Wolfe E-duality for E-differentiable vector optimization problems in E-invex with inequality and equality constraints

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Abstract: In this paper, the class of E-differentiable vector optimization problems with both inequality and equality constraints is considered. For such (not necessarily) differentiable vector optimization problems, The so-called scalar and vector Wolfe E-dual problems are defined for the considered E-differentiable multiobjective programming problem with both inequality and equality constraints and several E-dual theorems are established also under (generalized) E-invexity hypotheses.

Key–Words: E-invex set, E-invex function, E-differentiable function, Wolfe E-duality.

1 Introduction

Multiobjective optimization problems or vector optimization problems involving more than one objective function to be optimized simultaneously. Many real life problems arising in several field of science, engineering, economics, logistics, etc, are associated with mathematical optimization problems. The concept of invexity was first introduced by Hanson [13] as a broad generalization of convexity for differentiable real-valued functions defined on R^n. Hanson proved that both Karush-Kuhn-Tucker sufficiency results and Wolfe weak duality, in differentiable mathematical programming problems, hold with the invexity assumption. Jeyakumar and Mond [14] generalized Hansons definition to the vectorial case. They defined V-invexity of differentiable vector-valued functions which preserve the sufficient optimality conditions and duality results as in the scalar case and avoid the major difficulty of verifying that the inequality holds for the same function η for invex functions in multiobjective programming problems. Ben-Isreal and Mond [5] have defined quasi-invex function as a generalization of invex functions. Luc and Malivert [16] have extended the study of invexity to set-valued maps and vector optimization problems with set-valued data. Bazaraa et al. [6] have studied necessary conditions for optimality in a nonlinear vector optimization problem. Jeyakumar [15] defined generalized invexity for nonsmooth scalar-valued functions, established an equivalence of saddle points and optima, and studied duality results for nonsmooth problems. The concept of invexity for multiobjective nonlinear programming problems have been introduced and studied extensively in the literature (see, for example, [5], [8], [9], [12], [16], and others).

Dorn [11] has been formulated dual theorems for a class of convex programs for the primal problem of minimizing a convex functions, the duality relationship was established for a class of quadratic programs. Wolfe [19] has been formulated a dual problem for the mathematical programming problem of minimizing a convex function under convex constraints, this concept has been developed in the last decades in both differentiable and nondifferentiable case. Craven [10] has been introduced a modified wolfe dual for weak vector minimization.

Recently, the concepts of E-convex sets and E-convex functions were introduced by Youness [21]. This kind of generalized convexity is based on the effect of an operator $E : R^n \to R^n$ on the sets and the domains of functions. However, some results and proofs presented by Youness [21] were incorrect as it was pointed out by Yang [20]. Further, Megahed et al. [18] presented the concept of an E-differentiable convex function which transforms a (not necessarily) differentiable convex function to a differentiable function based on the effect of an operator $E : R^n \to R^n$.

Later, Abdulaleem [1] introduced a new concept of generalized convexity for not necessarily differentiable vector optimization problems. For E-differentiable functions and called them E-invex with respect to η. The concept of E-invexity is an extension of the concept of E-differentiable E-convexity introduced by Youness [21] and Megahed et al. [18] and invexity introduced by Hanson [13].

The main purpose of this paper is to use an E-differentiable E-invexity notion to establish the so-called Wolfe E-duality results for a new class of E-differentiable E-invex vector optimization problems. For the considered nonsmooth vector optimization problem, we study both a scalar and vector E-duality in the sense of Wolfe. By utilizing the concept of nonsmooth E-invexity, we prove various E-duality theorems between the nonconvex E-differentiable vector optimization problem and its E-duals in the sense of Wolfe.

2 Preliminaries

Let R^n be the n-dimensional Euclidean space and R^n_+ be its nonnegative orthant. The following convention for equalities and inequalities will be used in the paper.

For any vectors $x = (x_1, x_2, ..., x_n)^T$ and $y = (y_1, y_2, ..., y_n)^T$ in R^n, we define:

(i) $x = y$ if and only if $x_i = y_i$ for all $i = 1, 2, ..., n$;

(ii) $x > y$ if and only if $x_i > y_i$ for all $i = 1, 2, ..., n$;

(iii) $x \geqq y$ if and only if $x_i \geqq y_i$ for all $i = 1, 2, ..., n$;

(iv) $x \geq y$ if and only if $x \geqq y$ and $x \neq y$.

Definition 1 *[18] Let $E : R^n \to R^n$, and $f : M \to R$ be a (not necessarily) differentiable function at a given point u. It is said that f is an E-differentiable function at u if and only if $f \circ E$ is a differentiable function at u (in the usual sense) and, moreover,*

$$(f \circ E)(x) = (f \circ E)(u) + \nabla (f \circ E)(u)(x - u) + \theta (u, x - u) \|x - u\|, \quad (1)$$

where $\theta (u, x - u) \to 0$ as $x \to u$.

Definition 2 *[1] Let $E : R^n \to R^n$. A set $M \subseteq R^n$ is said to be an E-invex set (with respect to $\eta : M \times M \to R^n$) if and only if there exists a vector-valued function $\eta : M \times M \to R^n$ such that the relation*

$$E(u) + \lambda\eta (E(x), E(u)) \in M$$

holds for all $x, u \in M$ and any $\lambda \in [0, 1]$.

Definition 3 *[1] Let $E : R^n \to R^n$ and $f : M \to R^k$ be an E-differentiable function on a nonempty open set $M \subset R^n$. It is said that f is E-invex with respect to η at $u \in M$ on M if, there exists $\eta : M \times M \to R^n$ such that, for all $x \in M$,*

$$f_i(E(x)) - f_i(E(u)) \geqq \nabla f_i(E(u))\eta(E(x), E(u)), \ i = 1, ..., k. \quad (2)$$

If inequalities (2) hold for any $u \in M$, then f is E-invex with respect to η on M.

Remark 4 *From Definition 3, there are special cases:*

a) If f is a differentiable function and $E(x) \equiv x$ (E is an identity map), then the definition of an E-invex function reduces to the definition of an invex function introduced by Hanson [13] in the scalar case.

b) If $\eta : M \times M \to R^n$ is defined by $\eta(x, u) = x - u$, then we obtain the definition of an E-differentiable E-convex vector-valued function introduced by Megahed et al. [7].

c) If f is differentiable, $E(x) = x$ and $\eta(x, u) = x - u$, then the definition of an E-invex function reduces to the definition of a differentiable convex vector-valued function.

d) If f is differentiable and $\eta(x, u) = x - u$, then we obtain the definition of a differentiable E-convex function introduced by Youness [8].

Definition 5 *[1] Let $E : R^n \to R^n$ and $f : M \to R^k$ be an E-differentiable function on a nonempty open set $M \subset R^n$. It is said that f is strictly E-invex with respect to η at $u \in M$ on M if, there exists $\eta : M \times M \to R^n$ such that, for all $x \in M$ with $E(x) \neq E(u)$, the inequalities*

$$f_i(E(x)) - f_i(E(u)) > \nabla f_i(E(u))\eta(E(x), E(u)), \ i = 1, ..., k, \quad (3)$$

hold. If inequalities (3) are fulfilled for any $u \in M$ ($E(x) \neq E(u)$), then f is strictly E-invex with respect to η on M.

Definition 6 *[1] Let $E : R^n \to R^n$ and $f : M \to R^k$ be an E-differentiable function on a nonempty open set $M \subset R^n$. It is said that f is quasi-E-invex with respect to η at $u \in M$ on M if, there exists $\eta : M \times M \to R^n$ such that, for all $x \in M$ and $i = 1, ..., k$,*

$$f_i(E(x)) - f_i(E(u)) \leqq 0 \Rightarrow \nabla f_i(E(u))\eta(E(x), E(u)) \leqq 0. \quad (4)$$

If (4) holds for any $u \in M$, then f is quasi-E-invex with respect to η on M.

Consider the following (not necessarily differentiable) multiobjective programming problem (VP) with both inequality and equality constraints:

$$\begin{aligned} &\text{minimize } f(x) = \left(f_1(x), ..., f_p(x)\right) \\ &\text{subject to } g_j(x) \leqq 0, \ j \in J = \{1, ..., m\}, \\ &\qquad h_t(x) = 0, \ t \in T = \{1, ..., q\}, \\ &\qquad x \in X, \end{aligned} \quad \text{(VP)}$$

where X is nonempty open convex subset of R^n, $f_i : X \to R$, $i \in I = \{1, ..., p\}$, $g_j : X \to R$, $i \in I$, $h_t : X \to R$, $j \in J$, are real-valued functions defined on X. We shall write $g := (g_1, ..., g_m) : X \to R^m$ and $h := \left(h_1, ..., h_q\right) : X \to R^q$ for convenience.

For the purpose of simplifying our presentation, we will next introduce some notation which will be used frequently throughout this paper. Let

$$\Omega := \left\{x \in X : g_j(x) \leqq 0,\ \ j \in J, h_t(x) = 0,\ \ t \in T\right\}$$

be the set of all feasible solutions of (VP). Further, by $J(x)$, the set of inequality constraint indices that are active at a feasible solution x, that is, $J(x) = \left\{j \in J : g_j(x) = 0\right\}$.

For such multicriterion optimization problems, the following concepts of (weak) Pareto optimal solutions are defined as follows:

Definition 7 *A feasible point $\overline{x}$ is said to be a weak Pareto (weakly efficient) solution for (VP) if and only if there exists no feasible point x such that*

$$f(x) < f(\overline{x}).$$

Definition 8 *A feasible point $\overline{x}$ is said to be a Pareto (efficient) solution for (VP) if and only if there exists no feasible point x such that*

$$f(x) \leq f(\overline{x}).$$

Let $E : R^n \to R^n$ be a given one-to-one and onto operator. Throughout the paper, we shall assume that the functions constituting the considered multiobjective programming problem (VP) are E-differentiable at any feasible solution.

Now, for the considered multiobjective programming problem (VP), we define its associated differentiable vector optimization problem as follows:

$$\begin{aligned} &\text{minimize } f(E(x)) = \left(f_1(E(x)), ..., f_p(E(x))\right) \\ &\text{subject to } g_j(E(x)) \leqq 0,\ \ j \in J = \{1, ..., m\}, \\ &\qquad h_t(E(x)) = 0,\ \ t \in T = \{1, ..., q\}, \\ &\qquad x \in X. \end{aligned} \qquad (\mathrm{VP}_E)$$

We call the problem (VP_E) an E-vector optimization problem. Let

$$\Omega_E := \left\{x \in X : g_j(E(x)) \leqq 0,\ \ j \in J,\right.$$
$$\left. h_t(E(x)) = 0,\ \ t \in T\right\}$$

be the set of all feasible solutions of (VP_E). Since the functions constituting the problem (VP) are assumed to be E-differentiable at any feasible solution of (VP), by Definition 1, the functions constituting the E-vector optimization problem (VP_E) are differentiable at any its feasible solution (in the usual sense). Further, by $J_E(x)$, the set of inequality constraint indices that are active at a feasible solution x, that is, $J_E(x) = \left\{j \in J : \left(g_j \circ E\right)(x) = 0\right\}$.

Lemma 9 *[2] Let $E : R^n \to R^n$ be a one-to-one and onto and $\Omega_E = \left\{z \in X : \left(g_j \circ E\right)(z) \leqq 0,\ \ j \in J,\ \ (h_t \circ E)(z) = 0\right.$ $\left.,\ \ t \in T\right\}$. Then $E(\Omega_E) = \Omega$.*

Lemma 10 *[2] Let $\overline{x} \in \Omega$ be a weak Pareto solution (Pareto solution) of the considered multiobjective programming problem (VP). Then, there exists $\overline{z} \in \Omega_E$ such that $\overline{x} = E(\overline{z})$ and $\overline{z}$ is a weak Pareto (Pareto) solution of the E-vector optimization problem (VP_E).*

Lemma 11 *[2] Let $\overline{z} \in \Omega_E$ be a weak Pareto (Pareto) solution of the E-vector optimization problem (VP_E). Then $E(\overline{z})$ is a weak Pareto solution (Pareto solution) of the considered multiobjective programming problem (VP).*

Remark 12 *As it follows from Lemma 11, if $\overline{z} \in \Omega_E$ is a weak Pareto (Pareto) solution of the E-vector optimization problem (VP_E), then $E(\overline{z})$ is a weak Pareto solution (Pareto solution) of the considered multiobjective programming problem (VP). We call $E(\overline{z})$ a weak E-Pareto (E-Pareto) solution of the problem (VP).*

Now, under E-invexity hypotheses, we prove a (weak) Pareto optimal solution in problem (VP_E) (and, thus, a (weak) E-Pareto solution of the considered multiobjective programming problem (VP)).

Theorem 13 *Let $E : R^n \to R^n$ be an operator such that $E(\overline{x}) \in \Omega$ and the functions f_i, $i \in I$, g_j, $j \in J$, h_t, $t \in T^+$, $-h_t$, $t \in T^-$, are an E-invex E-differentiable at $\overline{x}$. If there exist Lagrange multipliers $\overline{\lambda} \in R^p$, $\overline{\mu} \in R^m$, $\overline{\xi} \in R^s$ such that*

$$\sum_{i=1}^{p} \overline{\lambda}_i \nabla f_i(E(\overline{x})) + \sum_{j=1}^{m} \overline{\mu}_j \nabla g_j(E(\overline{x})) + \sum_{t=1}^{s} \overline{\xi}_t \nabla h_t(E(\overline{x})) = 0, \tag{5}$$

$$\sum_{j=1}^{m} \overline{\mu}_j g_j(E(\overline{x})) = 0,\ \ j \in J(E(\overline{x})), \tag{6}$$

Then $\overline{x}$ is a (weak) Pareto optimal solution in problem (VP_E) (and, thus, $E(\overline{x})$ be a (weak) E-Pareto solution of the considered multiobjective programming problem (VP)).

Proof: Suppose that $\overline{x}$ is not an (weak) Pareto optimal solution of the problem (VP$_E$). Then, by Definition 8, there exists $x \in \Omega_E$ such that $f(E(x)) \leq f(E(\overline{x}))$, $\overline{\lambda} \in R^p$ we have

$$\sum_{i=1}^{p} \overline{\lambda_i} (f_i \circ E)(x) < \sum_{i=1}^{p} \overline{\lambda_i} (f_i \circ E)(\overline{x}) \tag{7}$$

holds. Since the functions f_i, $i \in I$, g_j, $j \in J$, h_t, $t \in T^+$, $-h_t$, $t \in T^-$, are an E-invex E-differentiable at $\overline{x}$, by Proposition 3, the inequalities

$$f_i(E(x)) - f_i(E(\overline{x})) \geqq \nabla f_i(E(\overline{x}))\eta(E(x), E(\overline{x})), \ i \in I, \tag{8}$$

$$g_j(E(x)) - g_j(E(\overline{x})) \geqq \nabla g_j(E(\overline{x}))\eta(E(x), E(\overline{x})), \ j \in J, \tag{9}$$

$$h_t(E(x)) - h_t(E(\overline{x})) \geqq \nabla h_t(E(\overline{x}))\eta(E(x), E(\overline{x})), \ t \in T^+, \tag{10}$$

$$-h_t(E(x)) + h_t(E(\overline{x})) \geqq$$
$$-\nabla h_t(E(\overline{x}))\eta(E(x), E(\overline{x})), \ \ t \in T^-(E(\overline{x})), \tag{11}$$

hold, respectively. Multiplying inequalities (8)-(11) by the corresponding Lagrange multipliers, respectively, we obtain

$$\sum_{i=1}^{p} \overline{\lambda}_i (f_i \circ E)(x) - \sum_{i=1}^{p} \overline{\lambda}_i (f_i \circ E)(\overline{x}) \geqq$$
$$\sum_{i=1}^{p} \overline{\lambda}_i \nabla (f_i \circ E)(\overline{x})\, \eta(E(x), E(\overline{x})), \ i \in I, \tag{12}$$

$$\sum_{j=1}^{m} \overline{\mu}_i \left(g_j \circ E\right)(x) - \sum_{j=1}^{m} \overline{\mu}_i \left(g_j \circ E\right)(\overline{x}) \geqq$$
$$\sum_{j=1}^{m} \overline{\mu}_i \nabla \left(g_j \circ E\right)(\overline{x})\, \eta(E(x), E(\overline{x})), \ j \in J(E(\overline{x})), \tag{13}$$

$$\sum_{t=1}^{s} \overline{\xi}_i (h_t \circ E)(x) - \sum_{t=1}^{s} \overline{\xi}_i (h_t \circ E)(\overline{x}) \geqq$$
$$\sum_{t=1}^{s} \overline{\xi}_i \nabla (h_t \circ E)(\overline{x})\, \eta(E(x), E(\overline{x})), \ t \in T^+(E(\overline{x})), \tag{14}$$

$$-\sum_{t=1}^{s} \overline{\xi}_i (h_t \circ E)(x) + \sum_{t=1}^{s} \overline{\xi}_i (h_t \circ E)(\overline{x}) \geqq$$
$$-\sum_{t=1}^{s} \overline{\xi}_i \nabla (h_t \circ E)(\overline{x})\, \eta(E(x), E(\overline{x})), \ t \in T^-(E(\overline{x})), \tag{15}$$

Adding both sides of the above inequalities, we obtain that the following inequality

$$\begin{gathered} \textstyle\sum_{i=1}^{p} \overline{\lambda}_i (f_i \circ E)(x) - \sum_{i=1}^{p} \overline{\lambda}_i (f_i \circ E)(\overline{x}) \\ + \textstyle\sum_{j=1}^{m} \overline{\mu}_i \left(g_j \circ E\right)(x) - \sum_{j=1}^{m} \overline{\mu}_i \left(g_j \circ E\right)(\overline{x}) \\ + \textstyle\sum_{t=1}^{s} \overline{\xi}_i (h_t \circ E)(x) - \sum_{t=1}^{s} \overline{\xi}_i (h_t \circ E)(\overline{x}) \\ \geqq \Big[\textstyle\sum_{i=1}^{p} \overline{\lambda}_i \nabla (f_i \circ E)(\overline{x}) + \sum_{j=1}^{m} \overline{\mu}_i \nabla \left(g_j \circ E\right)(\overline{x}) \\ + \textstyle\sum_{t=1}^{s} \overline{\xi}_i \nabla (h_t \circ E)(\overline{x}) \Big] \eta(E(x), E(\overline{x})) \end{gathered} \tag{16}$$

Thus, by (5), (6), we have the following inequality

$$\sum_{i=1}^{p} \overline{\lambda}_i (f_i \circ E)(x) \geqq \sum_{i=1}^{p} \overline{\lambda}_i (f_i \circ E)(\overline{x}) -$$
$$\sum_{j=1}^{m} \overline{\mu}_i \left(g_j \circ E\right)(x) \geqq \sum_{i=1}^{p} \overline{\lambda}_i (f_i \circ E)(\overline{x}) \tag{17}$$

holds, contradicting the inequality (7). Thus, $\overline{x}$ is an (weak) Pareto optimal solution of the problem (VP$_E$). Further, by 10, it follows that $E(\overline{x})$ is a weak E-Pareto solution of the problem (VP). □

Theorem 14 *[1] (E-Karush-Kuhn-Tucker necessary optimality conditions). Let $\overline{x} \in \Omega_E$ be a weak Pareto solution of the E-vector optimization problem (VP$_E$) (and, thus, $E(\overline{x})$ be a weak E-Pareto solution of the considered multiobjective programming problem (VP)). Further, f, g, h are E-differentiable at $\overline{x}$ and the E-Guignard constraint qualification be satisfied at $\overline{x}$. Then there exist Lagrange multipliers $\overline{\lambda} \in R^p$, $\overline{\mu} \in R^m$, $\overline{\xi} \in R^s$ such that*

$$\sum_{i=1}^{p} \overline{\lambda}_i \nabla f_i(E(\overline{x})) + \sum_{j=1}^{m} \overline{\mu}_j \nabla g_j(E(\overline{x})) + \sum_{t=1}^{s} \overline{\xi}_t \nabla h_t(E(\overline{x})) = 0, \tag{18}$$

$$\sum_{j=1}^{m} \overline{\mu}_j g_j(E(\overline{x})) = 0, \ \ j \in J(E(\overline{x})), \tag{19}$$

$$\overline{\lambda} \geq 0, \overline{\mu} \geqq 0. \tag{20}$$

3 Scalar E-Wolfe duality result

In this section, a scalar dual problem in the sense of Wolfe is considered for the class of E-differentiable E-invex vector optimization problems with inequality and equality constraints. Let $E : R^n \to R^n$ be a given operator. Consider the following dual problem in the sense of Wolfe related to the considered vector optimization problem (VP):

$$\psi_E(y,\lambda,\mu,\xi) = \sum_{i=1}^{p} \lambda_i (f_i \circ E)(y) + \sum_{j=1}^{m} \mu_j \left(g_j \circ E\right)(y) + \sum_{t=1}^{s} \xi_t (h_t \circ E)(y) \to \max$$

$$\text{s.t. } \sum_{i=1}^{p} \lambda_i \nabla (f_i \circ E)(y) + \sum_{j=1}^{m} \mu_j \nabla \left(g_j \circ E\right)(y) + \sum_{t=1}^{s} \xi_t \nabla (h_t \circ E)(y) = 0, \qquad (\mathrm{WD}_E)$$
$$\lambda \in R^p, \lambda \geq 0, \mu \in R^m, \mu \geqq 0, \xi \in R^s.$$

where all functions are defined in the similar way as for the considered vector optimization problem (VP). Further, let

$$\Gamma_E = \Big\{ (y,\lambda,\mu,\xi) \in R^n \times R^p \times R^m \times R^q : \sum_{i=1}^{p} \lambda_i \nabla (f_i \circ E)(y) + \sum_{j=1}^{m} \mu_j \nabla (g_j \circ E)(y) + \sum_{t=1}^{q} \xi_t \nabla (h_t \circ E)(y) = 0,\ \lambda \geq 0, \mu \geqq 0 \Big\}.$$

be the set of all feasible solutions of the problem (WD_E). Further, $Y_E = \{y \in X : (y,\lambda,\mu,\xi) \in \Gamma_E\}$. We call the scaler dual problem (WD_E) Wolfe scaler E-dual problem or scaler E-dual problem in the sense of Wolfe.

Now, under E-invexity hypotheses, we prove duality results between the E-vector problems (VP_E) and (WD_E) and, thus, E-duality results between the problems (VP) and (WD_E).

Theorem 15 *(Weak duality between (VP_E) and (WD_E) and also weak E-duality between (VP) and (WD_E)). Let z and (y,λ,μ,ξ) be any feasible solutions of the problems (VP_E) and (WD_E), respectively. Assume, moreover, that each objective function f_i, $i \in I$, is E-invex at y on $\Omega_E \cup Y_E$, each constraint function g_j, $j \in J$, is an E-invex function at y on $\Omega_E \cup Y_E$, the functions h_t, $t \in T^+(y)$ and the functions $-h_t$, $t \in T^-(y)$, are E-invex at y on $\Omega_E \cup Y_E$. Then*

$$(f \circ E)(z) \geqq \psi_E(y,\lambda,\mu,\xi). \qquad (21)$$

In other words, E-weak duality holds between the problems (VP) and (WD_E), that is, for any feasible solutions x and (y,λ,μ,ξ) of the problems (VP) and (WD_E), respectively, the following relation

$$f(x) \geqq \psi_E(y,\lambda,\mu,\xi) \qquad (22)$$

is true.

Proof: f_i, $i \in I(y)$, are an E-invex function at y on $\Omega_E \cup Y_E$, the constraint functions g_j, $j \in J(y)$, are an E-invex function at y on $\Omega_E \cup Y_E$, the functions h_t, $t \in T^+(y)$ and the function $-h_t$, $t \in T^-(y)$, are an E-invex functions at y on $\Omega_E \cup Y_E$. Then, by Definition 3, the following inequalities

$$(f_i \circ E)(z) - (f_i \circ E)(y) \geqq \nabla (f_i \circ E)(y)\, \eta(E(z), E(y)),\ i \in I(E(y)), \qquad (23)$$

$$\left(g_j \circ E\right)(z) - \left(g_j \circ E\right)(y) \geqq \nabla \left(g_j \circ E\right)(y)\, \eta(E(z), E(y)),\ j \in J(E(y)), \qquad (24)$$

$$(h_t \circ E)(z) - (h_t \circ E)(y) \geqq \nabla (h_t \circ E)(y)\, \eta(E(z), E(y)),\ t \in T^+(E(y)), \qquad (25)$$

$$-(h_t \circ E)(z) + (h_t \circ E)(y) \geqq -\nabla (h_t \circ E)(y)\, \eta(E(z), E(y)),\ t \in T^-(E(y)) \qquad (26)$$

hold, respectively. Multiplying both sides of the above inequalities by the associated Lagrange multipliers, respectively, we obtain

$$\lambda_i (f_i \circ E)(z) - \lambda_i (f_i \circ E)(y) \geqq \lambda_i \nabla (f_i \circ E)(y)\, \eta(E(z), E(y)),\ i \in I(E(y)), \qquad (27)$$

$$\mu_j \left(g_j \circ E\right)(z) - \mu_j \left(g_j \circ E\right)(y) \geqq \mu_j \nabla \left(g_j \circ E\right)(y)\, \eta(E(z), E(y)),\ j \in J(E(y)), \qquad (28)$$

$$\xi_t (h_t \circ E)(z) - \xi t (h_t \circ E)(y) \geqq \xi_t \nabla (h_t \circ E)(y)\, \eta(E(z), E(y)),\ t \in T^+(E(y)), \qquad (29)$$

$$-\xi_t (h_t \circ E)(z) + \xi_t (h_t \circ E)(y) \geqq -\xi_t \nabla (h_t \circ E)(y)\, \eta(E(z), E(y)),\ t \in T^-(E(y)). \qquad (30)$$

We denote by

$$\hat{\lambda}_i = \frac{\lambda_i}{\sum_{i \in I(E(y))} \lambda_i + \sum_{j \in J(E(y))} \mu_j + \sum_{t \in T^+(E(y))} \xi_t - \sum_{t \in T^-(E(y))} \xi_t}, \qquad (31)$$

$$\hat{\mu}_j = \frac{\mu_j}{\sum_{i \in I(E(y))} \lambda_i + \sum_{j \in J(E(y))} \mu_j + \sum_{t \in T^+(E(y))} \xi_t - \sum_{t \in T^-(E(y))} \xi_t}, \qquad (32)$$

$$\hat{\xi}_t^+ = \frac{\xi_t}{\sum_{i \in I(E(y))} \lambda_i + \sum_{j \in J(E(y))} \mu_j + \sum_{t \in T^+(E(y))} \xi_t - \sum_{t \in T^-(E(y))} \xi_t}, \qquad (33)$$

$$\hat{\xi}_t^- = \frac{-\xi_t}{\sum_{i \in I(E(y))} \lambda_i + \sum_{j \in J(E(y))} \mu_j + \sum_{t \in T^+(E(y))} \xi_t - \sum_{t \in T^-(E(y))} \xi_t}. \qquad (34)$$

Note that $0 \leqq \hat{\lambda}_i \leqq 1$, $i \in I(E(y))$, but at least one $\hat{\lambda}_i > 0$ for some $i \in I(E(y))$, $0 \leqq \hat{\mu}_j \leqq 1$, $j \in J(E(y))$,

$0 \leqq \hat{\xi}_t^+ \leqq 1, t \in T^+(E(y)), 0 \leqq \hat{\xi}_t^- \leqq 1, t \in T^-(E(y))$, and, moreover

$$\sum_{i\in I(E(y))} \hat{\lambda}_i + \sum_{j\in J(E(y))} \hat{\mu}_j + \sum_{t\in T^+(E(y))} \hat{\xi}_t^+ + \sum_{t\in T^-(E(y))} \hat{\xi}_t^- = 1. \tag{35}$$

Taking into account Equations (31)-(34) in the inequalities (27)-(30), we get, respectively,

$$\hat{\lambda}_i (f_i \circ E)(z) - \hat{\lambda}_i (f_i \circ E)(y) \geqq$$

$$\hat{\lambda}_i \nabla (f_i \circ E)(y) \eta (E(z), E(y)), i \in I(E(y)), \tag{36}$$

$$\hat{\mu}_j \left(g_j \circ E\right)(z) - \hat{\mu}_j \left(g_j \circ E\right)(y) \geqq$$

$$\hat{\mu}_j \nabla \left(g_j \circ E\right)(y) \eta (E(z), E(y)), j \in J(E(y)), \tag{37}$$

$$\hat{\xi}_t^+ (h_t \circ E)(z) - \hat{\xi}_t^+ (h_t \circ E)(y) \geqq$$

$$\hat{\xi}_t^+ \nabla (h_t \circ E)(y) \eta (E(z), E(y)), t \in T^+(E(y)), \tag{38}$$

$$-\hat{\xi}_t^- (h_t \circ E)(z) + \hat{\xi}_t^- (h_t \circ E)(y) \geqq$$

$$-\hat{\xi}_t^- \nabla (h_t \circ E)(y) \eta (E(z), E(y)), t \in T^-(E(y)). \tag{39}$$

Adding both sides of the inequalities (36)-(39), and then adding both sides of the obtained inequalities, we get

$$\sum_{i\in I(y)} \hat{\lambda}_i (f_i \circ E)(z) - \sum_{i\in I(y)} \hat{\lambda}_i (f_i \circ E)(y) +$$

$$\sum_{j\in J(y)} \hat{\mu}_j \left(g_j \circ E\right)(z) - \sum_{j\in J(y)} \hat{\mu}_j \left(g_j \circ E\right)(y) +$$

$$\sum_{t\in T^+(y)} \hat{\xi}_t^+ (h_t \circ E)(z) - \sum_{t\in T^+(y)} \hat{\xi}_t^+ (h_t \circ E)(y) -$$

$$\sum_{t\in T^-(y)} \hat{\xi}_t^- (h_t \circ E)(z) + \sum_{t\in T^-(y)} \hat{\xi}_t^- (h_t \circ E)(y) \geqq$$

$$\sum_{i\in I(y)} \hat{\lambda}_i \nabla (f_i \circ E)(y) \eta (E(z), E(y)) +$$

$$\sum_{j\in J(y)} \hat{\mu}_j \nabla \left(g_j \circ E\right)(y) \eta (E(z), E(y)) +$$

$$\sum_{t\in T^+(y)} \hat{\xi}_t^+ \nabla (h_t \circ E)(y) \eta (E(z), E(y)) -$$

$$\sum_{t\in T^-(y)} \hat{\xi}_t^- \nabla (h_t \circ E)(y) \eta (E(z), E(y)). \tag{40}$$

Using Equations (31)-(34) in the above inequality , we get

$$\sum_{i\in I(y)} \hat{\lambda}_i (f_i \circ E)(z) + \sum_{j\in J(y)} \hat{\mu}_j \left(g_j \circ E\right)(z) +$$

$$\sum_{t\in T^+(y)} \hat{\xi}_t^+ (h_t \circ E)(z) + \sum_{t\in T^-(y)} \hat{\xi}_t^- (h_t \circ E)(z) \geqq$$

$$\sum_{i\in I(y)} \hat{\lambda}_i (f_i \circ E)(y) + \sum_{j\in J(y)} \hat{\mu}_j \left(g_j \circ E\right)(y) +$$

$$\sum_{t\in T^+(y)} \hat{\xi}_t^+ (h_t \circ E)(y) + \sum_{t\in T^-(y)} \hat{\xi}_t^- (h_t \circ E)(y) \tag{41}$$

From the feasibility of x in problem (VP), it follows that

$$\sum_{i\in I(y)} \hat{\lambda}_i (f_i \circ E)(z) \geqq \sum_{i\in I(y)} \hat{\lambda}_i (f_i \circ E)(y) +$$

$$\sum_{j\in J(y)} \hat{\mu}_j \left(g_j \circ E\right)(y) + \sum_{t\in T^+(y)} \hat{\xi}_t^+ (h_t \circ E)(y) +$$

$$\sum_{t\in T^-(y)} \hat{\xi}_t^- (h_t \circ E)(y). \tag{42}$$

Taking into account the Lagrange multipliers equal to 0, we obtain

$$\sum_{i=1}^{p} \lambda_i (f_i \circ E)(z) \geqq \sum_{i=1}^{p} \lambda_i (f_i \circ E)(y) +$$

$$\sum_{j=1}^{m} \mu_j \left(g_j \circ E\right)(y) + \sum_{t=1}^{s} \xi_t (h_t \circ E)(y). \tag{43}$$

By the definition of the scalar Lagrange function ψ_E, we have that the inequality

$$\lambda (f \circ E)(z) \geqq \psi_E (y, \lambda, \mu, \xi).$$

holds, this means that the proof of weak duality theorem between the E-vector optimization problems (VP$_E$) and (WD$_E$) is completed. Then, the weak E-duality theorem between the problems (VP) and (WD$_E$), that is, the relation (22) follows directly from Lemma 9. Thus, the proof of this theorem is completed. □

Theorem 16 *(Strong duality between (VP$_E$) and (WD$_E$) and also strong E-duality between (VP) and (WD$_E$)). Let $\overline{x} \in \Omega_E$ be a (weak) Pareto solution of the E-vector optimization problem (VP$_E$) and the E-Guignard constraint qualification (GCQ$_E$) be satisfied at $\overline{x}$. Then there exist $\overline{\lambda} \in R^p$, $\overline{\mu} \in R^m$, $\overline{\xi} \in R^q$ such that $(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi})$ is feasible for the problem (WD$_E$) and the objective functions of (VP$_E$) and (WD$_E$) are equal at these points. If also all hypotheses of the weak duality theorem (Theorem 15) are satisfied, then $(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi})$ is a (weak) efficient solution of maximum type for the problem (WD$_E$).*

In other words, in such a case, $E(\overline{x}) \in \Omega$ is a (weak) E-Pareto solution of the multiobjective programming problem (VP) and the strong E-duality holds between the problems (VP) and (WD$_E$).

Proof: By assumption, $\overline{x} \in \Omega_E$ is a (weak) Pareto optimal solution of problem (VP$_E$) and the E-Guignard constraint qualification (GCQ$_E$) is satisfied at $\overline{x}$. Then, there exist Lagrange multiplier $\overline{\lambda} \in R^p, \overline{\mu} \in R^m, \overline{\xi} \in R^s$ such that the E-Karush-Kuhn-Tucker necessary optimality conditions (18)-(20) are satisfied at $\overline{x}$. Thus, the feasibility of $(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi})$ in problem (WD$_E$) follows directly from these conditions. By the weak duality theorem (Theorem 15), it follows that the inequality $\overline{\lambda}(f \circ E)(\overline{x}) \geqq \psi_E(y, \lambda, \mu, \xi)$ is satisfied for any feasible point (y, λ, μ, ξ) in dual problem (WD$_E$). Using the E-Karush-Kuhn-Tucker necessary optimality condition (19) together with the feasibility of $\overline{x}$ in problem (VP$_E$), we get the inequality

$$\begin{array}{c} \sum_{i=1}^{p} \overline{\lambda}_i (f_i \circ E)(\overline{x}) + \sum_{j=1}^{m} \overline{\mu}_j \left(g_j \circ E\right)(\overline{x}) \\ + \sum_{t=1}^{s} \overline{\xi}_t (h_t \circ E)(\overline{x}) \geqq \sum_{i=1}^{p} \lambda_i (f_i \circ E)(y) \\ + \sum_{j=1}^{m} \mu_j \left(g_j \circ E\right)(y) + \sum_{t=1}^{s} \xi_t (h_t \circ E)(y) \end{array} \tag{44}$$

is satisfied for any feasible point (y, λ, μ, ξ) in dual problem (WD$_E$). Hence, by (44), it follows that $(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi})$ is a weak efficient point of maximum type for Wolfe scaler E-dual problem (WD$_E$). The strong E-duality holds between the problems (VP) and (WD$_E$) follows directly from Lemma 10. Namely, $E(\overline{x})$ is a (weak) E-Pareto solution of the vector optimization problem (VP) and then $(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi})$ is a (weak) efficient solution of maximum type for the problem (WD$_E$). □

Theorem 17 *(Converse duality between (VP$_E$) and (WD$_E$) and also converse E-duality between (VP) and (WD$_E$)). Let $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right)$ be a (weak) efficient solution of a maximum type in E-Wolfe dual problem (WD$_E$) such that $\overline{x} \in \Omega_E$. Moreover, assume that the objective functions f_i, $i \in I$, are (strictly) E-invex at $\overline{x}$ on $\Omega_E \cup Y_E$, the constraint functions g_j, $j \in J$, are E-invex at $\overline{x}$ on $\Omega_E \cup Y_E$, the functions h_t, $t \in T^+(E(\overline{x}))$ and the functions $-h_t$, $t \in T^-(E(\overline{x}))$, are E-invex at $\overline{x}$ on $\Omega_E \cup Y_E$. Then $\overline{x}$ is a (weak) Pareto solution of the problem (VP$_E$) and, thus, $E(\overline{x})$ is a (weak) E-Pareto solution of the problem (VP).*

Proof: Proof of this theorem follows directly from Theorem 15. □

4 Vector Wolfe E-duality results

In this section, a vector dual problem in the sense of Wolfe is considered for the class of E-invex vector optimization problems with inequality and equality constraints. Let $E : R^n \to R^n$ be a given one-to-one and onto operator. Consider the following dual problem in the sense of Wolfe related to the considered vector optimization problem (VP):

$$\text{maximize } \psi_E(y, \mu, \xi) = (f \circ E)(y) + \left[\sum_{j=1}^{m} \mu_j \left(g_j \circ E\right)(y) + \sum_{t=1}^{q} \xi_t (h_t \circ E)(y)\right] e$$

$$\text{s.t. } \sum_{i=1}^{p} \lambda_i \nabla (f_i \circ E)(y) + \sum_{j=1}^{m} \mu_j \nabla \left(g_j \circ E\right)(y) + \sum_{t=1}^{q} \xi_t \nabla (h_t \circ E)(y) = 0, \tag{WD$_E$}$$

$$\lambda \in R^p, \lambda \geq 0, \lambda e = 1, \; e = (1, 1, ..., 1)^T \in R^p,$$
$$\mu \in R^m, \mu \geqq 0, \xi \in R^q,$$

where all functions are defined in the similar way as for the considered vector optimization problem (VP) and $e = (1, ..., 1) \in R^p$. Further, let

$$\Gamma_E = \Big\{ (y, \lambda, \mu, \xi) \in R^n \times R^p \times R^m \times R^q : \sum_{i=1}^{p} \lambda_i \nabla (f_i \circ E)(y) + \sum_{j=1}^{m} \mu_j \nabla (g_j \circ E)(y) + \sum_{t=1}^{q} \xi_t \nabla (h_t \circ E)(y) = 0, \; \lambda \geq 0, \; \lambda e = 1, \mu \geqq 0 \Big\}.$$

be the set of all feasible solutions of the problem (WD$_E$). Further, $Y_E = \{y \in X : (y, \lambda, \mu, \xi) \in \Gamma_E\}$. We call the vector dual problem (WD$_E$) Wolfe vector E-dual problem or vector E-dual problem in the sense of Wolfe.

Now, under E-invexity hypotheses, we prove duality results between the E-vector problems (VP$_E$) and (WD$_E$) and, thus, E-duality results between the problems (VP) and (WD$_E$).

Theorem 18 *(Weak duality between (VP$_E$) and (WD$_E$) and also weak E-duality between (VP) and (WD$_E$)). Let z and (y, λ, μ, ξ) be any feasible solutions of the problems (VP$_E$) and (WD$_E$), respectively. Assume, moreover, that each objective function f_i, $i \in I$, is E-invex at y on $\Omega_E \cup Y_E$, each constraint function g_j, $j \in J$, is an E-invex function at y on $\Omega_E \cup Y_E$, the functions h_t, $t \in T^+(E(y))$ and the functions $-h_t$, $t \in T^-(E(y))$, are E-invex at y on $\Omega_E \cup Y_E$. Then*

$$(f \circ E)(z) \nless \psi_E(y, \mu, \xi). \tag{45}$$

In other words, E-weak duality holds between the problems (VP) and (WD$_E$), that is, for any feasible *solutions x and (y,λ,μ,ξ) of the problems (VP) and (WD_E), respectively, the following relation*

$$f(x) \nless \psi_E(y,\mu,\xi) \tag{46}$$

is true.

Proof: Suppose, contrary to the result, that

$$(f\circ E)(z) < \psi_E(y,\mu,\xi)$$

Thus,

$$(f_i\circ E)(z) < (f_i\circ E)(y) + \Big[\sum_{j=1}^{m}\mu_j\left(g_j\circ E\right)(y) + \sum_{t=1}^{q}\xi_t(h_t\circ E)(y)\Big]e,\ i\in I.$$

Multiplying by λ_i and then adding both sides of the above inequalities and taking that $\sum_{i=1}^{p}\lambda_i = 1$, we get the inequality

$$\sum_{i=1}^{p}\lambda_i(f_i\circ E)(z) < \sum_{i=1}^{p}\lambda_i(f_i\circ E)(y)+\sum_{j=1}^{m}\mu_j\left(g_j\circ E\right)(y) + \sum_{t=1}^{q}\xi_t(h_t\circ E)(y)$$

holds. From the feasibility of z for the problem (VP_E), it follows that

$$\sum_{i=1}^{p}\lambda_i(f_i\circ E)(z)+\sum_{j=1}^{m}\mu_j\left(g_j\circ E\right)(z)+\sum_{t=1}^{q}\xi_t(h_t\circ E)(z) < \sum_{i=1}^{p}\lambda_i(f_i\circ E)(y)+\sum_{j=1}^{m}\mu_j\left(g_j\circ E\right)(y)+ \sum_{t=1}^{q}\xi_t(h_t\circ E)(y). \tag{47}$$

By assumption, z and (y,λ,μ,ξ) are feasible solutions for the problems (VP_E) and (WD_E), respectively. Since the functions f_i, $i\in I$, g_j, $j\in J$, h_t, $t\in T^+$, $-h_t$, $t\in T^-$, are E-invex on $\Omega_E\cup Y_E$, by Definition 3, the inequalities

$$(f_i\circ E)(z)-(f_i\circ E)(y) \geqq \nabla(f_i\circ E)(y)\,\eta(E(z),E(y)),\ i\in I, \tag{48}$$

$$\left(g_j\circ E\right)(z)-\left(g_j\circ E\right)(y) \geqq \nabla\left(g_j\circ E\right)(y)\,\eta(E(z),E(y)),\ j\in J_E(y), \tag{49}$$

$$(h_t\circ E)(z)-(h_t\circ E)(y) \geqq \nabla(h_t\circ E)(y)\,\eta(E(z),E(y)),\ t\in T^+(E(y)), \tag{50}$$

$$-(h_t\circ E)(z)+(h_t\circ E)(y) \geqq -\nabla(h_t\circ E)(y)\,\eta(E(z),E(y)),\ t\in T^-(E(y)) \tag{51}$$

hold, respectively. Multiplying inequalities (48)-(51) by the corresponding Lagrange multiplier and then adding both sides of the resulting inequalities, we obtain that the inequality

$$\sum_{i=1}^{p}\lambda_i(f_i\circ E)(z)-\sum_{i=1}^{p}\lambda_i(f_i\circ E)(y)+\sum_{j=1}^{m}\mu_i\left(g_j\circ E\right)(z) - \sum_{j=1}^{m}\mu_i\left(g_j\circ E\right)(y)+\sum_{t=1}^{q}\xi_i(h_t\circ E)(z)-\sum_{t=1}^{q}\xi_i(h_t\circ E)(y)$$

$$\geqq \Big[\sum_{i=1}^{p}\lambda_i\nabla(f_i\circ E)(y)+\sum_{j=1}^{m}\mu_i\nabla\left(g_j\circ E\right)(y)+ \sum_{t=1}^{q}\xi_i\nabla(h_t\circ E)(y)\Big]\eta(E(z),E(y))$$

holds. Thus, by (47), it follows that the inequality

$$\Big[\sum_{i=1}^{p}\lambda_i\nabla(f_i\circ E)(y)+\sum_{j=1}^{m}\mu_i\nabla\left(g_j\circ E\right)(y) + \sum_{t=1}^{q}\xi_i\nabla(h_t\circ E)(y)\Big]\eta(E(z),E(y)) < 0$$

holds, contradicting the first constraint of the Wolfe vector E-dual problem (WD_E). This means that the proof of weak duality theorem between the E-vector optimization problems (VP_E) and (WD_E) is completed. Then, the weak E-duality theorem between the problems (VP) and (WD_E), that is, the relation (46) follows directly from Lemma 9. Thus, the proof of this theorem is completed. □

If stronger E-invexity hypotheses are imposed on the functions constituting the considered vector optimization problems, then the stronger weak duality result is satisfied.

Theorem 19 *(Weak duality between (VP_E) and (WD_E) and also weak E-duality between (VP) and (WD_E)). Let z and (y,λ,μ,ξ) be any feasible solutions of the problems (VP_E) and (WD_E), respectively. Assume, moreover, that each objective function f_i, $i\in I$, is strictly E-invex at y on $\Omega_E\cup Y_E$, each constraint function g_j, $j\in J$, is an E-invex function at y on $\Omega_E\cup Y_E$, the functions h_t, $t\in T^+(E(y))$ and the functions $-h_t$, $t\in T^-(E(y))$, are E-invex at y on $\Omega_E\cup Y_E$. Then*

$$(f\circ E)(z) \nleqq \psi_E(y,\mu,\xi). \tag{52}$$

In other words, weak E-duality holds between the problems (VP) and (WD_E), that is, for any feasible solutions x and (y, λ, μ, ξ) of the problems (VP) and (WD_E), respectively,

$$f(x) \nleq \psi_E(y, \mu, \xi). \tag{53}$$

Remark 20 *As it follows from the proofs of Theorems 18 and 19, the assumption of E-invexity of constraints functions can be weakened. Indeed, these results can be established if each constraint functions g_j, $j \in J$, h_t, $t \in T^+(y)$ and the functions $-h_t$, $t \in T^-(y)$, are assumed to be quasi E-invex at y on $\Omega_E \cup Y_E$.*

Theorem 21 *(Strong duality between (VP_E) and (WD_E) and also strong E-duality between (VP) and (WD_E)). Let $\overline{x} \in \Omega_E$ be a (weak) Pareto solution of the E-vector optimization problem (VP) and the E-Guignard constraint qualification (GCQ_E) be satisfied at $\overline{x}$. Then there exist $\overline{\lambda} \in R^p$, $\overline{\mu} \in R^m$, $\overline{\xi} \in R^q$ such that $(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi})$ is feasible for the problem (WD_E) and the objective functions of (VP_E) and (WD_E) are equal at these points. If also all hypotheses of the weak duality theorem (Theorem 18 or Theorem 19) are satisfied, then $(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi})$ is a (weak) efficient solution of maximum type for the problem (WD_E).*
In other words, in such a case, $E(\overline{x}) \in \Omega$ is a (weak) E-Pareto solution of the multiobjective programming problem (VP) and the strong E-duality holds between the problems (VP) and (WD_E).

Proof: By assumption, $\overline{x} \in \Omega_E$ is a weak Pareto solution for the problem (VP_E) and the E-Guignard constraint qualification (GCQ_E) is satisfied at $\overline{x}$. Then, there exist Lagrange multiplier $\overline{\lambda} \in R^p$, $\overline{\mu} \in R^m$, $\overline{\xi} \in R^q$ such that the E-Karush-Kuhn-Tucker necessary optimality conditions (18)-(20) are satisfied at $\overline{x}$. Thus, the feasibility of $(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi})$ in the problem (WD_E) follows directly from these conditions. Therefore, the objective functions for the problems (VP_E) and (WD_E) are equal at $\overline{x}$ and $(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi})$, respectively. By the weak duality theorem (Theorem 18 or Theorem 19), it follows that the inequality $(f \circ E)(\overline{x}) \nless \psi_E(y, \mu, \xi)$ (or $(f \circ E)(\overline{x}) \nleq \psi_E(y, \mu, \xi)$) is satisfied for any feasible point (y, λ, μ, ξ) of Wolfe vector E-dual problem (WD_E). Using the E-Karush-Kuhn-Tucker necessary optimality conditions (19) and (20) we get, for any feasible point (y, λ, μ, ξ) of the problem (WD_E), that

$$(f \circ E)(\overline{x}) + \left[\sum_{j=1}^{m} \mu_j \left(g_j \circ E\right)(\overline{x}) + \sum_{t=1}^{q} \xi_t (h_t \circ E)(\overline{x})\right] e$$

$$\nless (f \circ E)(y) + \left[\sum_{j=1}^{m} \mu_j \left(g_j \circ E\right)(y) + \sum_{t=1}^{q} \xi_t (h_t \circ E)(y)\right] e. \tag{54}$$

Hence, by (54), it follows that $(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi})$ is a weak efficient point of maximum type for Wolfe vector E-dual problem (WD_E). The strong E-duality holds between the problems (VP) and (WD_E) follows directly from Lemma 10. Namely, $E(\overline{x})$ is a (weak) E-Pareto solution of the vector optimization problem (VP) and then $(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi})$ is a (weak) efficient solution of maximum type for the problem (WD_E). □

Theorem 22 *(Converse duality between (VP_E) and (WD_E) and also converse E-duality between (VP) and (WD_E)). Let $\left(\overline{x}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right)$ be a (weak) efficient solution of a maximum type in the vector E-Wolfe dual problem (WD_E) such that $\overline{x} \in \Omega_E$. Moreover, assume that the objective functions f_i, $i \in I$, are (strictly) E-invex at $\overline{x}$ on $\Omega_E \cup Y_E$, the constraint functions g_j, $j \in J$, are E-invex at $\overline{x}$ on $\Omega_E \cup Y_E$, the functions h_t, $t \in T^+(E(\overline{x}))$ and the functions $-h_t$, $t \in T^-(E(\overline{x}))$, are E-invex at $\overline{x}$ on $\Omega_E \cup Y_E$. Then $\overline{x}$ is a (weak) Pareto solution of the problem (VP_E) and, thus, $E(\overline{x})$ is a (weak) E-Pareto solution of the problem (VP).*

Proof: Proof of this theorem follows directly from Theorem 18 (or Theorem 19). □

Theorem 23 *(Restricted converse duality between (VP_E) and (WD_E) and also restricted converse E-duality between (VP) and (WD_E)). Let $\overline{x}$ and $\left(\overline{y}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right)$ be feasible solutions for the problems (VP_E) and (WD_E), respectively, such that*

$$(f \circ E)(\overline{x}) < (f \circ E)(\overline{y}) + \left[\sum_{j=1}^{m} \overline{\mu}_j \left(g_j \circ E\right)(\overline{y}) + \sum_{t=1}^{q} \overline{\xi}_t (h_t \circ E)(\overline{y})\right] e. \quad (\leq) \tag{55}$$

Moreover, assume that the objective functions f_i, $i \in I$, are (strictly) E-invex at $\overline{y}$ on $\Omega_E \cup Y_E$, the constraint functions g_j, $j \in J$, are E-invex at $\overline{y}$ on $\Omega_E \cup Y_E$, the functions h_t, $t \in T^+(E(\overline{y}))$ and functions $-h_t$, $t \in T^-(E(\overline{y}))$, are E-invex at $\overline{y}$ on $\Omega_E \cup Y_E$. Then $\overline{x} = \overline{y}$, that is, $\overline{x}$ is a (weak) Pareto solution of the problem (VP_E) and $\left(\overline{y}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right)$ is a (weak) efficient point of maximum type for the problem (WD_E). In other words, $E(\overline{x})$ is a weak E-Pareto (E-Pareto) solution of the problem (VP) and $\left(\overline{y}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right)$ is a (weak) efficient solution of maximum type for the problem (WD_E).

Proof: Note that, by (55), it follows that

$$(f_i \circ E)(\overline{x}) < (f_i \circ E)(\overline{y}) + \sum_{j=1}^{m} \overline{\mu}_j \left(g_j \circ E\right)(\overline{y}) +$$

$$\sum_{t=1}^{q} \overline{\xi}_t (h_t \circ E)(\overline{y}),\ i \in I. \tag{56}$$

Multiplying each inequality (56) by $\overline{\lambda}_i$, $i \in I$, and then adding both sides of the resulting inequalities, we get

$$\sum_{i=1}^{p} \overline{\lambda}_i (f \circ E)(\overline{x}) < \sum_{i=1}^{p} \overline{\lambda}_i (f \circ E)(\overline{y}) +$$

$$\Big[\sum_{j=1}^{m} \overline{\mu}_j \left(g_j \circ E\right)(\overline{y}) + \sum_{t=1}^{q} \overline{\xi}_t (h_t \circ E)(\overline{y}) \Big] \sum_{i=1}^{p} \lambda_i. \tag{57}$$

Since $\sum_{i=1}^{p} \lambda_i = 1$, (57) implies

$$\sum_{i=1}^{p} \overline{\lambda}_i (f_i \circ E)(\overline{x}) < \sum_{i=1}^{p} \overline{\lambda}_i (f_i \circ E)(\overline{y}) +$$

$$\sum_{j=1}^{m} \overline{\mu}_j \left(g_j \circ E\right)(\overline{y}) + \sum_{t=1}^{q} \overline{\xi}_t (h_t \circ E)(\overline{y}). \tag{58}$$

Now, we proceed by contradiction. Suppose, contrary to the result, that $\overline{x} \neq \overline{y}$. By assumption, the functions f_i, $i \in I$, g_j, $j \in J(E(\overline{y}))$, h_t, $t \in T^+(E(\overline{y}))$, and $-h_t$, $t \in T^-(E(\overline{y}))$ are E-invex at $\overline{y}$ on $\Omega_E \cup Y$. Then, by Definition 3, the inequalities

$$(f_i \circ E)(\overline{x}) - (f_i \circ E)(\overline{y}) \geqq$$

$$\nabla (f_i \circ E)(\overline{y})\, \eta(E(\overline{x}), E(\overline{y})),\ i \in I, \tag{59}$$

$$\left(g_j \circ E\right)(\overline{x}) - \left(g_j \circ E\right)(\overline{y}) \geqq$$

$$\nabla \left(g_j \circ E\right)(\overline{y})\, \eta(E(\overline{x}), E(\overline{y})),\ j \in J(E(\overline{y})), \tag{60}$$

$$(h_t \circ E)(\overline{x}) - (h_t \circ E)(\overline{y}) \geqq$$

$$\nabla (h_t \circ E)(\overline{y})\, \eta(E(\overline{x}), E(\overline{y})),\ t \in T^+(E(\overline{y})), \tag{61}$$

$$-(h_t \circ E)(\overline{x}) + (h_t \circ E)(\overline{y}) \geqq$$

$$-\nabla (h_t \circ E)(\overline{y})\, \eta(E(\overline{x}), E(\overline{y})),\ t \in T^-(E(\overline{y})) \tag{62}$$

hold, respectively. Multiplying inequalities (59)-(62) by the corresponding Lagrange multipliers and then adding both sides of the resulting inequalities, we get

$$\sum_{i=1}^{p} \overline{\lambda}_i (f_i \circ E)(\overline{x}) - \sum_{i=1}^{p} \overline{\lambda}_i (f_i \circ E)(\overline{y}) + \sum_{j=1}^{m} \overline{\mu}_i \left(g_j \circ E\right)(\overline{x})$$

$$- \sum_{j=1}^{m} \overline{\mu}_i \left(g_j \circ E\right)(\overline{y}) + \sum_{t=1}^{q} \overline{\xi}_i (h_t \circ E)(\overline{x}) -$$

$$\sum_{t=1}^{q} \overline{\xi}_i (h_t \circ E)(\overline{y}) \geqq$$

$$\Big[\sum_{i=1}^{p} \overline{\lambda}_i \nabla (f_i \circ E)(\overline{y}) + \sum_{j=1}^{m} \overline{\mu}_i \nabla \left(g_j \circ E\right)(\overline{y}) +$$

$$\sum_{t=1}^{q} \overline{\xi}_i \nabla (h_t \circ E)(\overline{y}) \Big] \eta(E(\overline{x}), E(\overline{y})) \tag{63}$$

By (63) and the first constraint of (WD_E), it follows that

$$\sum_{i=1}^{p} \overline{\lambda}_i (f_i \circ E)(\overline{x}) + \sum_{j=1}^{m} \overline{\mu}_j \left(g_j \circ E\right)(\overline{x}) + \sum_{t=1}^{q} \overline{\xi}_t (h_t \circ E)(\overline{x})$$

$$\geqq \sum_{i=1}^{p} \overline{\lambda}_i (f_i \circ E)(\overline{y}) + \sum_{j=1}^{m} \overline{\mu}_j \left(g_j \circ E\right)(\overline{y}) + \sum_{t=1}^{q} \overline{\xi}_t (h_t \circ E)(\overline{y}).$$

Hence, by $\overline{x} \in \Omega_E$, we get that the following inequality

$$\sum_{i=1}^{p} \overline{\lambda}_i (f_i \circ E)(\overline{x}) \geqq \sum_{i=1}^{p} \overline{\lambda}_i (f_i \circ E)(\overline{y}) + \sum_{j=1}^{m} \overline{\mu}_j \left(g_j \circ E\right)(\overline{y}) +$$

$$\sum_{t=1}^{q} \overline{\xi}_t (h_t \circ E)(\overline{y}). \tag{64}$$

holds, contradicting (58). Then, $\overline{x} = \overline{y}$ and this means, by weak duality (Theorem 18) that $\overline{x}$ is a weak Pareto solution of the problem (VP_E) and $\left(\overline{y}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right)$ is a weak efficient solution of maximum type for the problem (WD_E). Further, by Lemma 10, it follows that $E(\overline{x})$ is a weak E-Pareto solution of the problem (VP_E) and $\left(\overline{y}, \overline{\lambda}, \overline{\mu}, \overline{\xi}\right)$ is a weak efficient solution of maximum type for the problem (WD_E). Thus, the proof of this theorem is completed. □

5 Concluding remarks

In this paper, the class of E-differentiable vector optimization problems with both inequality and equality constraints has been considered. For such (not necessarily) differentiable vector optimization problems. The so-called scalar and vector Wolfe E-dual problems have been defined for the considered E-differentiable E-invexity multiobjective programming problem with both inequality and equality constraints and several E-dual theorems have been established also under (generalized) E-invexity hypotheses.

However, some interesting topics for further research remain. It would be of interest to investigate whether it is possible to prove similar results for other classes of E-differentiable vector optimization problems. We shall investigate these questions in subsequent papers.

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