Integral-Type Feedback Controller and Application to the Stabilization of Heat Equation with Boundary Input Delay

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Abstract: In this paper, we consider the stabilization of Heat-equation with boundary delayed control by a new kind of feedback controller. The new feedback controller is of the integral form in spatial variable, which is called the integral-type feedback controller. Our goal of the present paper is to select appropriate kernel functions such that the closed-loop system is exponentially stable. Here we mainly give a method of selecting kernel functions. To prove the stability of the closed-loop system, we design a target system which is exponentially stable, and then construct a revertible and bounded linear transformation that establishes the equivalence between the target system and the closed-loop system.

Key–Words: Integral-type feedback controller, heat equation, exponential stabilization, delayed control.

1 Introduction

In this paper, we consider the stabilization problem of a heat equation with delayed control, whose dynamic behaviour is governed by the partial differential equation:

$$\begin{cases} w_t(x,t) = w_{xx}(x,t) + \rho w(x,t) \\ w_x(0,t) = 0 \\ w_x(1,t) = u(t-\tau) \\ w(x,0) = w_0(x) \\ u(s-\tau) = h(s), s \in [0,\tau] \end{cases} \quad (1)$$

where w is the deflection apart equilibrium of the heat equation, $w_t(x,t) = \frac{\partial w}{\partial t}$, $w_x(x,t) = \frac{\partial w}{\partial x}$, $\rho w(x,t)$ is a source term with $\rho > 0$, τ is the delay time, $u(t)$ is control function defined on $[-\tau, \infty]$, $w_0(x)$ is the initial state and $h(s)$ is history function.

Since the heat equation is an extensive used model, the control and stabilization problems of the heat equations have been a hot topic in the control field. Among a great deal literature, some of them paid their attention to the internal control, while some of them focused on the boundary control due to its easy realization. Especially, with robot development, coupling a classical finite dimensional system to a heat equation becomes an interesting applicative situations. For example, Krstic [1, Chapter 15, pp.253] regarded the finite dimensional systems as a dynamic controller; Instead, Daafouz et al. [2] regarded the ordinary differential equations (ODEs) as a component for the PDEs system. Essentially speaking, the key point is to design controller and then analyze the stability of the closed loop system whatever it is. For instance, Li et al in [3] studied the rapid stabilization of heat equation in non-cylindrical domain; Zhao and Wang in [4] studied the stabilization of a Heat-ODE coupled system; and Baudouin et al in [5] investigated the stability analysis of a system coupled to a heat equation. Liu et al [6] studied controller design via Backstepping Method.

Note that the papers mentioned above do not consider the delay problems of control input. For the systems with delayed control, the stabilization problems have been investigated via many methods, such as Lyapunov function method [7–9], Backstepping method [1][10], pole assignment approach[11]. Some of these methods have been extended to study the stabilization problem of system described by PDEs with delay input or output [12–16]. In particular, recent several years, a new approach of dynamic feedback controller is used to study the stabilization problem of the system with delayed control, e.g., see [17–20]. However, there are many difficulties in stability analysis of the closed-loop systems due to the dynamic feedback controller.

In the present paper our goal is to introduce a new kind of feedback controller for the system described by PDEs with delayed input. Indeed, we can regard the system with delayed control as a coupled system. For example, employing the idea in [19], we can introduce a new variable

$$z(s,t)=u(t+s-\tau), s\in[0,\tau].$$

Then the system (1) can be rewritten into an equivalent PDE-PDE coupled system

$$\begin{cases} z_t(s,t)=z_s(s,t),\ \ s\in(0,\tau),\\ w_t(x,t)=w_{xx}(x,t)+\rho w(x,t),\ x\in(0,1),\\ z(\tau,t)=u(t),\ \ t>0,\\ w_x(0,t)=0,\ \ t>0,\\ w_x(1,t)=z(0,t),\ \ t>0,\\ w(x,0)=w_0(x),\\ z(s,0)=z_0(s)=h(s),\ \ s\in[0,\tau] \end{cases} \tag{2}$$

From the above equations we see that the control is applied to the boundary of transport equation, and both equations are coupled at the endpoint.

In the present paper we want to stabilize system (2) by using an integral-type feedback controller of the form

$$\begin{aligned} u(t)&=K(z,w)\\ &=\int_0^\tau p(\tau-r)z(r,t)dr+\int_0^1\gamma(\tau,x)w(x,t)dx. \end{aligned}$$

Since p and γ are unknown, the integral-type controller is called the parameterization controller.

With such a control, the closed-loop system corresponding to (2) is

$$\begin{cases} z_t(s,t)=z_s(s,t), s\in(0,\tau),\\ w_t(x,t)=w_{xx}(x,t)+\rho w(x,t), x\in(0,1),\\ z(\tau,t)=K(z,w),\\ w_x(0,t)=0,\\ w_x(1,t)=z(0,t),\\ w(x,0)=w_0(x), x\in[0,1],\\ z(s,0)=z_0(s), s\in[0,\tau]. \end{cases} \tag{3}$$

Designing the feedback controller means that we need to select appropriate kernel functions p and γ such that the corresponding closed-loop system is exponentially stable.

Observe that the integral term $K(z,w)$ is on the boundary in (3). Directly analyzing the system is difficult even if the functions p and γ have been selected appropriately. The main contribution of the present paper is to give an approach of selecting kernel functions, with which the closed-loop system (3) is always exponentially stable.

The rest of the paper is organized as follows. In section 2, we design the controller which is integral-type. In section 3, we construct a linear bounded transformation $\mathbb{T}$ that maps the solution of controlled system to the solution of the target system. In section 4, we find the inverse transformation that maps the solution of the target system to the solution of controlled system. In section 5, we conclude the paper.

2 Design of Parameterization Controllers

2.1 Stabilization of unstable heat equation

Firstly we consider a stabilization problem of unstable heat equation

$$\begin{cases} w_t(x,t)=w_{xx}(x,t)+\rho w(x,t),\ \ x\in(0,1),\\ w_x(0,t)=0,\ \ w_x(1,t)=U(t),\\ w(x,0)=w_0(x),\ \ x\in[0,1]. \end{cases} \tag{4}$$

Set $q(x,y)$ satisfies the following equation

$$\begin{cases} (\lambda+\rho)q(x,y)+q_{yy}(x,y)-q_{xx}(x,y)=0,\\ \quad 0\le y\le x\le 1\\ 2[q_x(x,x)+q_y(x,x)]+(\lambda+\rho)=0, x\in(0,1)\\ q_y(x,0)=0. \end{cases}$$

where $\lambda>0$. According to [21, Theorem 10,pp.2197], this equation has a solution

$$q(x,y)=-(\lambda+\rho)x\frac{I_1(\sqrt{(\lambda+\rho)(x^2-y^2)})}{\sqrt{(\lambda+\rho)(x^2-y^2)}}.$$

Taking control $U(t)$

$$U(t)=-\frac{\lambda+\rho}{2}w(1,t)+\int_0^1 q_x(1,y)w(y,t)dy \tag{5}$$

and the backstepping transformation

$$H(x,t)=w(x,t)-\int_0^x q(x,y)w(y,t)dy, \tag{6}$$

then $H(x,t)$ satisfies the following equation

$$\begin{cases} H_t(x,t)=H_{xx}(x,t)-\lambda H(x,t),\ x\in(0,1),\\ H_x(0,t)=H_x(1,t)=0,\\ H(x,0)=H_0(x)\\ =w_0(x)-\int_0^x q(x,y)w_0(y)dy. \end{cases} \tag{7}$$

Clearly, $H(x,t)$ decays exponentially at rate λ in space $L^2[0,1]$.

Similar to [21], we can prove that (7) is equivalent to the equation

$$\begin{cases} w_t(x,t)=w_{xx}(x,t)+\rho w(x,t),\ \ x\in(0,1),\\ w_x(0,t)=0,\\ w_x(1,t)=-\frac{\lambda+\rho}{2}w(1,t)\\ \qquad +\int_0^1 q_x(1,y)w(y,t)dy,\\ w(x,0)=w_0(x),\ \ x\in[0,1]. \end{cases} \tag{8}$$

The inverse transformation is given by

$$w(x,t)=H(x,t)-\int_0^x k(x,y)H(y,t)dy, \tag{9}$$

where

$$k(x,y)=(\lambda+\rho)x\frac{J_1(\sqrt{(\lambda+\rho)(x^2-y^2)})}{\sqrt{(\lambda+\rho)(x^2-y^2)}}.$$

2.2 Designing of Controller for (3)

For the system (3) we take the state space as

$$\mathcal{H} = L^2[0,\tau] \times L^2[0,1].$$

Note that the boundary condition in (8) can be written as

$$\begin{aligned} w_x(1,t) &= -\frac{\lambda+\rho}{2} w(1,t) + \int_0^1 q_x(1,y) w(y,t) dy \\ &= \int_0^1 \left[-\frac{\lambda+\rho}{2}\delta(y-1) + q_x(1,y) \right] w(y,t) dy. \end{aligned}$$

To design the feedback controller, we consider the following equation

$$\left\{ \begin{array}{l} \gamma_s(s,x) = \gamma_{xx}(s,x) + \rho\gamma(s,x), \ \ x \in (0,1), \\ \gamma_x(s,0) = 0, \ \ \gamma_x(s,1) = 0, \\ \gamma(0,x) = -\frac{\lambda+\rho}{2}\delta(x-1) + q_x(1,x). \end{array} \right. \tag{10}$$

Let $p(s) = \gamma(s,1)$. With such choices, the controller in (3) takes the form

$$\begin{aligned} u(t) &= K(z,w) = \int_0^\tau \gamma(\tau - r, 1) z(r,t) dr \\ &\quad + \int_0^1 \gamma(\tau, y) w(y,t) dy \end{aligned} \tag{11}$$

2.3 Stable target system

To show the stability of the system (3) with control (11), we choose the following system as a target system

$$\left\{ \begin{array}{l} v_t(s,t) = v_s(s,t), \ \ s \in (0,\tau) \\ v(\tau,t) = 0, \\ w_t(x,t) = w_{xx}(x,t) + \rho w(x,t), \ \ x \in (0,1), \\ w_x(0,t) = 0, \\ w_x(1,t) = v(0,t) - \frac{\lambda+\rho}{2} w(1,t) \\ \quad + \int_0^1 q(1,y) w(y,t) dy, \\ w(x,0) = w_0(x), x \in [0,1], \\ v(s,0) = v_0(s), \ \ s \in [0,\tau] \end{array} \right. \tag{12}$$

As to (12), we have the following result.

Theorem 1 *In space $\mathcal{H}$, the system (12) is exponentially stable. In particular, it decays at rate λ.*

Note that when $t > \tau$, (12) is the same as (8) where $v = 0$. So the result of Theorem 1 is obvious.

Since the stability of (12) is known, in the next step, we will construct a transformation $\mathbb{T}$ that establishes the equivalence between systems (3) and (12).

3 Constructing of Bounded Linear Transformation

In this section, we will construct a bounded and revertible linear transformation $\mathbb{T}$ to establish the equivalence between systems (3) and (12). In what follows, we always discuss the problem in the space $\mathcal{H}$. Let $\gamma(s,x)$ be a solution to (10) and $p(s) = \gamma(s,1)$.

Let $\mathcal{H}$ be defined as before. We define a linear operator $\mathbb{T}$ on $\mathcal{H}$ by

$$\begin{aligned} \begin{pmatrix} v(s) \\ w(x) \end{pmatrix} &= \mathbb{T} \begin{pmatrix} z(s) \\ w(x) \end{pmatrix} \\ &= \begin{pmatrix} 1 - \widehat{p} & -\widehat{\gamma} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z(s) \\ w(x) \end{pmatrix} \end{aligned} \tag{13}$$

where $\widehat{p}$ and $\widehat{\gamma}$ are defined as follows.

$$\left\{ \begin{array}{ll} \widehat{p}(z) = \int_0^s \gamma(s-r,1) z(r) dr, & \forall z \in L^2[0,\tau], \\ \widehat{\gamma}(w) = \int_0^1 \gamma(s,x) w(x) dx, & \forall w \in L^2[0,1]. \end{array} \right. \tag{14}$$

In the following subsections we will show the $\mathbb{T}$ is the desired transformation.

3.1 $\mathbb{T}$ maps the system (3) to system (12)

Firstly we show the operator $\mathbb{T}$ maps a solution of system (3) to a solution of (12).

Theorem 2 *Suppose that $(z(s,t), w(x,t)) \in \mathcal{H}$ is a classical solution of the system (3). Let*

$$\begin{pmatrix} v(s,t) \\ w(x,t) \end{pmatrix} = \mathbb{T} \begin{pmatrix} z(s,t) \\ w(x,t) \end{pmatrix} \tag{15}$$

Then $(v(s,t), w(x,t))$ is a solution of (12).

Proof: Let $(z(s,t), w(x,t))$ be the classical solution of system (3). Let

$$(v(s,t), w(x,t))^T = \mathbb{T}(z(s,t), w(x,t))^T$$

We will verify that $(v(s,t), w(x,t))$ is a solution of (12). According to (13) we have

$$\begin{aligned} v(s,t) &= z(s,t) - \int_0^s \gamma(s-r,1) z(r) dr \\ &\quad - \int_0^1 \gamma(s,x) w(x,t) dx. \end{aligned}$$

First we verify $v(s,t)$ satisfies the equation and boundary condition. Since

$$\begin{aligned} v_s(s,t) &= z_s(s,t) - \gamma(s,1) z(0,t) \\ - \int_0^s \gamma(r,1) z_s(s-r,t) dr &- \int_0^1 \gamma_s(s,x) w(x,t) dx \end{aligned}$$

we have

$$\begin{aligned}
v_t(s,t) &= z_t(s,t) - \int_0^s \gamma(r,1) z_t(s-r,t)dr \\
&\quad - \int_0^1 \gamma(s,x) w_t(x,t)dx \\
&= z_s(s,t) - \int_0^s \gamma(r,1) z_s(s-r,t)dr \\
&\quad - \int_0^1 \gamma(s,x)[w_{xx}(x,t) + \rho w(x,t)]dx \\
&= v_s(s,t) + \gamma(s,1)z(0,t) + \\
&\quad + \int_0^1 [\gamma_s(s,x) - \rho\gamma(s,x)] w(x,t)dx \\
&\quad - \int_0^1 \gamma(s,x) w_{xx}(x,t)dx \\
&= v_s(s,t) + \gamma(s,1)z(0,t) + \int_0^1 \gamma_{xx}(s,x) w(x,t)dx \\
&\quad - \int_0^1 \gamma(s,x) w_{xx}(x,t)dx \\
&\quad \text{using boundary conditions of } w(x,t) \text{ and } \gamma(s,x) \\
&= v_s(s,t).
\end{aligned}$$

Note that

$$z(\tau,t) = \int_0^\tau \gamma(\tau - r,1) z(r,t)dr + \int_0^1 \gamma(\tau,x) w(x,t)dx$$

it holds that

$$\begin{aligned}
v(\tau,t) &= z(\tau,t) - \int_0^\tau \gamma(\tau - r,1) z(r,t)dr \\
&\quad - \int_0^1 \gamma(\tau,x) w(x,t)dx = 0.
\end{aligned}$$

Therefore $v(s,t)$ satisfies the differential equation and boundary condition in (12).

Next, we verify $w(x,t)$ satisfies the differential equation and boundary conditions in (12). In fact, we only need to check the boundary condition at $x = 1$. Since $w_x(1,t) = z(0,t)$ in (3), and

$$v(0,t) = z(0,t) - \int_0^1 \gamma(0,x) w(x,t)dx,$$

using the initial value condition in (10), we have

$$\begin{aligned}
w_x(1,t) &= v(0,t) + \int_0^1 \gamma(0,x) w(x,t)dx \\
&= v(0,t) - \frac{\lambda+\rho}{2} w(1,t) + \int_0^1 q(1,x) w(x,t)dx.
\end{aligned}$$

Therefore $w(x,t)$ satisfies the differential equation and boundary condition in (12).

Finally, we determine the initial condition in (12). Since $w(x,0) = w_0(x)$ and

$$\begin{aligned}
v(s,t) &= z(s,t) - \int_0^s \gamma(s-r,1) z(r,t)dr \\
&\quad - \int_0^1 \gamma(s,x) w(x,t)dx,
\end{aligned}$$

we have

$$\begin{aligned}
v_0(s) &= v(s,0) = z_0(s) - \int_0^s \gamma(s-r,1) z_0(r)dr \\
&\quad - \int_0^1 \gamma(s,x) w_0(x)dx.
\end{aligned}$$

That is, $(v_0(s), w_0(x))^T = \mathbb{T}(z_0(s), w_0(x))^T$ □

3.2 The boundedness of Transformation $\mathbb{T}$

In this subsection, we will study the boundedness of $\mathbb{T}$ in $\mathcal{H}$. Firstly we consider the solvability of (10).

Note that the initial value condition of (10) is composed of two parts: one is the delta function $\delta(x-1)$, the other is a differentiable function $q_x(1,x)$. So we can decompose $\gamma(s,x)$ into two parts $\gamma^0(s,x)$ and $\gamma^1(s,x)$, where

$$\begin{cases}
\gamma^0_s(s,x) = \gamma^0_{xx}(s,x) + \rho\gamma^0(s,x), x \in (0,1), \\
\gamma^0_x(s,0) = 0, \ \ \gamma^0_x(s,1) = 0, \ \ s > 0 \\
\gamma^0(0,x) = -\frac{\lambda+\rho}{2}\delta(x-1), \ \ x \in [0,1]
\end{cases} \tag{16}$$

and

$$\begin{cases}
\gamma^1_s(s,x) = \gamma^1_{xx}(s,x) + \rho\gamma^1(s,x), x \in (0,1), \\
\gamma^1_x(s,0) = 0, \ \ \gamma^1_x(s,1) = 0, \ \ s > 0, \\
\gamma^1(0,x) = q_x(1,x), \ \ x \in [0,1].
\end{cases} \tag{17}$$

Let us consider the boundary eigenvalue problem:

$$-\varphi''_n(x) = \lambda_n \varphi_n(x), \quad \varphi'_n(0) = \varphi'_n(1) = 0. \tag{18}$$

Obviously $\lambda_n = (n\pi)^2, n \geq 0$ and $\varphi_0(x) = 1$ and $\varphi_n(x) = \sqrt{2}\cos n\pi x$ satisfy $\int_0^1 |\varphi_n(x)|^2 dx = 1$. So $\{\varphi_n(x), n \geq 0\} = \{1, \sqrt{2}\cos n\pi x, n \geq 1\}$ forms an orthogonal basis for $L^2[0,1]$. Especially, $\varphi_n^2(1) = 2$.

Since the equation (16) has the singular initial conditions, the following theorem gives its solvability.

Theorem 3 *The singular initial and boundary value problem:*

$$\begin{cases}
\gamma^0_s(s,x) = \gamma^0_{xx}(s,x) + \rho\gamma^0(s,x), x \in (0,1), \\
\gamma^0_x(s,0) = 0, \ \ \gamma^0_x(s,1) = 0, s > 0, \\
\gamma^0(0,x) = -\frac{\lambda+\rho}{2}\delta(x-1), x \in [0,1]
\end{cases}$$

has a unique solution:

$$\gamma^0(s,x) = -\frac{\lambda+\rho}{2}\sum_{n=0}^{\infty} e^{(-\lambda_n+\rho)s}\varphi_n(1)\varphi_n(x). \quad (19)$$

Proof Let $\{\varphi_n(x), n \geq 0\} = \{1, \sqrt{2}\cos n\pi x, n \geq 1\}$ be the solutions of boundary eigenvalues (18). Clearly, the family $\{\varphi_n(x), n \geq 0\}$ forms an orthogonal basis in $L^2[0,1]$. Let $\gamma^0(s,x)$ be defined as (19). For any $s > 0$ the series converges and is differentiable. In addition,

$$\begin{aligned} &\gamma^0_s(s,x) \\ = & -\frac{\lambda+\rho}{2}\sum_{n=0}^{\infty} e^{(-\lambda_n+\rho)s}(-\lambda_n+\rho)\varphi_n(1)\varphi_n(x) \\ = & -\frac{\lambda+\rho}{2}\sum_{n=0}^{\infty} e^{(-\lambda_n+\rho)s}\varphi_n(1)[\varphi''_n(x)+\rho\varphi_n(x)] \\ = & \ \gamma^0_{xx}(s,x)+\rho\gamma^0(s,x) \end{aligned}$$

and

$$\gamma^0_x(s,x) = -\frac{\lambda+\rho}{2}\sum_{n=0}^{\infty} e^{(-\lambda_n+\rho)s}\varphi_n(1)\varphi'_n(x)$$

hence it holds

$$\gamma^0_x(s,0) = \gamma^0_x(s,1) = 0.$$

To verify the initial value condition, for any $f \in C^1[0,1]$, there is a series expression

$$f(x) = \sum_{n=0}^{\infty} a_n(f)\varphi_n(x)$$

that converges absolutely in pointwise. Furthermore

$$\begin{aligned} &\int_0^1 \gamma^0(s,x)f(x)dx \\ &= -\frac{\lambda+\rho}{2}\sum_{n=0}^{\infty} e^{(-\lambda_n+\rho)s}\varphi_n(1)\int_0^1 \varphi_n(x)f(x)dx \\ &= -\frac{\lambda+\rho}{2}\sum_{n=0}^{\infty} e^{(-\lambda_n+\rho)s}\varphi_n(1)a_n(f), \end{aligned}$$

so we have

$$\begin{aligned} &\lim_{s\to 0}\int_0^1 \gamma^0(s,x)f(x)dx \\ &= -\frac{\lambda+\rho}{2}\sum_{n=0}^{\infty} a_n(f)\varphi_n(1) = -\frac{\lambda+\rho}{2}f(1). \end{aligned}$$

Therefore, the initial condition is satisfied. □

The following theorem gives the boundedness of transformation $\mathbb{T}$.

Theorem 4 *Let $\gamma(s,x)$ is the solution of (10). Then the following assertions are true*

1) There is a positive constant M_1 such that the following inequality holds true for $\forall g \in L^2[0,1]$,

$$\int_0^\tau \left|\int_0^1 \gamma(s,x)g(x)dx\right|^2 ds \leq M_1\int_0^1 |g(x)|^2dx.$$

2) There is a positive constant M_2 such that the following inequality holds true for $\forall z \in L^2[0,\tau]$,

$$\int_0^\tau \left|\int_0^s \gamma(s-r,1)z(r)dr\right|^2 ds \leq M_2\int_0^\tau |z(r)|^2 dr.$$

Proof: Let $\gamma^0(s,x)$ and $\gamma^1(s,x)$ be the solution (16) and (17) respectively. Obviously $\gamma^1(s,x)$ is continuous and differentiable in s and x.

1) For any $g \in L^2[0,1]$,

$$\begin{aligned} \int_0^1 \gamma(s,x)g(x)dx = &\int_0^1 \gamma^0(s,x)g(x)dx \\ &+\int_0^1 \gamma^1(s,x)g(x)dx \end{aligned}$$

and

$$\begin{aligned} &\int_0^\tau \left|\int_0^1 \gamma^1(s,x)g(x)dx\right|^2 ds \\ &\leq \int_0^\tau\int_0^1 |\gamma^1(s,x)|^2dxds\int_0^1 |g(x)|^2dx, \end{aligned}$$

we only need to prove

$$\int_0^1 \gamma^0(s,x)g(x)dx \in L^2[0,\tau].$$

Set $a_n(g) = \int_0^1 \varphi_n(x)g(x)dx$. Since

$$\begin{aligned} &\int_0^1 \gamma^0(s,x)g(x)dx \\ &= -\frac{\lambda+\rho}{2}\sum_{n=0}^{\infty} e^{(-\lambda_n+\rho)s}\varphi_n(1)a_n(g), \end{aligned}$$

we have

$$\begin{aligned} &\int_0^\tau \left|\int_0^1 \gamma^0(s,x)g(x)dx\right|^2 ds \\ \leq & \frac{(\lambda+\rho)^2}{2}\int_0^\tau \sum_{n=1}^{\infty}|e^{(-\lambda_n+\rho)s}|^2\sum_{n=1}^{\infty}|a_n(g)|^2ds \\ \leq & \frac{(\lambda+\rho)^2}{2}\int_0^1 |g(x)|^2dx\int_0^\tau e^{2\rho s}ds \\ \leq & \frac{(\lambda+\rho)^2}{2}\frac{e^{2\rho\tau}-1}{2\rho}\int_0^1 |g(x)|^2dx. \end{aligned}$$

So the first assertion holds true.

2) Note that

$$\int_0^s \gamma(s-r,1)z(r)dr = \int_0^s \gamma^0(s-r,1)z(r)dr + \int_0^s \gamma^1(s-r,1)z(r)dr.$$

We only need to prove $\int_0^s \gamma^0(s-r,1)z(r)dr \in L^2[0,\tau]$. Since

$$\begin{aligned}&\int_0^s \gamma^0(s-r,1)z(r)dr\\ &= -\frac{(\lambda+\rho)}{2}\sum_{n=0}^{\infty}\varphi_n^2(1)\int_0^s e^{(-\lambda_n+\rho)(s-r)}z(r)dr\\ &= -\frac{(\lambda+\rho)}{2}\sum_{n=0}^{\infty}a_n(s)\varphi_n(1)\end{aligned}$$

while $a_n(s) = \varphi_n(1)\int_0^s e^{(-\lambda_n+\rho)(s-r)}z(r)dr$ satisfies the equation

$$\begin{cases} a_n'(s) + (\lambda_n-\rho)a_n(s) = z(s)\varphi_n(1), s>0, \\ a_n(0)=0 \end{cases} \tag{20}$$

and the function defined by

$$w(x,s) = \sum_{n=0}^{\infty} a_n(s)\varphi_n(x)$$

is a solution of the following equation

$$\begin{cases} w_t(x,s) = w_{xx}(x,s) + \rho w(x,s), x\in(0,1), \\ w_x(0,s) = 0 \\ w_x(1,s) = z(t) \\ w(x,0) = 0, \end{cases} \tag{21}$$

so we have

$$\int_0^s \gamma^0(s-r,1)z(r)dr = -\frac{(\lambda+\rho)}{2}w(1,s).$$

By the well-posedness of the system (21) with output $y(s) = w(1,s)$, we can get

$$\int_0^\tau |y(s)|^2 ds \le M_2 \int_0^\tau |z(s)|^2 ds.$$

Therefore, the second assertion holds true. □

4 Constructing of the Inverse Transformation of $\mathbb{T}$

In the previous section, we have proved that $\mathbb{T}$ is a bounded and linear operator on $\mathcal{H}$. In this section we will construct directly the inverse transformation of $\mathbb{T}$.

We observe from definition of $\mathbb{T}$ that

$$v = (1-\widehat{p})(z) - \widehat{\gamma}(w) \in L^2[0,\tau],\ \ \forall (z,w)\in\mathcal{H},$$

it is equivalent to

$$(1-\widehat{p})(z) = v + \widehat{\gamma}(w) \in L^2[0,\tau].$$

Note that

$$(1-\widehat{p})(z) = z(s) - \int_0^s \gamma(s-r,1)z(r)dr$$

is bounded and revertible in $L^2[0,\tau]$. So we have

$$z = (1-\widehat{p})^{-1}(v) + (1-\widehat{p})^{-1}\widehat{\gamma}(w)$$

Based on this observation, we can construct the inverse transformation of $\mathbb{T}$. Let $\mathcal{H}$ be defined as before. We define a linear operator $\mathbb{S}$ in $\mathcal{H}$.

$$\begin{aligned}\begin{pmatrix} z(s) \\ w(x)\end{pmatrix} &= \mathbb{S}\begin{pmatrix} v(s) \\ w(x)\end{pmatrix}\\ &= \begin{pmatrix} 1-\widehat{p}_1 & -\widehat{\gamma}_1 \\ 0 & 1\end{pmatrix}\begin{pmatrix} v(s) \\ w(x)\end{pmatrix}\end{aligned} \tag{22}$$

where operators $\widehat{p}_1 = 1-(1-\widehat{p})^{-1}$ and $\widehat{\gamma}_1 = (1-\widehat{p})^{-1}\widehat{\gamma}$ are the following form

$$\begin{cases} \widehat{p}_1(v) = \int_0^s \widetilde{p}(s-r)z(r)dr, & \forall v\in L^2[0,\tau], \\ \widehat{\gamma}_1(w) = \int_0^1 \widetilde{\gamma}(s,x)w(x)dx, & \forall w\in L^2[0,1]. \end{cases} \tag{23}$$

From the definition of $\mathbb{S}$, $\mathbb{S}$ is formally the inverse transformation of $\mathbb{T}$. However, we need to prove that $\mathbb{S}$ maps (12) to (3) and $\mathbb{S}$ is bounded on $\mathcal{H}$.

4.1 $\mathbb{S}$ maps the system (12) to (3)

In this subsection we will determine $\widetilde{p}$ and $\widetilde{\gamma}$ such that $\mathbb{S}$ maps (12) to (3) and $\mathbb{S}$ is bounded on $\mathcal{H}$.

Theorem 5 *Let $\mathbb{S}$ be defined by (22), and let $(v(s,t),w(x,t))$ be a classical solution of (12). Set $(z(s,t),w(x,t))^T = \mathbb{S}(v(s,t),w(x,t))^T$. Then $(z(s,t),w(x,t))$ is a solution of (3) if and only if the function $\widetilde{\gamma}(s,x)$ satisfies the following equations*

$$\begin{cases} \widetilde{\gamma}_s(s,x) = \widetilde{\gamma}_{xx}(s,x) + \rho\widetilde{\gamma}(s,x) \\ \qquad + \widetilde{\gamma}(s,1)q(1,x), x\in(0,1), s>0, \\ \widetilde{\gamma}_x(s,0) = 0,\ \ s>0, \\ \widetilde{\gamma}_x(s,1) = -\frac{\lambda+\rho}{2}\widetilde{\gamma}(s,1), \\ \widetilde{\gamma}(0,x) = \frac{\lambda+\rho}{2}\delta(x-1) - q_x(1,x), \end{cases} \tag{24}$$

and

$$\widetilde{p}(s) = \widetilde{\gamma}(s,1). \tag{25}$$

Proof: According (22) and (23), we have

$$\begin{aligned} z(s,t) &= v(s,t) - \int_0^s \widetilde{p}(s-r)v(r,t)dr \\ &- \int_0^1 \widetilde{\gamma}(s,x)w(x,t)dx \end{aligned}$$

Since $(v(s,t), w(x,t))$ is the classical solution of (12), we can calculate

$$z_s(s,t) = v_s(s,t) - \widetilde{p}(s)v(0,t) - \int_0^s \widetilde{p}(r)v_s(s-r,t)dr - \int_0^1 \widetilde{\gamma}_s(s,x)w(x,t)dx$$

and

$$\begin{aligned} & z_t(s,t) = v_t(s,t) - \int_0^s \widetilde{p}(r)v_t(s-r,t)dr - \int_0^1 \widetilde{\gamma}(s,x)w_t(x,t)dx \\ =\ & v_s(s,t) - \int_0^s \widetilde{p}(r)v_s(s-r,t)dr \\ & - \int_0^1 \widetilde{\gamma}(s,x)[w_{xx}(x,t) + \rho w(x,t)]dx \\ =\ & z_s(s,t) + \widetilde{p}(s)v(0,t) + \int_0^1 \widetilde{\gamma}_s(s,x)w(x,t)dx \\ & - \int_0^1 [\rho\widetilde{\gamma}(s,x) + \widetilde{\gamma}_{xx}(s,x)]w(x,t)dx \\ & -\widetilde{\gamma}(s,x)w_x(x,t)|_0^1 + \widetilde{\gamma}_x(s,x)w(x,t)|_0^1 \\ =\ & z_s(s,t) + [\widetilde{p}(s) - \widetilde{\gamma}(s,1)]v(0,t) \\ & + \int_0^1 [\widetilde{\gamma}_s(s,x) - \widetilde{\gamma}_{xx}(s,x) - \rho\widetilde{\gamma}(s,x)]w(x,t)dx \\ & -\widetilde{\gamma}(s,1)\int_0^1 q_x(1,x)w(x,t)dx - \gamma_x(s,0)w(0,t) \\ & + \left[\gamma_x(s,1) + \frac{\lambda+\rho}{2}\widetilde{\gamma}(s,1)\right] w(1,t). \end{aligned}$$

If $(z(x,t), w(x,t))$ is a solution of (3), then $z_t(s,t) = z_s(s,t)$, and hence

$$\begin{cases} \widetilde{\gamma}_s(s,x) = \widetilde{\gamma}_{xx}(s,x) + \rho\widetilde{\gamma}(s,x) + \widetilde{\gamma}(s,1)q_x(1,x), \\ \widetilde{\gamma}_x(s,0) = 0, \\ \widetilde{\gamma}_x(s,1) = -\frac{\lambda+\rho}{2}\widetilde{\gamma}(s,1), \\ \widetilde{\gamma}(s,1) = \widetilde{p}(s). \end{cases}$$

In addition,

$$\begin{aligned} & z(\tau,t) = v(\tau,t) - \int_0^\tau \widetilde{p}(\tau-s)v(s,t)ds - \int_0^1 \widetilde{\gamma}(\tau,x)w(x,t)dx \\ =\ & -\int_0^\tau \widetilde{p}(\tau-s)v(s,t)ds - \int_0^1 \widetilde{\gamma}(\tau,x)w(x,t)dx \end{aligned}$$

Let us recall the boundary condition in (3) and (12), i.e., $w_x(1,t) = z(0,t)$ and

$$w_x(1,t) = v(0,t) - \tfrac{\lambda+\rho}{2}w(1,t) + \int_0^1 q_x(1,x)w(x,t)dx,$$

that imply

$$\begin{aligned} z(0,t) &= v(0,t) - \frac{\lambda+\rho}{2}w(1,t) \\ &+ \int_0^1 q_x(1,x)w(x,t)dx \\ &= v(0,t) - \int_0^1 \widetilde{\gamma}(0,x)w(x,t)dx. \end{aligned}$$

Therefore, we have

$$\widetilde{\gamma}(0,x) = \frac{\lambda+\rho}{2}\delta(x-1) - q_x(1,x).$$

The equations (24) and (25) follow. □

4.2 Boundedness of $\mathbb{S}$

Similarly we can prove the following result.

Theorem 6 *Let $\widetilde{\gamma}(s,x)$ be a solution of (24) and $\widetilde{p}(s) = \widetilde{\gamma}(s,1)$. Let $\mathbb{S}$ be defined as (22) and (23). Then $\mathbb{S}$ is a bounded linear operator on $\mathcal{H}$.*

5 Conclusion

In this paper, we propose a new approach of designing feedback controller for the system described by PDEs with delayed control, and apply it to study stabilization problem of heat equation with the Neumann boundary delayed control. The approach we used is different from the dynamic feedback control. The advantage of the approach is the stability of the closed-loop system can be settled by the process of designing controller, which can be seen from the stable target system. Here a technique trick is to construct a bounded and revertible linear transformation $\mathbb{T}$ that establishes the equivalence of the target system and the closed-loop system. In this approach, an important technique is to select the kernel function equations and initial value conditions. Different selection will lead to different target system and transformation $\mathbb{T}$.

Although the model we studied is similar to that studied in [1, Chapter 18], the approach we used is entirely different. The approach used in [1, Chapter 18] has restriction on the models, however, our approach can be used to any system with delayed control. Moreover, we have given an approach of constructing the target system and transformation $\mathbb{T}$, including $\mathbb{T}^{-1}$. In the future, when the target system is given, we will explore a universal method to construct transformation $\mathbb{T}$ and transformation $\mathbb{T}^{-1}$.

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