# On the Solution of the Fredholm Equation of the Second Kind

I.G.BUROVA, N.S.DOMNIN, A.E.VEZHLEV, A.V.LEBEDEVA, A.N.PAKULINA
St. Petersburg State University
7/9 Universitetskaya nab., St.Petersburg, 199034
RUSSIA
i.g.burova@spbu.ru, burovaig@mail.ru http://www.math.spbu.ru/user/burova/

*Abstract:* - The present paper is devoted to the application of local polynomial integro-differential splines to the solution of integral equations, in particular, to the solution of the integral equations of Fredholm of the second kind. To solve the Fredholm equation of the second kind, we apply local polynomial integro-differential splines of the second and third order of approximation. To calculate the integral in the formulae of a piecewise quadratic integro-differential spline and piecewise linear integro-differential spline, we propose the corresponding quadrature formula. The results of the numerical experiments are given.

*Key-Words:* - polynomial splines, polynomial integro-differential splines, Fredholm equation

## 1 Introduction

Integral equations often arise in different applications. Astrophysics, the theory of elasticity, hydrodynamics, geology, etc. problems are formulated in terms of integral equations. Solutions of certain problems of mathematical physics are often reduced to the solution of integral equations. The solution of the Fredholm equations of the second kind is the most studied. Well known methods for solving the Fredholm integral equations of the second kind, such as the method of quadratures, the Galerkin method, the method of least squares, the collocation method, the method of replacing the kernels with the degenerate one. Nevertheless, in connection with the emerging needs for constructing methods of high accuracy, many researchers again resort to modernizing the known methods for solving integral equations and construction the new ones. In paper [1] a collocation method with high precision by using the polynomial basis functions is proposed to solve the Fredholm integral equation of the second kind with a weakly singular kernel. The authors introduce the polynomial basis functions and use them to reduce the given equation to a system of linear algebraic equation. In paper [2] the authors consider the Legendre multi-Galerkin methods to solve the Fredholm integral equations of the second kind with a weakly singular kernel and the corresponding eigenvalue problem. In study [3] a numerical scheme for approximating the solutions of nonlinear system of fractional-order Volterra-Fredholm integral differential equations (VFIDEs) has been proposed. The proposed method is based on the orthogonal functions defined over (0,1), combined with their operational matrices of integration and fractional-order differentiation. The main characteristic behind this approach is that it reduces such problems to a linear system of algebraic equations.

Paper [4] presents the method for approximation of two-dimensional function integrals. Then, by combining this approximation with Bernstein collocation method for numerical solution of two-dimensional nonlinear Fredholm integral equations, the kernels double integrals of the integral equations were approximated. The combination of the two-dimensional functions numerical integration method with numerical solution of integral equations method (in both methods, Bernstein polynomials were used) resulted in an increase of convergence speed and accuracy of the method.

In paper [5] a new and efficient method is presented for solving three-dimensional Volterra-Fredholm integral equations of the second kind, first kind and even singular type of these equations. Here, the authors discuss three variable Bernstein polynomials and their properties. This method has several advantages in reducing computational burden with good degree of accuracy. Furthermore, the authors obtain an error bound for this method.

Paper [6] presents a computational technique based on a special family of the Mëuntz-Legendre polynomials to solve a class of Volterra-Fredholm integral equations. The proposed method reduces the integral equation into algebraic equations via the Chebyshev-Gauss-Lobatto points, so that the system matrix coefficients are obtained by the least squares approximation method. The useful properties of the Jacobi polynomials are exploited to analyze the approximation error.

In paper [7] the authors study the numerical approximation of functional integral equations, a class of nonlinear Fredholm-type integral equations of the second kind, by the collocation method with piecewise continuous basis functions. The resulting nonlinear algebraic system is solved with the Picard iteration method.

Paper [8] compares the performance of the Legendre wavelets (LWs) with integer and noninteger orders for solving fractional nonlinear Fredholm integro-differential equations (FNFIDEs). The generalized fractional-order Legendre wavelets (FLWs) are formulated and the operational matrix of fractional derivative in the Caputo sense is obtained. Based on the FLWs, the operational matrix and the Tau method an efficient algorithm is developed for FNFIDEs. The FLWs basis leads to more efficient and accurate solutions of the FNFIDE than the integer-order Legendre wavelets.

For solving a Fredholm integral equation of the second kind, the authors of the paper [9] approximate its kernel by two types of bivariate spline quasiinterpolants, namely the tensor product and the continuous blending sum of univariate spline quasi-interpolants. The authors give the construction of the approximate solutions, and we prove some theoretical results related to the approximation errors of these methods. Everyone knows about the complicated solution of the nonlinear Fredholm integro-differential equation in general. Hence, often, authors attempt to obtain the approximate solution.

In paper [10] a numerical method for the solutions of the nonlinear Fredholm integro-differential equation (NFIDE) of the second kind in the complex plane is presented. In fact, by using the properties of Rationalized Haar (RH) wavelet, authors try to give the solution of the problem.

At present, the theory of approximation by local interpolation splines continues to evolve. Approximation with local polynomial and local non-polynomial splines of the Lagrange types can be used in many applications. Approximation with the use of these splines is constructed on each mesh interval separately as a linear combination of the products of the values of the function at the grid nodes and basic functions. The basis functions are defined as a solution of a system of linear algebraic equations (approximation relations). The approximation relations are formed from the conditions of accuracy of approximation on the functions forming the Chebyshev system.

The constructed basic splines provide an approximation of the prescribed order which is equal to the number of equations in the system, or, in other words, it is equal to the number of grid intervals in the support of the basic splines. Using basic splines, one can construct continuous types of approximation.

Integro-differential splines were considered by the authors earlier in the following papers (see [12–19]), which compete with existing polynomial and nonpolynomial splines of the Lagrange type. The main features of integro-differential splines are the following: the approximation is constructed separately for each grid interval (or elementary rectangular); the approximation constructed as the sum of products of the basic splines and the values of function in nodes and/or the values of integrals of this function over subintervals. Basic splines are determined by using a solving system of equations which are provided by the Chebyshev system of functions. It is known that when integrals of the function over the intervals are equal to the integrals of the approximation of the function over the intervals then the approximation has some physical parallel. Recently, scientists from China joined in the construction and research of the properties of new integro-differential splines [20].

In this paper, the one-dimensional polynomial basic splines of the third or the second order approximation are constructed when the values of the function are known in each point of interpolation. For the construction of the spline, we could also use quadrature with the appropriate order of approximation. These basic splines can be used to solve various problems, including the approximation of a function of one and several variables; the construction of quadrature and cubature formulas; the solution of boundary value problems; the solution of the Fredholm equation, and the Cauchy problem. Currently there are papers in which certain types of splines are used to solve the Fredholm equation and the Heat equation (see [11, 21-31]).

In this paper we consider the solution of the Fredholm equation using polynomial integro-differential splines of the third order approximation and the second order approximation. When integro-differential splines are applied to the solution of integral equations, we replace the integral in the formula of the integro-differential spline with the corresponding quadrature formula. The results of numerical experiments are given in every section.

## 2 Option 1. Construction of a solution of the Fredholm equation with the use of quadratic polynomial splines

Suppose that $a$, $b$ are real numbers and $n$ is a natural number. We construct on the interval $[a,b]$ a uniform grid $\{x_j\}_{j=0}^{n}$ with step $h=\frac{b-a}{n}$.

First we prove two auxiliary statements.

*Lemma 1.* Let function $u(x)$ be such that $u\in C^3[x_{j-1},x_{j+1}]$. The following formula is valid:

$$\int_{x_j}^{x_{j+1}} u(x)dx \approx \frac{h}{12}\big(5u(x_{j+1})+8u(x_j)-u(x_{j-1})\big). \quad (1)$$

*Proof.* We put $\int_{x_j}^{x_{j+1}} u(x)dx \approx \int_{x_j}^{x_{j+1}} v(x)dx$,
where

$$v(x)=u(x_{j-1})w_{j-1}(x)+u(x_j)w_j(x)+u(x_{j+1})w_{j+1}(x),\ x\in[x_j,x_{j+1}], \quad (2)$$

Determining the basic splines $w_{j-1}(x)$, $w_j(x)$, $w_{j+1}(x)$ solving the system of equations $v(x)=u(x), u(x)=1,x,x^2$, we obtain

$$w_{j-1}(x)=\frac{(x-x_j)(x-x_{j+1})}{(x_{j-1}-x_j)(x_{j-1}-x_{j+1})},$$

$$w_j(x)=\frac{(x-x_{j-1})(x-x_{j+1})}{(x_j-x_{j-1})(x_j-x_{j+1})},$$

$$w_{j+1}(x)=\frac{(x-x_{j-1})(x-x_j)}{(x_{j+1}-x_{j-1})(x_{j+1}-x_j)}.$$

After integration we obtain formula (1).
The proof is complete.

*Remark 1.* It is not difficult to obtain the following relation with the help of (2)

$$|u(x)-v(x)| \le \frac{\sqrt{3}}{27}h^3\|u'''\|_{[x_{j-1},x_{j+1}]},\ x\in[x_j,x_{j+1}].$$

Using (1) it can be shown that

$$\left|\int_{x_j}^{x_{j+1}} u(x)dx-\frac{h}{12}\big(5u(x_{j+1})+8u(x_j)-u(x_{j-1})\big)\right|\le K_1h^4\parallel u'''\parallel_{[x_{j-1},x_{j+1}]},\ K_1>0.$$

*Lemma 2.* Let function $u(x)$ be such that $u\in C^3[x_j,x_{j+2}]$. The following formula is valid:

$$\int_{x_j}^{x_{j+1}} u(x)dx \approx \frac{h}{12}\big(5u(x_j)+8u(x_{j+1})-u(x_{j+2})\big). \quad (3)$$

*Proof.* We put $\int_{x_j}^{x_{j+1}} u(x)dx \approx \int_{x_j}^{x_{j+1}} v(x)dx$,
where

$$v(x)=u(x_j)w_j(x)+u(x_{j+1})w_{j+1}(x)+u(x_{j+2})w_{j+2}(x),\ x\in[x_j,x_{j+1}]. \quad (4)$$

Determining the basic splines $w_j(x)$, $w_{j+1}(x)$, $w_{j+2}(x)$ solving the system of equations $v(x)=u(x), u(x)=1,x,x^2$, we obtain

$$w_j(x)=\frac{(x-x_{j+2})(x-x_{j+1})}{(x_j-x_{j+2})(x_j-x_{j+1})},$$

$$w_{j+1}(x)=\frac{(x-x_{j+2})(x-x_j)}{(x_{j+1}-x_{j+2})(x_{j+1}-x_j)},$$

$$w_{j+2}(x)=\frac{(x-x_{j+1})(x-x_j)}{(x_{j+2}-x_{j+1})(x_{j+2}-x_j)}.$$

After integration we obtain formula (3).
The proof is complete.

*Remark.* It is not difficult to obtain the following relation with the help of (4)

$$|u(x)-v(x)| \le \frac{\sqrt{3}}{27}h^3\|u'''\|_{[x_j,x_{j+2}]},\ x\in[x_j,x_{j+1}].$$

Using (3) it can be shown that

$$\left|\int_{x_j}^{x_{j+1}} u(x)dx-\frac{h}{12}\big(5u(x_j)+8u(x_{j+1})-u(x_{j+2})\big)\right|\le K_2h^4\parallel u'''\parallel_{[x_j,x_{j+2}]},\ K_2>0.$$

Consider the Fredholm equation of the second kind

$$\varphi(x)-\int_a^b K(x,s)\,\varphi(s)\,ds=f(x). \quad (5)$$

We construct an approximate solution of the integral equation by applying quadratic polynomial splines as follows.
First we represent the integral in (5) in the following form:

$$\int_a^b K(x,s)\,\varphi(s)\,ds=\int_a^{b-h} K(x,s)\,\varphi(s)\,ds+\int_{b-h}^{b} K(x,s)\,\varphi(s)\,ds. \quad (6)$$

We make the following transformation in the first integral of right side of (6). First, we replace the function $\varphi(s)$, $s\in[x_j,x_{j+1}]$, by $\psi(s)$:

$$\psi(s)=\varphi(x_j)\omega_j(s)+\varphi(x_{j+1})\omega_{j+1}(s)+\int_{x_{j+1}}^{x_{j+2}}\varphi(t)dt\,\omega_j^{<1>}(s). \quad (7)$$

Here $\omega_j(s)$, $\omega_{j+1}(s)$, $\omega_j^{<1>}(s)$ are the continuous integro-differential splines which will be defined later. Now we can rewrite (7) using (1) in the form

$$\psi(s)\approx\varphi(x_j)\omega_j(s)+\varphi(x_{j+1})\omega_{j+1}(s)+\frac{h}{12}\left(5\varphi(x_{j+2})+8\varphi(x_{j+1})-\varphi(x_j)\right)\omega_j^{<1>}(s). \quad (8)$$

*Lemma 3*. Suppose $\varphi(x)$ be such that $\varphi\epsilon C^3[x_j,x_{j+2}]$ and $\psi(x)$ is given by (8). The following formulae are valid:

$$\omega_j(s)=\frac{(s-h-jh)(3s-5h-3jh)}{5h^2}, \quad (9)$$

$$\omega_{j+1}(s)=-\frac{(s-jh)(9s-14h-9jh)}{5h^2}, \quad (10)$$

$$\omega_j^{<1>}(s)=\frac{6(s-h-jh)(s-jh)}{5h^3}. \quad (11)$$

*Proof*. Using $\psi(s)=\varphi(s),\varphi(s)=1,s,s^2$, formula (8) and the Taylor expansion, it is not difficult to obtain the relations (9), (10), (11). The proof is complete.

*Remark*. If $s\in[x_j,x_{j+1}]$, $t\in[0,1]$, $s=x_j+th$, then the basic splines (9), (10), (11) can be written in the form:

$$\omega_j(x_j+th)=\frac{(t-1)(3t-5)}{5},$$

$$\omega_{j+1}(x_j+th)=-\mathrm{t}(9\mathrm{t}-14)/5,$$

$$\omega_j^{<1>}(x_j+th)=\frac{6\mathrm{t}(\mathrm{t}-1)}{5\mathrm{h}}.$$

Plots of the functions $\omega_j(x_j+th)$, $\omega_{j+1}(x_j+th)$, $\omega_j^{<1>}(x_j+th)$ are shown on Fig. 1.

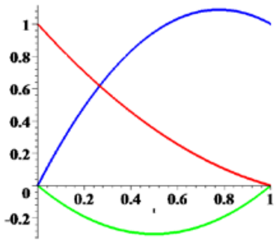


Fig.1. Plots of the functions $\omega_j(x_j+th)$, $\omega_{j+1}(x_j+th)$, $\omega_j^{<1>}(x_j+th)$

In the second integral of (6) we apply the following transformation using integro-differential splines.

We replace the function $\varphi(s)$, $s\in[x_j,x_{j+1}]$, by $\psi(s)$:

$$\psi(s)=\varphi(x_j)\mu(s)+\varphi(x_{j+1})\mu_{j+1}(s)+\int_{x_{j-1}}^{x_j}\varphi(t)dt\,\mu_j^{<-1>}(s). \quad (12)$$

Here $\mu_j(s)$, $\mu_{j+1}(s)$, $\mu_j^{<-1>}(s)$ are the continuous integro-differential splines which will be defined later. Now we can rewrite (12) using (3) in the form

$$\psi(s)\approx\varphi(x_j)\mu_j(s)+\varphi(x_{j+1})\mu_{j+1}(s)+\frac{h}{12}\left(5\varphi(x_{j-1})+8\varphi(x_j)-\varphi(x_{j+1})\right)\mu_j^{<-1>}(s). \quad (13)$$

*Lemma 4*. Suppose $\psi(s)$ be such that $\psi\in C^3[x_j,x_{j+2}]$ and $\psi(s)$ is given by (12). The following formulae are valid:

$$\mu_j(s)=-\frac{(9s+5h-9jh)(s-h-jh)}{5h^2}, \quad (14)$$

$$\mu_{j+1}(s)=\frac{(3s+2h-3jh)(s-jh)}{5h^2}, \quad (15)$$

$$\mu_j^{<-1>}(s)=\frac{6(s-h-jh)(s-jh)}{5h^3}. \quad (16)$$

*Proof*. Using $\psi(s)=\varphi(s)$, $\varphi(s)=1,s,s^2$, formulae (13), and the Taylor expansion, it is not difficult to obtain the relations (14), (15), (16). The proof is complete.

*Remark*. If $s\in[x_j,x_{j+1}]$, $t\in[0,1]$, $s=x_j+th$, the basic splines (14), (15), (16) can be written in the form:

$$\mu_j(x_j+\mathrm{th})=-\frac{(9\mathrm{t}+5)(\mathrm{t}-1)}{5},$$

$$\mu_{j+1}(x_j+\mathrm{th})=\mathrm{t}(3\mathrm{t}+2)/5,$$

$$\mu_j^{<-1>}(x_j+\mathrm{th})=6\mathrm{t}(\mathrm{t}-1)/(5\mathrm{h}).$$

Plots of the functions $\mu_j(x_j+\mathrm{th})$, $\mu_{j+1}(x_j+\mathrm{th})$, $\mu_j^{<-1>}(x_j+\mathrm{th})$ are shown on Fig. 2.

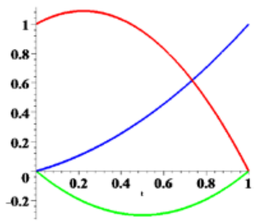


Fig.2. Plots of the functions $\mu_j(x_j+\mathrm{th})$, $\mu_{j+1}(x_j+\mathrm{th})$, $\mu_j^{<-1>}(x_j+\mathrm{th})$

It is not difficult to obtain the following relation:

$|\varphi(x)-\psi(x)| \le Kh^3 \ \| \varphi''' \|_{[x_{j-1},x_{j+1}]}, x \in [x_j, x_{j+1}]$, $K$>0.

Using (8), (9)-(11), (13), (14)-(16) and the following notations:

$$A_j^{<l>}(x) = \int_{x_j}^{x_{j+1}} K(x,s) \left(\omega_j(s) - \frac{h}{12}\omega_j^{<1>}(s)\right) ds,$$

$$B_j^{<l>}(x) = \int_{x_j}^{x_{j+1}} K(x,s) \left(\omega_{j+1}(s) + \frac{2\,h}{3}\omega_j^{<1>}(s)\right) ds,$$

$$C_j^{<l>}(x) = \frac{5\,h}{12}\int_{x_j}^{x_{j+1}} K(x,s)\ \omega_j^{<1>}(s) ds,$$

$$A_{n-1}^{<r>}(x) = \frac{5\,h}{12}\int_{x_{n-1}}^{x_n} K(x,s)\ \mu_{n-1}^{<-1>}(s) ds,$$

$$B_{n-1}^{<r>}(x) = \int_{x_{n-1}}^{x_n} K(x,s) \left(\mu_{n-1}(s) + \frac{2\,h}{3}\mu_{n-1}^{<-1>}(s)\right) ds,$$

$$C_{n-1}^{<r>}(x) = \int_{x_{n-1}}^{x_n} K(x,s) \left(\mu_n(s) - \frac{h}{12}\mu_{n-1}^{<-1>}(s)\right) ds,$$

we get the following system of equations for calculating $\tilde{\varphi}(x_i) \approx \varphi(x_i)$, $i = 0, \dots, n$:

$$\tilde{\varphi}(x_i) - \sum_{j=0}^{n-2}( \tilde{\varphi}(x_j)\, A_j^{<l>}(x_i) + \tilde{\varphi}(x_{j+1}) B_j^{<l>}(x_i) + \tilde{\varphi}(x_{j+2}) C_j^{<l>}(x_i)) - (\tilde{\varphi}(x_{n-2}) A_{n-1}^{<r>}(x_i) + (\tilde{\varphi}(x_{n-1}) B_{n-1}^{<r>}(x_i) + (\tilde{\varphi}(x_n) C_{n-1}^{<r>}(x_i)) = f(x_i).$$

### 2.1 Numerical results

Here we present some numerical results. Fig.4 shows the error of numerical solution of equation (5) when $K(x,s) = x^2 s^2, \varphi(x) = x^{\frac{3}{2}} \sin(x)$, here $f(x)$ was constructed using $K(x,s)$ and $\varphi(s)$, $h = 0.1, a = 0, b = 1$.

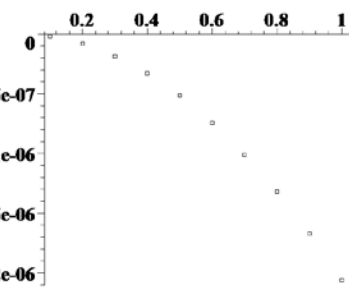


Fig 4. Plot of the error of numerical solution when $K(x,s) = x^2 s^2, \varphi(x) = x^{3/2}\ \sin(x), h = 0.1$

Fig.5 shows the error of numerical solution of equation (5) when $h = 0.01$, $K(x,s) = e^x \cos(s)$, $\varphi(x) = x^{\frac{3}{2}} \sin(x), a = 0, b = 1$.

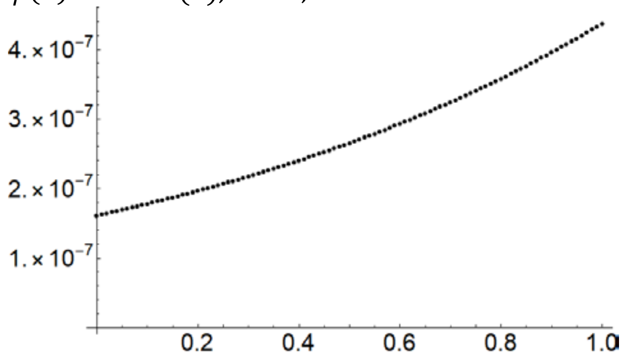


Fig 5. Plot of the error of numerical solution when $K(x,s) = e^x \cos(s), \varphi(x) = x^{\frac{3}{2}} \sin(x), h = 0.01$

In Table 1 one can see absolute values of the difference between the exact solutions and solutions, obtained with the method being suggested in this section, when $a = 0, b = 1$, with $n = 10$ and $n = 100$, *Digits*=15. Here $f(x)$ was constructed using $K(x,s)$ and $\varphi(s)$.

Table 1. Absolute values of errors of approximation when $n = 10$ and $n = 100$.

| $K(x, s)$, $\varphi(x)$ | $n$ = 10 | $n$ = 100 |
|---|---|---|
| $K = x^2 \cdot s^2$, $\varphi = x^{3/2}\sin(x)$ | $0.21\cdot 10^{-5}$ | $0.24\cdot 10^{-7}$ |
| $K = e^x \cdot \cos(s)$, $\varphi = x^{3/2}\sin(x)$ | $0.37\cdot 10^{-3}$ | $0.44\cdot 10^{-6}$ |
| $K = xs, \varphi = 1/(1+25\,x^2)$ | $0.60\cdot 10^{-4}$ | $0.61\cdot 10^{-8}$ |

Table 2 shows the condition numbers for solved systems of linear algebraic equations when $n$ = 10 and $n$ = 100.

Table 2. The condition numbers when $n$ = 10 and $n$ = 100.

| $K(x, s)$, $\varphi(x)$ | $n$ = 10 | $n$ = 100 |
|---|---|---|
| $K(x, s) = x^2 \cdot s^2$, $\varphi(x) = x^{3/2} \sin(x)$ | 1.795 | 1.879 |
| $K(x,s) = e^x \cdot \cos(s)$, $\varphi(x) = x^{3/2} \sin(x)$ | 21.661 | 23.025 |
| $K(x, s) = x \cdot s$, $\varphi(x) = 1/(1+25x^2)$ | 2.508 | 2.613 |

## 3 Option 2: Construction of a solution of the Fredholm equation with the use of linear polynomial splines

In this section let us take $\tilde{\varphi}(s)$in the form:

$$\tilde{\varphi}(s) = \varphi(x_j)\omega_j(s) + \int_{x_j}^{x_{j+1}} \varphi(s) ds\, \omega_j^{<0>}(s), \quad (17)$$

$s \in [x_j, x_{j+1}]$,

where $\omega_j(s), \omega_j^{<0>}(s)$ are the basis integro-differential splines which we obtain later. Using Lemma 1 and Lemma 2 we obtain for $s \in [x_j, x_{j+1}]$:

$$\tilde{\varphi}(s) \approx \varphi(x_j)\omega_j(s) + \frac{h}{12}(5\varphi(x_j) + 8\varphi(x_{j+1}) - \varphi(x_{j+2}))\omega_j^{<0>}(s), \quad (18)$$
$$j = 0, \dots, n-2,$$

$$\tilde{\varphi}(s) \approx \varphi(x_{n-1})\tilde{\omega}_{n-1}(s) + \frac{h}{12}(5\varphi(x_n) + 8\varphi(x_{n-1}) - \varphi(x_{n-2}))\tilde{\omega}_{n-1}^{<0>}(s).$$
(19)

*Lemma 5* Suppose $\tilde{\varphi}(s)$ be such that $\tilde{\varphi} \in \mathrm{C}^2[x_j, x_{j+1}]$ and $\tilde{\varphi}(s)$ is given by (17). The following formulae are valid:

$$\omega_j(s) = \frac{12s - 5x_j - 8x_{j+1} + x_{j+2}}{7x_j - 8x_{j+1} + x_{j+2}}, \quad (20)$$

$$\omega_j^{<0>}(s) = \frac{s - x_j}{-7x_j + 8x_{j+1} - x_{j+2}}, \quad (21)$$

$$\tilde{\omega}_j(s) = \frac{12s + x_{j-1} - 8x_j - 5x_{j+1}}{-x_{j-1} + 4x_j - 5x_{j+1}}, \quad (22)$$

$$\tilde{\omega}_j^{<0>}(s) = \frac{s - x_j}{x_{j-1} - 4x_j + 5x_{j+1}}. \quad (23)$$

***Proof.*** Using $\tilde{\varphi}(s) = \varphi(s), \varphi(s) = 1, s$, where $s \in [x_j, x_{j+1}]$ and (17), (1), (3) and the Taylor expansion, it is not difficult to obtain relations (20)-(23). The proof is complete.

*Remark.* If $s \in [x_j, x_{j+1}], t \in [0,1], s = x_j + th$ the basic splines can be written in the form:

$$\omega_j(x_j + th) = \tilde{\omega}_j(x_j + th) = 1 - 2t,$$

$$\omega_j^{<0>}(x_j + th) = \tilde{\omega}_j^{<0>}(x_j + th) = t/6.$$

It is not difficult to obtain the following relation:

$$|\varphi(x) - \tilde{\varphi}(x)| \le Kh^2 \ \| u'' \|_{[x_{j-1}, x_{j+1}]},$$
$$x \in [x_j, x_{j+1}], \qquad K > 0.$$

Plots of the functions $\omega_j(s)$, $\omega_j^{<0>}(s)$ are shown on Fig. 6.

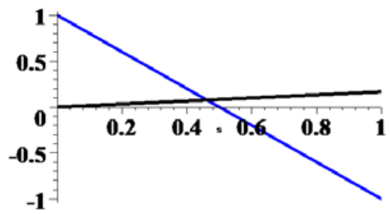

Fig 6. Plots of the functions $\omega_j(s)$, $\omega_j^{<0>}(s)$

Using (18)-(23) we get the system of equations for obtaining $\tilde{\varphi}(x_i) \approx \varphi(x_i), i = 0, \dots, n$:

$$\tilde{\varphi}(x_i) - \sum_{j=0}^{n-2} \tilde{\varphi}(x_j) A_j(x_i) - \sum_{j=0}^{n-2} \left(5\tilde{\varphi}(x_j) + 8\tilde{\varphi}(x_{j+1}) - \tilde{\varphi}(x_{j+2})\right) B_j(x_i) - \tilde{\varphi}(x_{n-1})\tilde{A}_{n-1}(x_i) - \left(5\tilde{\varphi}(x_n) + 8\tilde{\varphi}(x_{n-1}) - \tilde{\varphi}(x_{n-2})\right)\tilde{B}_{n-1}(x_i) = f(x_i),$$

where

$$A_j(x) = \int_{x_j}^{x_{j+1}} K(x,s)\omega_j(s)ds,$$

$$B_j(x) = \int_{x_j}^{x_{j+1}} K(x,s)\omega_j^{<0>}(s)ds,$$

$$\tilde{A}_j(x) = \int_{x_j}^{x_{j+1}} K(x,s)\tilde{\omega}_j(s)ds,$$

$$\tilde{B}_j(x) = \int_{x_j}^{x_{j+1}} K(x,s)\tilde{\omega}_j^{<0>}(s)ds.$$

### 3.1 Numerical results

Here we present some numerical results. In Table 3 one can see absolute values of the difference between the exact solutions and solutions, obtained with method being suggested in this section, when $a = 0, b = 1$, with $n = 10$ and $n = 100$, Digits=15, $f(x)$ was constructed using $K(x,s)$ and $\varphi(s)$.

Table 3. Absolute values of errors of approximation when $n = 10$ and $n = 100$.

| *K(x, s), φ(x)* | *n* = 10 | *n* = 100 |
|---|---|---|
| $K = x^2 \cdot s^2$, $\varphi = x^{3/2}\sin(x)$ | $0.32\cdot10^{-4}$ | $0.97\cdot10^{-8}$ |
| $K = e^x \cdot \cos(s), \varphi = x^{3/2}\sin(x)$ | $0.28\cdot10^{-3}$ | $0.35\cdot10^{-6}$ |
| $K = x\cdot s$, $\varphi = 1/(1+25x^2)$ | $0.20\cdot10^{-4}$ | $0.98\cdot10^{-8}$ |

Fig. 7 shows the error of numerical solution of equation (5) when $K(x,s) = x^2s^2, \varphi(x) = x^{\frac{3}{2}}\sin(x)$, $f(x)$ was constructed using $K(x,s)$ and $\varphi(s)$, $h = 0.1, a = 0, b = 1$. Fig. 8 shows the error of numerical solution of equation (5) when $h = 0.01$, $K(x,s) = e^x\cos(s)$, $\varphi(x) = x^{\frac{3}{2}}\sin(x), a = 0, b = 1$.

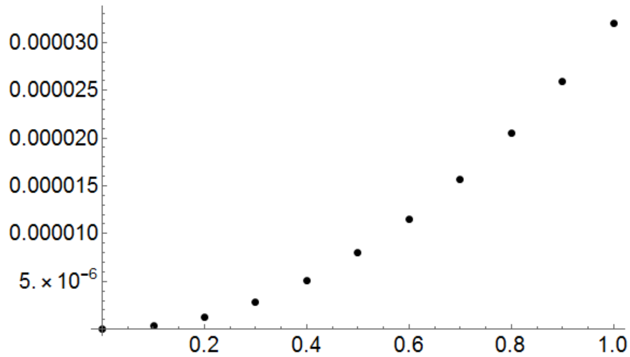


Fig 7. Plot of the error of numerical solution when $K(x,s) = x^2s^2, \varphi(x) = x^{3/2}\ \sin(x)$ , $h = 0.1$

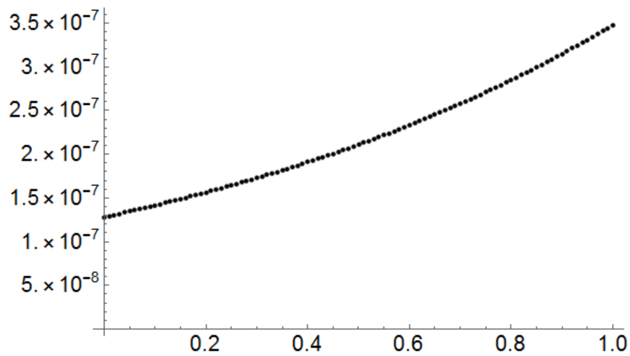


Fig 8. Plot of the error of numerical solution when $K(x,s) = e^x\cos(s), \varphi(x) = x^{\frac{3}{2}}\sin(x), h = 0.01$

Table 4 shows the condition numbers for solved systems of linear algebraic equations.

Table 4. The condition numbers when $n$ = 10 and $n$ = 100.

| *K(x, s), φ(x)* | *n* = 10 | *n* = 100 |
|---|---|---|
| $K(x,s)= x^2\cdot s^2$, $\varphi(x)=x^{3/2}\sin(x)$ | 1.284 | 1.254 |
| $K(x,s)=e^x\cdot\cos(s),\varphi(x)=x^{3/2}\sin(x)$ | 4.623 | 3.962 |
| $K(x,s)= x\cdot s$, $\varphi(x)=1/(1+25x^2)$ | 1.515 | 1.505 |

# 4 Comparison with classical methods for solving integral equations

In this section we compare the results of the numerical solution of integral equations by the methods proposed in sections 2 and 3 with the results of applying classical methods [32-36]. First consider the rule of the trapezium. The method is well known, but we recall briefly its construction. Let us divide the interval of integration $[a,b]$ into $n$ equal parts, thus $h = \frac{(b-a)}{n}$. We denote $x_k = a + kh, g_k = g(x_k)$.

Let us take a compound formula of trapezes:

$$\int_a^b g(x)dx \approx \frac{h}{2}g_0 + h\sum_{j=1}^{n-1} g_j + \frac{h}{2}g_n.$$

Applying this formula to equation (5) and rejecting an error we receive the equation in which $\tilde{\varphi}(x_k) \approx \varphi(x_k)$ is to be obtained:

$$\tilde{\varphi}(x) = \frac{h}{2}K(x,x_0)\tilde{\varphi}(x_0) + h\sum_{j=1}^{n-1} K(x,x_j)\tilde{\varphi}(x_j) + \frac{h}{2}K(x,x_n)\tilde{\varphi}(x_n) + f(x).$$

Now we can put $x = x_i$, $i = 0,1,\dots,n$. Denoting $\tilde{\varphi}_i = \tilde{\varphi}(x_i)$ we receive the system of equations (trapezoid method):

$$\tilde{\varphi}_i = \frac{h}{2}K_{i0}\tilde{\varphi}_0 + h\sum_{j=1}^{n-1} K_{ij}\,\tilde{\varphi}_j + \frac{h}{2}K_{in}\tilde{\varphi}_n + f_i,$$
$i = 0,1,\dots,n.$

Similarly we can apply a compound formula of Simpson ($n$ is even):

$$\int_a^b g(x)dx \approx \frac{h}{3}(g_0 + 4(g_1 + g_3 + \cdots + g_{n-1}) + 2(g_2 + g_4 + \cdots + g_{n-2}) + g_n)$$

In this case the system of the equations will have the form:

$$\tilde{\varphi}_i = \frac{h}{3}K_{i0}\tilde{\varphi}_0 + \frac{4h}{3}\sum_{j=1(2)}^{n-1} K_{ij}\,\tilde{\varphi}_j + \frac{2h}{3}\sum_{j=2(2)}^{n-2} K_{ij}\,\tilde{\varphi}_j + \frac{h}{3}K_{in}\tilde{\varphi}_n + f_i, \quad i = 0,1,\dots,n.$$

This system can be written in the form (Sympson method):

$$\tilde{\varphi}_i = \frac{h}{3}K_{i0}\tilde{\varphi}_0 + \frac{4h}{3}(K_{i1}\tilde{\varphi}_1 + K_{i3}\tilde{\varphi}_3 + \cdots + K_{in-1}\tilde{\varphi}_{n-1}) + \frac{2h}{3}(K_{i2}\tilde{\varphi}_2 + K_{i4}\tilde{\varphi}_4 + \cdots + K_{in-2}\tilde{\varphi}_{n-2}) + \frac{h}{3}K_{in}\tilde{\varphi}_n) + f_i, \ \ i = 0,1,\dots,n.$$

Table 5 shows the errors in absolute values of solution of the same integral equations using trapezoid method when $n = 10, 100$.

Table 5. Absolute values of errors of approximation when $n = 10$ and $n = 100$.

| *K(x, s)* | *φ(x)* | $n$ = 10 | $n$ = 100 |
|---|---|---|---|
| $x^2 \cdot s^2$ | $x^{3/2} sin(x)$ | $0.364 \cdot 10^{-2}$ | $0.363 \cdot 10^{-4}$ |
| $e^x \cdot cos(s)$ | $x^{3/2} sin(x)$ | $0.163 \cdot 10^{-2}$ | $0.159 \cdot 10^{-4}$ |
| $x \cdot s$ | $1/(1+25x^2)$ | $0.133 \cdot 10^{-2}$ | $0.129 \cdot 10^{-4}$ |

Table 6 shows the results of numerical solution of the same integral equations using Simpson rule when $n = 10, 100$.

Table 6. Absolute values of errors of approximation when $n = 10$ and $n = 100$.

| K(x, s) | φ(x) | $n$ = 10 | $n$ = 100 |
|---|---|---|---|
| $x^2 \cdot s^2$ | $x^{3/2}\sin(x)$ | $0.113 \cdot 10^{-4}$ | $0.110 \cdot 10^{-8}$ |
| $e^x \cdot \cos(s)$ | $x^{3/2}\sin(x)$ | $0.786 \cdot 10^{-4}$ | $0.180 \cdot 10^{-7}$ |
| $x \cdot s$ | $1/(1+25x^2)$ | $0.177 \cdot 10^{-3}$ | $0.126 \cdot 10^{-7}$ |

## 4 Conclusion

The quadratic polynomial integro-differential spline and linear polynomial integro-differential spline proposed in this paper showed the possibility of applying them to solving the Fredholm integral equation. In the proposed quadratic method, it is necessary to calculate the integrals $A_j^{<l>}(x)$, $B_j^{<l>}(x)$, $C_j^{<l>}(x)$, $A_{n-1}^{<r>}(x)$, $B_{n-1}^{<r>}(x)$, $C_{n-1}^{<r>}(x)$. In the proposed linear method, it is necessary to calculate the integrals. $A_j(x)$, $B_j(x)$, $\tilde{A}_j(x)$, $\tilde{B}_j(x)$.

It should be noted that for the application of classical methods, you need to be sure that not only the desired solution, but also the kernel has the necessary smoothness.

In future papers, the application of nonpolynomial splines to solve the Fredholm equation will be investigated.

*References:*

[1] G.Zeng, C.Chen, L.Lei X.Xu, A Modified Collocation Method for Weakly Singular Fredholm Integral Equations of Second Kind, *Journal of Computational Analysis and Applications*, Vol.27 No.7, 2019, pp. 1091-1102.

[2] B.L.Panigrahi, M. Mandal, G. Nelakanti, Legendre Multi-Galerkin Methods for Fredholm Integral Equations with Weakly Singular Kernel and the Corresponding Eigenvalue Problem, *Journal of Computational and Applied Mathematics*, Vol.346, 2019, pp. 224-236.

[3] J.Xie, M.Yi, Numerical Research of Nonlinear System of Fractional Volterra-Fredholm Integral-Differential Equations via Block-Pulse Functions and Error Analysis, *Journal of Computational and Applied Mathematics*, Vol.345, 2019, pp. 159-167.

[4] A.Babaaghaie, K.Maleknejad, A New Approach for Numerical Solution of Two-Dimensional Nonlinear Fredholm Integral Equations in the Most General Kind of Kernel, Based on Bernstein Polynomials and its Convergence Analysis, *Journal of Computational and Applied Mathematics*, Vol.344, 2018, pp. 482-494.

[5] K.Maleknejad, J.Rashidinia, T.Eftekhari, Numerical Solution of Three-Dimensional Volterra-Fredholm Integral Equations of the First and Second Kinds Based on Bernstein's Approximation, *Applied Mathematics and Computation*, Vol.339, 2018, pp. 272-285.

[6] N.Negarchi, K.Nouri, Numerical Solution of Volterra-Fredholm Integral Equations Using the Collocation Method Based on a Special Form of the Mëuntz-Legendre Polynomials, *Journal of Computational and Applied Mathematics*, Vol.344, 2018, pp. 15-24.

[7] A.M.Rocha, J.S.Azevedo, S.P.Oliveira, M.R. Correa, Numerical Analysis of a Collocation Method for Functional Integral Equations, *Applied Numerical Mathematics*, Vol.134, 2018, pp. 31-45.

[8] F.Mohammadi, J.A.T.Machado, A Comparative Study of Integer and Noninteger Order Wavelets for Fractional Nonlinear Fredholm Integro-Differential Equations, *Journal of Computational and Nonlinear Dynamics*, Vol.13, No 8, 2018.

[9] D.Barrera, F.Elmokhtari, D.Sbibih, Two Methods Based on Bivariate Spline Quasi-Interpolants for Solving Fredholm Integral Equations, *Applied Numerical Mathematics*, Vol.127, 2018, pp. 78-94.

[10] M.Erfanian, H.Zeidabadi, Solving of Nonlinear Fredholm Integrodifferential Equation in a Complex Plane with Rationalized Haar Wavelet Bases, *Asian-European Journal of Mathematics*, 2018. Article in Press. C.Allouch, P.Sablonni`ere, Iteration Methods for Fredholm Integral Equations of the Second Kind Based on Spline Quasi-Interpolants, *Mathematics and Computers in Simulation*, Vol. 99, 2014, pp. 19-27.

[11] I.G.Burova, O.V.Rodnikova, Integro-Differential Polynomial and Trigonometrical Splines and Quadrature Formulae, *WSEAS Transactions on Mathematics*, Vol.16, 2017, pp.11-18.
[12] I.G.Burova, A.G.Doronina, I.D.Miroshnichenko,A Comparison of Approximations with Left, Right and Middle Integro-Differential Polynomial Splines of the Fifth Order, *WSEAS Transactions on Mathematics*, Vol.16, 2017, pp. 339-349.
[13] I.G.Burova, S.V.Poluyanov, On Approximations by Polynomial And Trigonometrical Integro-Differential Splines, *International Journal of Mathematical Models and Methods in Applied Sciences*, Vol.10, 2016, pp. 190-199.
[14] I.G.Burova, A.G.Doronina, On Approximations by Polynomial and Nonpolynomial Integro-Differential Splines. *Applied Mathematical Sciences*, Vol.10, No 13-16, 2016, pp. 735-745.
[15] I.G.Burova, On Left Integro-Differential Splines and Cauchy Problem. *International Journal of Mathematical Models and Methods in Applied Sciences,* Vol.9, 2015, pp. 683-690.
[16] I.G. Burova, O.V.Rodnikova, Application of Integro-differential Splines to Solving an Interpolation Problem, *Computational Mathematics and Mathematical Physics,* Vol.54, No 12, 2014, pp. 1903-1914.
[17] I.G.Burova, S.V.Poluyanov, Construction of Meansquare Approximation with Integro-Differential Splines of Fifth Order and First Level, *Vestnik St. Petersburg University, Mathematics,* Vol.47, No 2, 2014, pp. 57-63.
[18] I.G.Burova, T.O.Evdokimova, On Construction Third Order Approximation Using Values of Integrals, *WSEAS Transactions on Mathematics,* Vol.13, 2014, pp. 676-683.
[19] F.-G.Lang, X.-P.Xu, On a New Approximation Method with Four Times Continuously Differentiable Quintic Integro-Differential Spline, *AENG International Journal of Applied Mathematics*, Vol. 47, No 4, 1, pp. 371-380.
[20] A.Bellour, D. Sbibih, A.Zidna, Two Cubic Spline Methods for Solving Fredholm Integral Equations, *Applied Mathematics and Computation*, Vol.276, 2016, pp. 1-11.
[21] Fengmin Chen, Patricia J. Y.Wong, Discrete Biquintic Spline Method for Fredholm Integral Equations of the Second Kind, *12th International Conference on Control, Automation, Robotics & Vision Guangzhou*, China, 5-7th December 2012 (ICARCV 2012)
[22] N.Ebrahimi, J.Rashidinia, Spline Collocation for Solving System of Fredholm and Volterra Integral Equations, *International Journal of Mathematical and Computational Sciences*, Vol.8, No 6, 2014.
[23] P.Kalyani, P.S.Ramachandra Rao, Numerical Solution of Heat Equation through Double Interpolation, *IOSR Journal of Mathematics (IOSR-JM)*, Vol.6, No 6, May. - Jun. 2013, pp. 58-62.
[24] Chafik Allouch, Paul Sablonnière, Driss Sbibih, Solving Fredholm Integral Equations by Approximating Kernels by Spline Quasi-Interpolants, *Numer Algor*, Vol.56, 2011, pp. 437-453.
[25] S.Saha Ray, P.K.Sahu, Application of Semiorthogonal B-Spline Wavelets for the Solutions of Linear Second Kind Fredholm Integral Equations, *Appl. Math. Inf. Sci*., Vol.8 No 3, 2014, pp.1179-1184.
[26] J.Rashidinia, E.Babolian, Z.Mahmoodi, Spline Collocation for Fredholm Integral Equations, Mathematical Sciences, 5 (2), 2011,pp. 147-158.
[27] P.S. Rama Chandra Rao, Solution of Fourth Order of Boundary Value Problems Using Spline Functions, *Indian Journal of Mathematics and Mathematical Sciences*, Vol. 2, No 1, 2006, pp. 47-56.
[28] P.S.Rama Chandra Rao, Solution of a Class of Boundary Value Problems using Numerical Integration, *Indian journal of mathematics and mathematical sciences*, Vol.2, No 2, 2006, pp. 137-146.
[29] A.S.V. Ravikanth, Numerical Treatment of Singular Boundary Value Problems, *Ph.D. Thesis*, National Institute of Technology, Warangal, India, 2002.
[30] J.H. Ahlberg, E. N. Nilson, J. L. Walsh,: *Theory of Splines and their Applications*, *Mathematics in Science and Engineering*, Chapt. IV. Academic Press, New York, 1967.
[31] S. Saha Ray and P. K. Sahu, Numerical Methods for Solving Fredholm Integral Equations of Second Kind, *Abstract and Applied Analysis*, Vol. 2013, Article ID 426916, 17 p.
[32] M. J. Emamzadeh and M. T. Kajani, Nonlinear Fredholm Integral Equation of the Second Kind with Quadrature Methods, *Journal of*

*Mathematical Extension*, Vol. 4, No. 2, 2010, pp. 51-58.
[33] L.M.Delves, J.Walsh, *Numerical Solution of Integral Equations*, Clarendon press, Oxford, 1974.
[34] K.Atkinson, A Survey of Numerical Methods for the Solution of Fredholm Integral *Equations of the Second Kind*, SIAM publ., Philadelphia, 1976.
[35] C.T.H.Baker, *The Numerical Treatment of Integral Equations*, Clarendon Press, Oxford, 1977.
[36] P.J.Davis, P.Rabinowitz, *Methods of Numerical Integration*. Academic Press, 1984.