Remark on boundary value problems arising in Ginzburg-Landau theory

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Abstract: The equation $x'' = -a(x - x^3) \quad (i)$ is considered together with the boundary conditions $x'(0) = 0,\ x'(1) = 0 \quad (ii), x'(0) = 0,\ x'(T) = 0 \quad (iii)$. The exact number of solutions for the boundary value problems (BVP) (i), (ii) and (i), (iii) is given. The problem of finding the initial values $x_0 = x(0)$ of solutions to the problem (i), (iii) is solved also.

Key–Words: Boundary value problem, Jacobian elliptic functions, cubic nonlinearity, phase trajectory, multiplicity of solutions

1 Introduction

The cubic complex Ginzburg - Landau equation

$$\partial_t A = A + (1 + ib)\triangle A + (1 + ic)|A|^2 A,$$

appears in numerous descriptions of remarkable physical phenomena. It suffices to mention the superconductivity theory [6]. A survey of multiple applications can be found in [2] and [11]. Stationary solutions of “real” Ginzburg - Landau equation [2]

$$\partial_t A = A + \triangle A - |A|^2 A$$

generate various problems for ordinary differential equations. It was mentioned in the work [9] that there are lacking in the literature the results on boundary value problems that arise in Ginzburg - Landau theory of superconductivity. Namely, the boundary value problem

$$x'' = -a(x - x^3), \tag{1}$$

$$x'(0) = 0, \quad x'(1) = 0 \tag{2}$$

was mentioned and the problem of finding the initial values x_0 of solutions of the problem (1),

$$x'(0) = 0, \quad x'(T) = 0, \tag{3}$$

that is interpreted as “eigenvalue” problem. Our intent is to fill this gap.

First, we analyze the problem and provide the exact number of solutions depending on the parameter a. Despite of the fact that the phase portrait of the equation is very well known we are unaware of the proof that exact number of solutions to the Neumann problem depends entirely on the properties of solutions of the linearized equation around the zero equilibrium point. To prove this, monotonicity of a period for closed trajectories lying in G (the region of a phase plane between two heteroclinic trajectories) must be proved first. Second, using the theory of Jacobian elliptic functions, we give explicit expressions for solutions of the problem. Avoiding cumbersome formulas provided by symbolic computation software. Generally it is known that solutions of the equation can be expressed in terms of the Jacobian elliptic functions. However, it is nor easy nor convenient to get the respective formulas explicitly. Only few sources provide the related information and generally are useless in specific contexts like the Neumann boundary value problem. Standard software provides general expressions that are not easy to use in order to get explicit analytical formulas for specific cases under consideration. Formulas for solutions of the quadratic equation are quite different for solutions that behave differently in different subsets of a phase plane. For instance, no solutions of the Neumann BVP can exist (if speaking about the trajectories of solutions) outside the region G. We made all the necessary computations for the Neumann problem by ourselves and the results look quite satisfactory. The explicit formula for solutions of the respective Cauchy problem was obtained.

In order to study the problem numerically approximations of the initial values for possible solutions are needed. We derived the equation for finding the initial values x_0 of solutions to the Neumann problem.

These formulas involve several Jacobian elliptic functions and can be used to effectively find the initial values of solutions. The initial conditions of the Neumann problem (1), (2) can be found easily now as zeros of some one argument function. The advantages of the proposed approach are demonstrated considering the example in the final section.

2 Phase portrait

First consider equation (1). The phase portrait is well-known. There are three critical points of equation (1) at $x_{1,3} = \pm 1$, $x_2 = 0$. The origin is a center and $x_{1,3} = \pm 1$ both are saddle points. Two heteroclinic trajectories connect the two saddle points, Fig. 1.

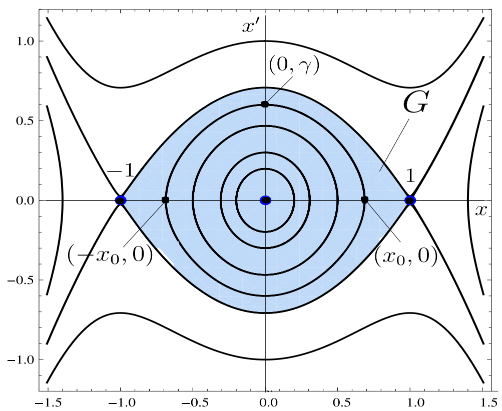

Figure 1: The phase portrait of equation (1), shaded is the region G

Any solution of (1) satisfies the “energy” relation

$$x'^2(t) = -ax^2(t) + \frac{1}{2}ax^4(t) + C, \qquad (4)$$

where C is an arbitrary constant. It is clear from the analysis of the phase portrait that solutions (trajectories) of the Neumann problem can exist only in the region (we denote it G) between the heteroclinic trajectories.

3 Monotonicity result

Denote open region bounded by the two heteroclinic trajectories connecting saddle points by G. Consider trajectories (closed curves) that fill the region G. The heteroclinic solution at infinity satisfies

$$\begin{aligned} 0 = x'^2(\infty) = -ax^2(\infty) + \tfrac{1}{2}ax^4(\infty) + C = \\ = (x^2(\infty) = 1) = -a + \tfrac{1}{2}a + C \end{aligned} \qquad (5)$$

and therefore the respective $C = \frac{1}{2}a$. Then any trajectory located in the region G satisfies the relation (4), where $|C| < \frac{1}{2}a$. Since any trajectory in the region G is closed it is convenient to consider the respective solutions in polar coordinates. Introduce polar coordinates by

$$x(t) = r(t)\sin\phi(t), \quad x'(t) = r(t)\cos\phi(t) \qquad (6)$$

Equation (1) written in polar coordinates turns to a system:

$$\begin{cases} \phi'(t) = a\sin^2\phi(t) - ar^2(t)\ \sin^4\phi(t) + \cos^2\phi(t), \\ r'(t) = \frac{1}{2}r(t)\sin 2\phi(t)\left(1 - a + a\, r^2(t)\ \sin^2\phi(t)\right). \end{cases} \qquad (7)$$

Consider any solution of equation (1) with the initial conditions $(x(t_0), x'(t_0)) \in G$. Let initial conditions be written as

$$\phi(t_0) = \phi_0, \quad r(t_0) = r_0, \quad (\phi_0, r_0) \in G, \quad r_0 > 0. \qquad (8)$$

Lemma 1 *The angular function $\phi(t)$ of any solution of (7), (8) is monotonically increasing.*

Proof: Consider the first equation of system (7) multiplied by r^2

$$\begin{aligned} r^2\phi'(t) = \\ = ar^2\sin^2\phi(t) - ar^4(t)\ \sin^4\phi(t) + r^2\cos^2\phi(t). \end{aligned} \qquad (9)$$

Returning to (x, y) coordinates

$$\begin{aligned} r^2\phi'(t) = ax^2(t) - ax^4(t) + y^2(t) = \\ = ax^2(t)(1 - x^2(t)) + y^2(t) > 0. \end{aligned} \qquad (10)$$

Since $x^2(t) < 1$ for any solution of (1) located in the domain G, the angular function $\phi(t)$ is increasing. □

Corollary 2 *Let $x(t)$ be a solution of equation (1) with the initial conditions in G. Then between any two consecutive zeros of $x(t)$ there is exactly one point of extremum.*

4 Existence and multiplicity theorem

We can prove the following result considering the boundary value problem (1), (3). Equation (1) has an integral

$$x'^2(t) = -ax^2(t) + \frac{1}{2}ax^4(t) + C, \qquad (11)$$

where C is an arbitrary constant. Solutions $x(t; x_0)$ of the Cauchy problem (1), (12)

$$x(0) = x_0, \quad x'(0) = 0, \quad 0 < x_0 < 1 \qquad (12)$$

satisfy the relation (11) where $C = ax_0^2 - \frac{1}{2}ax_0^4$. Denote a solution of the Cauchy problem (1), (12) by $x(t; x_0)$. The series of transformations

$$x'^2(t;x_0) = -ax^2(t;x_0) + \tfrac{1}{2}ax^4(t;x_0) + ax_0^2 - \tfrac{1}{2}ax_0^4, \quad \frac{dx}{dt} = \pm\sqrt{-ax^2(t;x_0) + \frac{1}{2}ax^4(t;x_0) + ax_0^2 - \frac{1}{2}ax_0^4}, \tag{13}$$

(notice that $x'(t; x_0) < 0$) and therefore

$$-\frac{dx}{\sqrt{-ax^2(t;x_0) + \frac{1}{2}ax^4(t;x_0) + ax_0^2 - \frac{1}{2}ax_0^4}} = dt \tag{14}$$

$$-\int_{x_0}^{0} \frac{dx}{\sqrt{-ax^2+\frac{1}{2}ax^4+ax_0^2-\frac{1}{2}ax_0^4}} = \int_{0}^{x_0} \frac{dx}{\sqrt{-ax^2+\frac{1}{2}ax^4+ax_0^2-\frac{1}{2}ax_0^4}} = \int_0^t dt = t \tag{15}$$

leads to

$$\int_0^{x_0} \frac{dx}{\sqrt{-ax^2+\frac{1}{2}ax^4-\frac{1}{2}ax_0^4+ax_0^2}} = \sqrt{\frac{1}{a}}\int_0^{x_0} \frac{dx}{\sqrt{(x_0^2-\frac{1}{2}x_0^4)-(x^2-\frac{1}{2}x^4)}} = \left|\xi = \frac{x}{x_0}\right| = \sqrt{\frac{1}{a}}\int_0^1 \frac{d\xi}{\sqrt{(1-\xi^2)-\frac{1}{2}x_0^2(1-\xi^4)}} = \int_0^{T_{x_0}} dt = T_{x_0}, \tag{16}$$

where T_{x_0} is the time needed to move on a phase plane from $(x_0, 0)$ to the vertical axis $x = 0$ (a quarter of a period). It follows then that T_{x_0} is increasing function of x_0.

Thus the following lemma was proved.

Lemma 3 *The function T_{x_0} monotonically increases (from $\frac{\pi}{2\sqrt{a}}$, this will be shown below) to $+\infty$ as x_0 changes from zero to 1.*

The exact number of solutions for problem (1), (3) is given by Theorem 4.

Theorem 4 *Let i be a positive integer such that*

$$\frac{i\,\pi}{\sqrt{a}} < T < \frac{(i+1)\pi}{\sqrt{a}}, \tag{17}$$

where T is the right end point of the interval in (3). The Neumann problem (1), (3) has exactly $2i$ nontrivial solutions such that $x(0) = x_0 \neq 0$, $x'(0) = 0$, $-1 < x_0 < 1$.

Proof: Consider solutions of the Cauchy problem (1), $x(0) = x_0,\ x'(0) = 0$, where $0 < x_0 < 1$. Solutions for x_0 small enough behave like solutions of the equation of variations $y'' = -a\,y$ around the trivial solution. The solution of the linearized equation is

$$y(t) = x_0 \cos\sqrt{a}t. \tag{18}$$

Due to the assumption $\frac{i\,\pi}{\sqrt{a}} < T < \frac{(i+1)\pi}{\sqrt{a}}$ solutions $y(t)$ along with solutions $x(t; x_0)$ (for small enough x_0) have exactly i extrema in the interval $(0, T)$ and $t = T$ is not an extremum point. These extrema due to Lemma 3 move monotonically to the right as x_0 increases. Solutions $x(t; x_0)$ with $0 < x_0 < 1$ and close enough to 1 have not extrema in $(0, 1]$ since the respective trajectories are close to the upper heteroclinic (and the "period" of a heteroclinic solution is infinite). Therefore there are exactly i solutions of the problem (1), (3). The additional i solutions are obtained considering solutions with $x_0 \in (-1, 0)$ due to symmetry arguments. Hence the proof. □

The phase plane analysis was used to study multiple solutions of BVP in [8], [12], [13].

The review of some aspects of multiple solutions of BVP is in [3]. The related results are in [4], [5].

5 Eigenvalue problem

Let us address the eigenvalue problem posed in [9]. Consider the Cauchy problem (1), (12):

$$x'' = -a(x-x^3),\ x(0) = x_0,\ x'(0) = 0,\ 0 < x_0 < 1.$$

Let a and T (in (3)) be given. We wish to find x_0 such that the respective solutions $x(t; x_0)$ of the above problem satisfy the boundary condition $x'(T) = 0$, i.e. $x(t; x_0)$ solve the Neumann problem (1), (3).

The following assertion provides the explicit formula for a solution of (1), (12).

Lemma 5 *The function*

$$x(t, a, x_0) = x_0\, cd\left(\sqrt{\frac{a(2-x_0^2)}{2}}t; k\right), \tag{19}$$

where $k = \sqrt{\frac{x_0^2}{2-x_0^2}}$, is a solution of the Cauchy problem (1), (12).

Proof: Consider equation (16) where the quarter of period T_{x_0} is given by formula

$$\begin{aligned} T_{x_0} &= \sqrt{\tfrac{1}{a}}\int_0^1 \frac{d\xi}{\sqrt{(1-\xi^2)-\frac{1}{2}x_0^2(1-\xi^4)}} = \\ &= \sqrt{\tfrac{1}{a}}\int_0^1 \frac{d\xi}{\sqrt{\frac{2-x_0^2}{2}(1-\xi^2)(1-\frac{x_0^2}{2-x_0^2}\xi^2)}} = \\ &= \sqrt{\tfrac{2}{a(2-x_0^2)}}\int_0^1 \frac{d\xi}{\sqrt{(1-\xi^2)(1-\frac{x_0^2}{2-x_0^2}\xi^2)}} = \\ &= \sqrt{\tfrac{2}{a(2-x_0^2)}}\int_0^1 \frac{d\xi}{\sqrt{(1-\xi^2)(1-k^2\xi^2)}} = \\ &= \sqrt{\tfrac{2}{a(2-x_0^2)}}K(k), \end{aligned} \tag{20}$$

where $0 < k^2 = \frac{x_0^2}{2-x_0^2} < 1$. Replacing $\xi = \sin\phi(t)$ we have

$$K(k) = \int_0^1 \frac{d\xi}{\sqrt{(1-\xi^2)(1-k^2\xi^2)}} = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1-k^2\sin\phi(t)}}. \tag{21}$$

Inverse function of function $F(\phi(t),k) = t$

$$F(\phi(t),k) = \int_0^{\phi(t)} \frac{d\psi}{\sqrt{1-k^2\sin^2\psi(t)}} = t \tag{22}$$

is Jacobian amplitude $\phi(t) = am(t,k)$ [14], [10], but $\sin\phi(t) = \sin am(t,k) = sn(t,k)$. For solutions of the problem (1), (3)

$$\begin{aligned} \sqrt{\tfrac{a(2-x_0^2)}{2}}\,t &= \int_{\phi(t)}^{\frac{\pi}{2}} \frac{d\psi}{\sqrt{1-k^2\sin^2\psi(t)}} = \\ &= \int_0^{\frac{\pi}{2}} \frac{d\psi}{\sqrt{1-k^2\sin^2\psi(t)}} - -\int_0^{\phi(t)} \frac{d\psi}{\sqrt{1-k^2\sin^2\psi(t)}} = \\ &= K(k) - \int_0^{\phi(t)} \frac{d\psi}{\sqrt{1-k^2\sin^2\psi(t)}}, \end{aligned} \tag{23}$$

$$\begin{aligned} F(\phi(t),k) &= \int_0^{\phi(t)} \frac{d\psi}{\sqrt{1-k^2\sin^2\psi(t)}} = \\ &= K(k) - \sqrt{\tfrac{a(2-x_0^2)}{2}}\,t, \end{aligned} \tag{24}$$

$$\phi(t) = am\left[K(k) - \sqrt{\frac{a(2-x_0^2)}{2}}\,t, k\right], \tag{25}$$

$$\begin{aligned} \sin\phi(t) &= \sin am\left[K(k) - \sqrt{\tfrac{a(2-x_0^2)}{2}}\,t, k\right] = \\ &= sn\left[K(k) - \sqrt{\tfrac{a(2-x_0^2)}{2}}\,t, k\right]. \end{aligned} \tag{26}$$

Note that the $x(t) = x_0\xi = x_0\sin\phi(t)$

$$x(t) = x_0\, sn\left[K(k) - \sqrt{\frac{a(2-x_0^2)}{2}}\,t, k\right]. \tag{27}$$

It can be shown by using the reduction formula $sn(K-u) = cd\,u$ ([10])

$$x(t) = x_0\, cd\left(\sqrt{\frac{a(2-x_0^2)}{2}}\,t, k\right). \tag{28}$$

□

Denote $f(t,a,x_0) = x'_t(t,a,x_0)$. This derivative can be computed and the following formula is valid. One obtains using [10, p.222] (and Wolfram Mathematica) that

$$\begin{aligned} f(t,a,x_0) &= x_0\, cd'_t\left(\sqrt{\tfrac{a(2-x_0^2)}{2}}\,t;k\right) = x_0(k^2-1)\times \\ &\times\sqrt{\tfrac{a(2-x_0^2)}{2}}\, nd\left(\sqrt{\tfrac{a(2-x_0^2)}{2}}t;k\right)\, sd\left(\sqrt{\tfrac{a(2-x_0^2)}{2}}t;k\right). \end{aligned} \tag{29}$$

We have arrived at the statement.

Lemma 6 *For given a and T the eigenvalue problem (1), (3) can be solved by solving the below equation with respect to x_0*

$$f(T,a,x_0) = 0. \tag{30}$$

Theorem 7 *A solution to the Neumann problem (1), (3) is given by (19) where x_0 is a solution of (30).*

Applications of the theory of Jacobian elliptic functions to various problems for ordinary differential equations can be found also in [2], [7].

6 Results for n-th order equation

Many of the facts that were established previously for the cubic case are valid also for cases of higher degree polynomials in the right side of equations. Consider equations of the type

$$\begin{aligned} &x'' = -a(x-x^3), \\ &x'' = -a[(x-x^3)+(x^5-x^7)], \\ &x'' = -a[(x-x^3)+(x^5-x^7)+(x^9-x^{11})], \\ &\ldots \\ &x'' = -a[(x-x^3)+(x^5-x^7)+(x^9-x^{11})+\ldots+ \\ &+(x^{2n-1}-x^{2n+1})]. \end{aligned} \tag{31}$$

All equations have only three critical points at $x_{1,3} = \pm 1$, $x_2 = 0$. The origin is a center and $x_{1,3} = \pm 1$ both are saddle points. Two heteroclinic trajectories connect the two saddle points. The phase portraits of these equations are similar to that for the cubic case as depicted in Fig. 1.

The analogues of Theorem 4 are valid also for the equations of type (31). We formulate the following theorem.

Theorem 8 *Let i be a positive integer such that*

$$\frac{i\,\pi}{\sqrt{a}} < T < \frac{(i+1)\pi}{\sqrt{a}}, \tag{32}$$

where T is the right end point of the interval in (3). The Neumann problem where equation is of type as in equation (31) with condition (3) has exactly $2i$ nontrivial solutions such that $x(0) = x_0 \neq 0$, $x'(0) = 0$, $-1 < x_0 < 1$.

Proof: The proof is similar to that for Theorem 4 because solutions of the Cauchy problem (1), $x(0) = x_0,\ x'(0) = 0$, where $0 < x_0 < 1$ for x_0 small enough behave like solutions of the equation of variations $y'' = -a\,y$ around the trivial solution. The solution of the linearized equation is as given in (18). Due to the assumption $\frac{i\,\pi}{\sqrt{a}} < T < \frac{(i+1)\pi}{\sqrt{a}}$ solutions $y(t)$ along with solutions $x(t; x_0)$ (for small enough x_0) have exactly i extrema in the interval $(0, T)$ and $t = T$ is not an extremum point. At the same time solutions with x_0 tending to the critical point at $x = 1$ have not points of extrema since the periods of such solutions tend to infinity as x_0 goes to 1. The extrema of solutions $x(t; x_0)$ leave the interval $(0, T)$ passing through the point T as x_0 goes to 1. Therefore there are exactly i solutions of the problem and the additional i solutions are obtained considering solutions with $x_0 \in (-1, 0)$ due to symmetry arguments. □

The value T_{x_0} defined in formula (20) seems to be monotonically increasing for all equations (31) as computations show. We believe that this can be proved similarly to the proof of Lemma 3.

7 Example

Consider equation (1) with $a = 121$:

$$x'' = -121(x - x^3). \tag{33}$$

Let the initial conditions be $x(0) = x_0,\ x'(0) = 0$, $0 < x_0 < 1$. Then the number of solutions satisfying the boundary conditions (2), $x(0) = 0$, $x'(1) = 0$, is three and, symmetrically, for initial conditions $x(0) = x_0$, $x'(0) = 0$, $-1 < x_0 < 0$ there are also three solutions to the problem (33), (2), totally six nontrivial solutions. By Theorem 4, this is the case for $T = 1$ and $i = 3$. Indeed, $\frac{3\pi}{11} < T = 1 < \frac{4\pi}{11}$) in the inequality (17).

Consider equation (30) with $a = 121$ and T=1, then

$$f(1, 121, x_0) = 11x_0(k^2 - 1)\sqrt{1 - \tfrac{1}{2}x_0^2}\times$$
$$\times\, nd\left(11\sqrt{1 - \tfrac{1}{2}x_0^2}; k\right)\ sd\left(11\sqrt{1 - \tfrac{1}{2}x_0^2}; k\right) = 0. \tag{34}$$

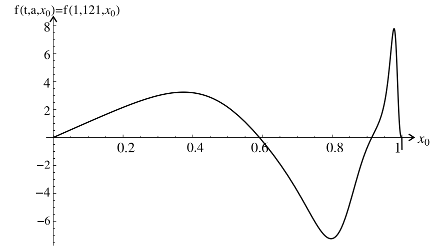

Figure 2: The graph of $f(1, 121, x_0)$

The graph of $f(1, 121, x_0)$ is depicted in Fig. 2. There are three zeros of (34) and, respectively, three initial values x_0 at $x_0 \approx 0.59$, $x_0 \approx 0.913$ and $x_0 \approx 0.998$. In Fig. 3 and Fig. 4 respectively the graphs of solutions $x(t)$ of the problem (33), (2) and its derivatives $x'(t)$ are depicted.

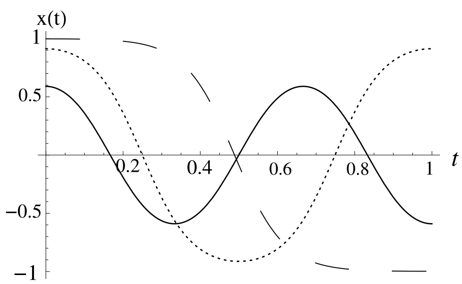

Figure 3: Graphs $x(t)$ for solutions of the problem (33), (2), $x_0 \approx 0.59$ (solid), $x_0 \approx 0.913$ (dashing tiny), $x_0 \approx 0.998$ (dashing large)

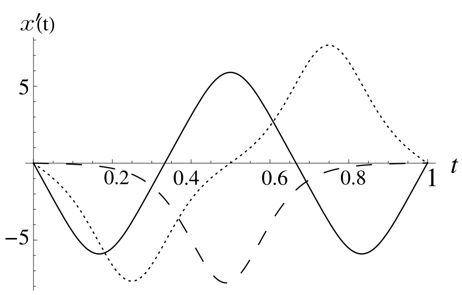

Figure 4: Graphs $x'(t)$ for solutions of the problem (33), (2), $x_0 \approx 0.59$ (solid), $x_0 \approx 0.913$ (dashing tiny), $x_0 \approx 0.998$ (dashing large)

For increasing T the number of zeros x_0 of equation (30) increases also and the number of solutions of the problem (33), (2) increases consequently. The examples for $T = 2$ and $T = 3$ are given in Fig. 5 and Fig. 6 respectively.

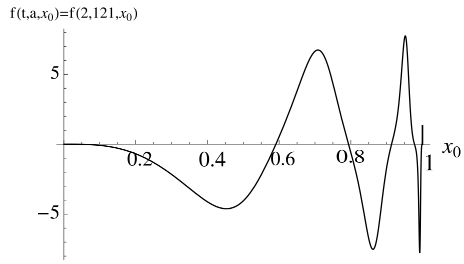

Figure 5: The graph of $f(2, 121, x_0)$, $T = 2$, the number of solutions is six

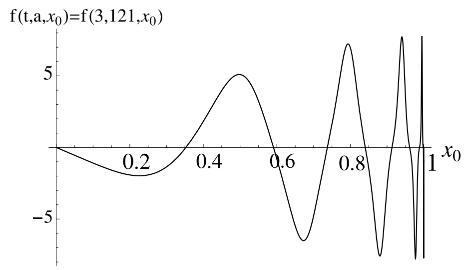

Figure 6: The graph of $f(3, 121, x_0)$, $T = 3$, there are ten solutions, because $\frac{10\,\pi}{11} < 3 < \frac{11\,\pi}{11} = \pi$

8 Conclusion

The number of solutions of the boundary value problem (1), (3) depends entirely on the coefficient a (for given T) and is known precisely (Theorem 4). The initial values x_0 for solutions of boundary value problem (1), (3) can be found by solving the equation (30) that is composed of certain Jacobian elliptic functions. Then the solutions of the Neumann problem (1), (3) are known analytically using the formula (19).

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