \omega b - Cl(A) \subseteq \delta Cl(x + A).

(2). Let \(x \in \omega b - Cl(A)\) and let \(W\) be an open set in \(X\) containing \(x\). Then \(x \in W \subseteq \text{Int}[Cl(W)]\).

so there exist \(\omega b - \text{open sets} U\) containing \(x\) in topological field \(K\) and \(V\) containing \(x\) in \(X\) such that \(U \cap V \subseteq \text{Int}[Cl(W)]\). Since \(x \in \omega b - Cl(A)\), there is some \(b \in A\) \(V\). Therefore it follows immediately \(\lambda x \in (\lambda A)I \subseteq (\lambda A)I \subseteq (U \cap V) \subseteq (\lambda A)I \subseteq \text{Int}[Cl(W)]\). Hence \(\lambda x \in \text{Int}[Cl(W)]\). This implies that \(x \in \text{Int}[Cl(A)]\).

Theorem 5.7. Let \(A\) be a subset of an almost \(\omega b\)-topological vector space \(X\). Then the following assertions hold:

1. \(\omega b - Cl(x + A) \subseteq x + \delta Cl(A)\) for each \(x \in X\).
2. \(\omega b - Cl(\lambda A) \subseteq \lambda \text{[Cl}(A)]\) for each non-zero scalar \(\lambda\).

Proof. (1). Let \(y \in \omega b - Cl(x + A)\) and let \(W\) be an open set in \(X\) containing \(-x + y\). Since \(X\) is an almost \(\omega b\)-topological vector space and \(W \subseteq \text{Int}[Cl(W)]\), there exist \(\omega b - \text{open sets} U\) and \(V\) in \(X\) such that \(-x \in U\), \(y \in V\) and \(U + V \subseteq \text{Int}[Cl(W)]\). Since \(y \in \omega b - Cl(x + A)\), there is some \(a \in (x + A)\) \(V\) and hence \(-x + a \in A\) \(V\) and \(U + V \subseteq \text{Int}[Cl(W)]\). Since \(y \in \omega b - Cl(x + A)\), there is some \(a \in (x + A)\) \(V\) and hence \(-x + y \in \text{Int}[Cl(W)]\). Therefore \(\lambda x \in \text{Int}[Cl(W)]\) for each non-zero scalar \(\lambda\).

(2). Let \(x \in \omega b - Cl(\lambda A)\) and \(W\) be an open set in \(X\) containing \(\lambda x\). So there exist \(\omega b - \text{open sets} U\) in topological field \(K\) containing \(\lambda x\) and \(V\) in \(X\) containing \(x\) such that \(U \cap V \subseteq \text{Int}[Cl(W)]\). As \(x \in \omega b - Cl(\lambda A), (\lambda A)I \subseteq \text{Int}[Cl(W)]\). Then we have \(\lambda x \in A\) \(V\). Consequently, it shows that \(\lambda x \in \text{Int}[Cl(W)]\).

This implies \((x + A)I \text{Int}[Cl(W)] \neq \phi\). Therefore it follows that \(x + \omega b - Cl(A) \subseteq \delta Cl(x + A)\).

Theorem 5.8. Let \(A\) be an open set in an almost \(\omega b\)-topological vector space \(X\). Then the following assertions hold:

1. \(\omega b - Cl(x + A) \subseteq x + Cl(A)\) for each \(x \in X\).
2. \(\omega b - Cl(\lambda A) \subseteq \lambda \text{[Cl}(A)]\) for each non-zero scalar \(\lambda\).

Proof. (1). Let \(y \in \omega b - Cl(x + A)\) and let \(W\) be an open set in \(X\) containing \(-x + y\). Since \(X\) is an almost \(\omega b\)-topological vector space and \(W \subseteq \text{Int}[Cl(W)]\), there exist \(\omega b - \text{open sets} U\) and \(V\) in \(X\) such that \(-x \in U\), \(y \in V\) and \(U + V \subseteq \text{Int}[Cl(W)]\). Since \(y \in \omega b - Cl(x + A)\), there is some \(a \in (x + A)\) \(V\) and hence \(-x + a \in A\) \(V\) and \(U + V \subseteq \text{Int}[Cl(W)]\). Then we have \(\lambda x \in A\) \(V\). Therefore it follows that \(\lambda x \in \text{Int}[Cl(W)]\).

(2). Let \(y \in \omega b - Cl(\lambda A)\) and \(W\) be an open set in \(X\) containing \(\lambda x\). Then there exist \(\omega b - \text{open sets} U\) in topological field \(K\) containing \(\lambda x\) \(V\) in \(X\) containing \(y\) such that \(U \cap V \subseteq \text{Int}[Cl(W)]\). As \(y \in \omega b - Cl(\lambda A), (\lambda A)I \subseteq \text{Int}[Cl(W)]\). Therefore it follows that \(\lambda x \in \text{Int}[Cl(W)]\).

Theorem 5.9. Let \(A\) and \(B\) be subsets of an almost \(\omega b\)-topological vector space \(X\). Prove that
Let $x \in \omega b - Cl(A)$ and $y \in \omega b - Cl(B)$. Let $W$ be an open neighbourhood of $x + y$ in $X$. Since $W \subseteq Int[C(W)]$ and $Int[C(W)]$ is regular open, there exist $U, V \subseteq \omega b - O(X)$ such that $x \in U$, $y \in V$ and $U + V \subseteq \omega b - Cl(A)$ and $y \in \omega b - Cl(B)$, there are $a \in A$, $V$, $b \in B$, $V$. Thus $x + y \in \delta Cl(A + B)$. That is, $\omega b - Cl(A) + \omega b - Cl(B) \subseteq \delta Cl(A + B)$.

**Theorem 5.10.** Let $A$ be any subset of an almost $\omega b$–topological vector space $X$. Then the following assertions are true:

1. $\delta Int(x + A) \subseteq x + \omega b - Int(A)$ for each $x \in X$.
2. $x + \delta Int(A) \subseteq \omega b - Int(x + A)$ for each $x \in X$.

**Proof.** (1). Let $y \in \delta Int(x + A)$. We know that $\delta Int(x + A)$ is $\delta$–open. Then for each $y \in \delta Int(x + A)$, there exist a regular open set $W$ in $X$ containing $x$ and $y$, respectively, such that $y \in W \subseteq \delta Int(x + A)$. Since $\delta Int(x + A) \subseteq (x + A)$, $y = x + a$ for some $a \in A$. Since $X$ is an almost $\omega b$–topological vector space, then there exist $\omega b$–open sets $U$ and $V$ in $X$ containing $x$ and $a$, respectively, such that $U + V \subseteq W$. Thus it follows that $x + V \subseteq W \subseteq \delta Int(x + A) \subseteq x + A$. Hence we have $V \subseteq (x + A)$, $y = x + a$. Since $V$ is $\omega b$–open, then $V \subseteq \omega b - Int(A)$ and therefore $a \in \omega b - Int(A)$, $x + y \in \omega b - Int(A)$.

(2). Let $y \in x + \delta Int(A)$. Then there exists a regular open set $W$ in $X$ such that $x + y \in W \subseteq \delta Int(A) \subseteq A$. By definition of almost $\omega b$–topological vector spaces, there exist $\omega b$–open sets $U$ and $V$ in $X$ containing $x$ and $y$, respectively, such that $U + V \subseteq W$. Thus $-x + y \subseteq U + V \subseteq W \Rightarrow V \subseteq x + W \subseteq x + A$. Since $V$ is $\omega b$–open set. Thus, $y \in \omega b - Int(x + A)$. Hence it follows that $x + \delta Int(A) \subseteq \omega b - Int(x + A)$.

**Theorem 5.11.** Let $A$ be a subset of an almost $\omega b$–topological vector space $X$. Then the following assertions are true:

1. $\delta Int(\lambda A) \subseteq \lambda [\omega b - Int(A)]$, for any nonzero scalar $\lambda$.
2. $\lambda [\delta Int(A)] \subseteq \omega b - Int(\lambda A)$, for each nonzero scalar $\lambda$.

**Proof.** (1). Let $y \in \delta Int(\lambda A)$. Then there exists a regular open set $W$ in $X$ such that $y \in W \subseteq \delta Int(\lambda A)$. Then $y = \lambda a$ for some $a \in A$. By definition of almost $\omega b$–topological vector spaces, there exist $\omega b$–open sets $U$ in the topological field $K$ containing $\lambda$ and $V$ in $X$ containing $a$, respectively, such that $U.V \subseteq W$. Now $y = \lambda a \in \lambda V \subseteq U.V \subseteq W \subseteq \delta Int(\lambda A) \subseteq \lambda A$.

(2). Let $y \in \lambda [\delta Int(A)]$. Then $y = \lambda a$ for some $a \in \delta Int(A)$ and $\delta Int(A)$ is $\delta$–open in $X$. Thus there exists a regular open set $W$ in $X$ such that $a = \lambda^{-1} y \in W \subseteq \delta Int(A)$. By definition of almost $\omega b$–topological vector spaces, there exist $\omega b$–open sets $U$ in topological field $K$ containing $\lambda^{-1}$ and $V$ in $X$ containing $y$, respectively, such that $U.V \subseteq W$. Now $a = \lambda^{-1} y \in \lambda^{-1} V \subseteq U.V \subseteq W \subseteq \delta Int(A) \subseteq \lambda A$.

**Theorem 5.12.** Let $X$ be an almost $\omega b$–topological vector space. Then the following assertions are true:

1. The translation mapping $T_x : X \to X$ defined by $T_x(y) = x + y$, $\forall x, y \in X$, is almost $\omega b$–continuous.
(2). The multiplication mapping $M_\alpha: X \to X$ defined by $M_\alpha(y) = \lambda x$, $\forall x \in X$, is almost $\alpha$–continuous, where $\lambda$ is non-zero fixed scalar in $K$.

**Proof.** (1). Let $y \in X$ be an arbitrary element. Let $W$ be any regular open set in $X$ containing $T_y(y)$. Then, by definition of almost $\omega b$–topological vector spaces, there exist $\omega b$–open sets $U$ in $X$ containing $x$ and $V$ in $X$ containing $y$ such that $U + V \subseteq W$. This results in $x + V \subseteq W \Rightarrow T_y(V) \subseteq W$. This indicates that $T_y$ is almost $\omega b$–continuous at $y$ and hence $T_y$ is almost $\omega b$–continuous.

(2). Let $x \in X$ and $W$ be any regular open set in $X$ containing $x$. Then there exist $\omega b$–open sets $U$ in the topological field $K$ containing $\lambda$ and $V$ in $X$ containing $x$ such that $U + V \subseteq W$. This gives that $\lambda V \subseteq W$. This means that $M_\alpha(V) \subseteq W$ showing that $M_\alpha$ is almost $\omega b$–continuous at $x$. Since $x \in X$ was an arbitrary element, it follows that $M_\alpha$ is almost $\omega b$–continuous.

**6 Conclusion**

Topological vector spaces are a fundamental notion and play an important role in various advanced branches of mathematics like fixed point theory, operator theory, etc. This paper expounds $\omega b$–topological vector spaces and the almost $\omega b$–topological vector spaces which are basically a generalization of topological vector spaces. This paper makes us familiar with $\omega b$–topological vector spaces as well as with the origin of almost $\omega b$–topological vector spaces and the elementary concepts that are used to develop the theory of almost $\omega b$–topological vector spaces. We investigate several new properties and characterizations of $\omega b$–topological vector spaces and almost $\omega b$–topological vector spaces. We explore some more basic features of $\omega b$–topological vector spaces and almost $\omega b$–topological vector spaces and interprets their relationships with some well-known existing spaces.

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**References:**


