A way for finding the relation between of the degree and order for multistep method used to applied to solving of the Volterra integro-differential equation

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Abstract: -As is known there are some classes of numerical methods have constructed to solving initial value problem of the ODE, which fundamentally investigated by the many known scientist. Therefore the specialists tried to study many scientific and applied problems by using these methods. Here we have defined the direct way between the initial value problem for the Volterra integro-differential and Ordinary Differential Equations. By using this way have constructed the multistep methods with constant coefficients, which are applied to solving initial value problem for the Volterra integro-differential equations and have determined the necessary and sufficient conditions for its convergence. And also have proven that the constructed here methods are more accurately than the known, which is illustrated by the application of concrete method to solving of the model problem.

Key-Words: - Volterra Integro-Differential Equations, initial value problem of the ODE, multistep methods (MM), stability and degree for MM.

1 Introduction

As is known, the integro-differential equation with variable boundaries has been study for more than 100 years. The construction and application of such equations is connected with the name of Vito Volterra, who first and successfully applied integro-differential equations in solving many theoretical and practical problems (see [1, p.177-195]). Merits of Vito Volterra in the field of mathematical sciences are estimated by many serious organizations of different countries.

Numerical solution of the initial value problem for linear and nonlinear integro-differential equations has been investigated by many authors (see [2] - [7]). In these works, quadrature methods or some of its modifications were mainly used to calculation of the values of the integral. In the papers [8] - [10], for finding the numerical solution of the above mentioned problems, have proposed to use methods with new properties. These methods use a certain amount of computational work, which in the small differs from the amount of computational work using in solving similar problems of ODE. Therefore, here we consider to study of several multi-step methods with the constant coefficients and their application to solving of the following problem:

$$y'(x) = f(x, y) + \int_{x_0}^{x} K(x, s, y(s))ds,$$
$$y(x_0) = y_0, x \le s \le x \le X. \quad (1)$$

Assume that the equation (1) has a unique continuous solution defined on t he segment $[-X, X]$. To find the approximately values of the solution of problem (1) on some mesh points, let us divide the segment $[x_0, X]$ into N equal parts by the mesh points $x_i = x_0 + ih(i = 0,1,..., N)$. Here $0 < h$ - is a step size. Let us also denote by the y_i approximately values, but by the $y(x_i)$ through, the exact value of the solution of the problem (1) at the mesh points x_i $(i = 0,1,..., N)$, respectively.

We note that the problem (1) does not generalize all known initial problems for Volterra integro-differential equations, since the integral participating in the problem (1) has one variable boundary. However, the algorithm constructed here for solving the problem (1) can easily be adapted to solve linear and nonlinear integro-differential equations of Volterra type.

Many authors the solving of the problem (1) reduce it to solving following problem:

$$y'(x) = f(x,y) + v(x), y(x_0) = y_0, \quad x \le s \le x \le X, \tag{2}$$

where the function $v(x)$ is defined in the following form:

$$v(x) = \int_{x_0}^{x} K(x,s,y(s))ds \tag{3}$$

Suppose that the function $v(x)$ is known. Then, after applying the trapezoid method to solving of the problem (1), we have:

$$y_{n+1} = y_n + h(f_{n+1} + f_n)/2 + h(v_{n+1} + v_n)/2. \tag{4}$$

Obviously, for the investigation of this method, are sufficiently known the y_0 and $v(x_0)$. Thus, by using the value of v_n, one can find the value of v_{n+1}, and then the value of y_{n+1} calculate by using the formula (4).

It is easy to understand tha

$$v(x_{n+1}) = v(x_n) + \int_{x_0}^{x_{n+1}} K(x_{n+1},s,y(s))ds - $$
$$-\int_{x_0}^{x_n} K(x_n,s,y(s))ds = v(x_n) + \int_{x_0}^{x_{n+1}} K(x_n,s,y(s))ds + \tag{5}$$
$$+h\int_{x_0}^{x_n} K'_x(\zeta_n,s,y(s))ds.$$

Suppose that by the using some methods the solution of problem (1) is found, after taking into account (3), which we obtain some equality. Then by the differentiate the resulting equality, we have:

$$v'(x) = K(x,x,y(x)) + \int_{x_0}^{x} K'_x(\zeta_n,s,y(s))ds.$$

By putting $x = \zeta_n$ here, we can find the value $h\int_{x_0}^{x_n} K'_x(\zeta_n,s,y(s))ds$, and taking which into the equality of (5), we have:

$$v(x_{n+1}) = v(x_n) + \int_{x_n}^{\zeta_n} K(x_n,s,y(s))ds +$$
$$\int_{\zeta_n}^{x_{n+1}} K(x_{n+1},s,y(s))ds + hv'(\zeta_n) - hK(\zeta_n,\zeta_n,y(\zeta_n) \tag{6}$$

Here, by using the different formulas for calculating the integral and derivatives of the function $v(x)$, one can be obtained the different formulas. From the equality of (6) in one variant can be obtaining the following:

$$v_{n+1} = v_n + h(2K(x_{n+1},x_{n+1},y_{n+1}) + K(x_{n+1},x_n,y_n) + K(x_n,x_n,y_n))/4. \tag{7}$$

This scheme is obtained from the equality of (6) and has the order of accuracy of $O(h^2)$.

Note that by investigating the above scheme, it is possible to construct different methods having different order of accuracy. For example, the following methods (see for example [11]):

$$y_{n+2} = (y_n + y_{n+1})/2 + h(2K(x_{n+1},x_n,y_n) -$$
$$-K(x_n,x_n,y_n) - 4K(x_{n+1},x_{n+1},y_{n+1})$$
$$+4K(x_{n+2},x_{n+1},y_{n+1}) + 3K(x_{n+2},x_{n+2},y_{n+2}))/8,$$

$$y_{n+2} = (y_n + y_{n+1})/2 + h(K(x_{n+1},x_n,y_n) +$$
$$+3K(x_{n+1},x_{n+1},y_{n+1}) + 5K(x_{n+2},x_{n+1},y_{n+1}) +$$
$$+3K(x_{n+2},x_{n+2},y_{n+2}))/8,$$

$$y_{n+2} = y_n + h(2K(x_n,x_n,y_n) - K(x_{n+1},x_n,y_n) -$$
$$-2K(x_{n+1},x_{n+1},y_{n+1}) + 2K(x_{n+2},x_{n+1},y_{n+1})$$
$$+3K(x_{n+2},x_{n+2},y_{n+2}))/3.$$

All these methods are stable. The first two methods of them have the order of accuracy of $p = 3$, and the third method has of the order of $p = 4$, which is a modification of the Simpson method.

In the case when the function of $K(x,s,y)$ is not linear in y, then the methods of predictor-corrector can be used to applying of the methods (7) and (4) to solving of the considering problem.

If we generalize the methods (4) and (7), then we have:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h\sum_{i=0}^{k} \beta_i f_{n+i} + h\sum_{i=0}^{k} \beta_i v_{n+i}, (\gamma_i^{(j)} = 0, i > j) \tag{8}$$

$$\sum_{i=0}^{k} \hat{\alpha}_i v_{n+i} = h\sum_{i=0}^{k}\sum_{j=0}^{k} \gamma_i^{(j)} K(x_{n+j},x_{n+i},y_{n+i}). \tag{9}$$

Thus, the multi-step method with the constant coefficients was constructed for solving the problem (1). Remark, that the various modifications of the method of (8) have been investigated and applied to solving of the Volterra integral equations and also to solving of the problem of (1) (see e.g. [11] - [13]).

However, the necessary conditions imposed on the coefficients of the formula (8) have not been investigated in the above mentioned works. Therefore, here we consider the determined of the necessary conditions, which have imposed on the coefficients of formula (8), and the necessary conditions for its convergence.

2 The natural conditions imposed on the coefficients of the method (8)

The necessary conditions imposed on t he coefficients of multistep methods with constant coefficients which have applied to solving of the initial problem for an ODE are determined by Dahlquist (see [14], [15]). Similar conditions imposed on the coefficients of methods (8) and (9) can be defined in the following form if the method constructed by using the schemes (8) and (9) is convergent:

A. $\alpha_i, \beta_i^{(j)} (i, j = 0,1,...,k)$ coefficients are some real numbers, more over $\alpha_k \neq 0$.

B. The following characteristic polynomials:

$$\rho(\lambda) \equiv \sum_{i=0}^{k} \alpha_i \lambda^i, \gamma(\lambda) \equiv \sum_{i=0}^{k} \sum_{j=0}^{k} \beta_i^{(j)} \lambda^i,$$

$$\sigma(\lambda) \equiv \sum_{i=0}^{k} \beta_i \lambda^i$$

do not have common factors differ from the constant.

C. T he $\sigma(1) \neq 0$ and the p-order of accuracy of the method (8) satisfies the condition: $p \geq 1$.

The necessity of the condition of A is evident, since by using the methods (8) and (9) we calculate the values of the real functions. Therefore, let us consider to the necessity of condition of B and suppose converse. Then, receive that the polynomials $\rho(\lambda)$, $\sigma(\lambda)$ and $\gamma(\lambda)$ have a common factor, which denoted by function of $\psi(\lambda)$.

Obviously, by using the shift operator $E(Ey(x)) = y(x+h)$ and the polynomials $\rho(\lambda)$, $\sigma(\lambda)$, $\gamma(\lambda)$ the methods (8) and (9) can be written in the following form:

$$\rho(E)y_n - h\sigma(E)f_n - h\sigma(E)v_n = 0, \quad (10)$$

$$\rho(E)v_n - h\gamma(E)K(x_n, x_n, y_n) = 0. \quad (11)$$

These difference equations have the order of k. If take into account the assumptions in the equations (10) and (11), then we have:

$$\psi(E)(\rho_1(E)y_n - h\sigma_1(E)f_n - h\sigma_1(E)v_n) = 0, \quad (12)$$

$$\psi(E)(\rho_1(E)v_n - h\gamma_1(E)K(x_n, x_n, y_n)) = 0, \quad (13)$$

here

$$\sigma_1(\lambda) = \sigma(\lambda)/\psi(\lambda), \gamma_1(\lambda) = \gamma(\lambda)/\psi(\lambda),$$

$$\rho_1(\lambda) = \rho(\lambda)/\psi(\lambda).$$

By using the condition $\psi(\lambda) \neq const$, we get:

$$\rho_1(E)y_n - h\sigma_1(E)f_n - h\sigma_1(E)v_n = 0, \quad (14)$$

$$\rho_1(E)v_n - h\gamma_1(E)K(x_n, x_n, y_n) = 0. \quad (15)$$

If we take into account that the difference equations (12), (13) and equations (14), (15) are equivalent, we get that the solution of each difference equation (14) and (15) are solutions of the corresponding equations (12) and (13), and on the contrary. However, the order of each of the difference equations (14) and (15) is less than k. Consequently, they have a unique solution.

Then, from the equivalence of equations (14), (15) and equations (12), (13), follows that the equations (12) and (13) have the unique solutions in the case when the known initial values are less than k. But this contradicts to the theory of difference equations. Consequently, the assumption that the polynomials have a common factor does not hold.

Now we investigate the condition C.

If we pass to the limit for $h \to 0$ in equation (12), then received that:

$$\rho(1) = 0. \quad (16)$$

The condition (16) is usually called the necessary condition for convergence.

If condition (16) is taken into account in equations (12) and (13), then we have:

$$(E-1)\rho_1(E)y_n = h\sigma(E)f_n + h\sigma(E)v_n, \quad (17)$$

$$(E-1)\rho_1(E)v_n = h\gamma(E)K(x_n, x_n, y_n). \quad (18)$$

It can be proved that for sufficiently small h equation (18) can be rewritten in the following form:

$$\rho_1(E)(v_{i+1} - v_i) = h\sigma(E)K(x, x_i, y_i), \quad (19)$$

here x is a fixed point. If the interpolation theory applied to the function $K(x, x_n, y_n)$, then we can obtain the equation (18).

Assigning the values $i = 0,1,...,n-1$ to the variable i and sum up the obtained equalities in the equation (19) we receive:

$$\rho_1(E)(v_n - v_0) = \sigma(E)\sum_{i=0}^{n-1} K(x, x_i, y_i).$$

Here, passing to the limit for $h \to 0$, we have:

$$\rho_1(1)(v(x) - v_0) = \sigma(1)\int_{x_0}^{x} K(x, s, y(s))ds, \quad (20)$$

here $x = x_0 + nh$ is a fixed point.

We note that if $\sigma(1) = 0$, then from the equalities (20) we obtain that $v(x) = v_0 = const$, which does not hold. Therefore, $\sigma(1) \neq 0$.

Given that $v_0 = 0$, we get:

$$\rho_1(1) = \sigma(1). \quad (21)$$

Now, we apply the above using scheme to equation (17). Then we get that:

$$\rho_1(1)(y(x)-y(x_0))=\sigma(1)(\int_{x_0}^{x} f(s,y(s))ds+\int_{x_0}^{x} v(s)ds).$$

In this equality by taking into account equality of (21) obtain the following equation:

$$y(x)=y(x_0)+\int_{x_0}^{x} f(s,y(s))ds+\int_{x_0}^{x} v(s)ds.$$

We note that it follows from conditions (16) and (21) that, $p\ge 1$ (here the p designates the order of accuracy for the method (8)).

Thus we have proved that the condition C also holds, if the method constructed according to (8) and (9) is convergent. Now we consider the definition of the coefficients of the methods (8) and (9).

3 Subsection

In order to determine the coefficients of the methods (8) and (9), we consider the special case and put $K(x,s,y)\equiv\varphi(s,y)$. Then from (3) we have:

$$v'(x)=\varphi(x,y(x)), v(x_0)=0 .$$

Taking this into account, in the problem (2) we have:

$$y'(x)=f(x,y)+v(x), y(x_0)=y_0, \quad (22)$$

$$v'(x)=\varphi(x,y), v(x_0)=0. \quad (23)$$

Thus, the solving of the problem (1) was reduced to solving the initial value problem for the system of ordinary differential equations of the first order, to solving of which proposed, here following finite difference method:

$$\sum_{i=0}^{k}\alpha_i z_{n+i}=h\sum_{i=0}^{k}\beta_i z'_{n+i}. \quad (24)$$

Then we have:

$$\sum_{i=0}^{k}\alpha_i y_{n+i}=h\sum_{i=0}^{k}\beta_i f_{n+i}+h\sum_{i=0}^{k}\beta_i v_{n+i}, \quad (25)$$

$$\sum_{i=0}^{k}\alpha_i v_{n+i}=h\sum_{i=0}^{k}\beta_i \varphi_{n+i}. \quad (26)$$

Remark that the methods (25) and (26) coincide with the methods (8) and (9) if we take into account the following:

$$\sum_{j=0}^{k}\gamma_i^{(j)}=\beta_i, (i=0,1,2,...,k). \quad (27)$$

Thus, here proved that by using the same method (in our case, the method (24)), one can solve the problem (2) and calculate the values of the function. Thus, we prove that by using the same method one can solve initial value problems for integro-differential and ordinary differential equations. As such method, here have used the finite difference method (24), which is often called the multi-step method with the constant coefficients. It is easy to see that the accuracy of our method coincides with the accuracy of the finite-difference method determined by the formula (24).

It should be noted that a m ore accurate scheme can be used to calculate of the values of the function of $v(x)$ than the one proposed above. As follows from the above mentioned, the accuracy of the method applied to solving of problem (1) depends from the accuracy of the method, which have applied to solving of the problem (2). Consequently, the maximum accuracy of our method in this case will be determined by the laws of Dahlquist (see, for example, [14, 15]).

As is known, if the method has the maximum accuracy ($p=2k$, here p is the order of accuracy of the method and k is the order of the method (24)), then it is unique. However, in our case this is not so. This is due to the fact that the solution of system (27) is not unique. Note that in Dahlquist's work have used only the values of the coefficients $\beta_i (i=0,1,...,k)$, but here we use the $\beta_i (i=0,1,...,k)$ coefficients to determine the value of the coefficients $\gamma_i^{(j)}(i,j=0,1,...,k)$. Therefore, the constructed here methods with the maximum accuracy are not unique.

As is known some authors for solving the problem (1) are replace that to solving the initial value problem for the Volterra i ntegro-differential equations of the second order and to solving which have used the multistep second derivative methods by taking into account that these methods are more exactly than the methods of type (8). Remark, that the using these methods are difficult, which is related with the calculation of the following function:

$$y''(x)=f'_x(x,y)+f'_y(x,y)y'+$$

$$+K(x,x,y(x))+\int_{x_0}^{x} K(x,x,y(x)).$$

Therefore, here proposed the methods (8) and (9) to solving the problem (1). Now let us to determine the system of algebraic equations for the finding the values of coefficients $\alpha_i,\beta_i,\hat{\alpha}_i,\gamma_i^{(j)}(i,j=0,1,...,k)$.

For this aim one can be used the following linear system of algebraic equations fundamentally investigated by many authors (see for example [16-20]:

$$\sum_{i=0}^{k}\alpha_i = 0; \sum_{i=0}^{k} i\alpha_i = \sum_{i=0}^{k}\beta_i;$$
$$\sum_{i=0}^{k} i^l\alpha_i = \sum_{i=0}^{k} l i^{l-1}\beta_i (l = 2,3,..., p). \quad (28)$$

In usually the specialists are using the conception of stability and degree for the numerical method. Here, we use the same conception. It is to say that the integer p is called the degree for the method (24) or (25) if the following is holds:

$$\sum_{i=0}^{k}(\alpha_i y(x+ih)) - h\beta_i f(x_{n+i}, y(x_{n+i})) - h\beta_i v(x+ih) = $$
$$= O(h^{p+1}), h \to 0.$$

But the method (24) or (25) is stable if the roots of the polynomial $\rho(\lambda) = \alpha_k\lambda^k + \alpha_{k-1}\lambda^{k-1} + ... + \alpha_1\lambda + \alpha_0$ are in the unit circle on the boundary which there are no multiply roots. It is not difficult to understand that the method constructed by using the methods (8) and (9) is convergent if the methods (8) and (9) are stable.

It is follows remark that the order of the accuracy for the methods of type (8) depends from the order of accuracy of the methods of type (9). Therefore let us consider determination of the values of the coefficients of the method (9). For this aim let us consider computation of the following value:

$$v_{n+k} = \int_{x_0}^{x_{n+k}} K(x_{n+k}, s, y(s))ds. \quad (29)$$

As is known for the application of the method (8) in usually are assume that the value y_j and $v_j (j = 0,1,..., k-1)$ are known, therefor we to consider the calculation of the value v_{n+k}.It is obvious that the exactness of the value v_{n+k} depends from the degree of the method which has applied to calculation v_{n+k}.Thus, receive that the values of the degree for the method applied to calculation of the values of function $v(x)$ basically depends from the order of accuracy of the methods which is applied in calculation of the integral (29). Taking into account this phenomena, by the help Lagrange`s formula the function of $v(x)$ we rewrite in the following form:

$$v(x) = \sum_{j=0}^{k}\gamma^{(j)}\int_{x_0}^{x} K(x_{n+j}, s, y(s))ds +$$
$$+ R_n, x \in [x_n, x_{n+k}]. \quad (30)$$

Here R_n-remainder term, but $\gamma^{(j)}_{(x)}$-the Lagrange interpolation basis function (see [21,p.121]). It is not difficult to understand, that by application of any quadrature formula in calculation of the integral participate in (30), we receive the formula of (9).For the determine the values of the coefficients for the methods (8) and (9) in first let us determine the values of the coefficients $\alpha_i, \beta_i (i = 0,1,..., k)$ by using the system of (28) and after that by solving the system of (27) we can find the values of the coefficients $\gamma_i^{(j)} (i = 0,1,..., k)$.

Let us remark that in the construction of the method (9), the basic properties of the Volterra integral equation is not taking into account. If we use those properties in the formula (9), receive the following:

$$\sum_{i=0}^{k}\hat{\alpha}_i y_{n+i} = h\sum_{i=0}^{k}\sum_{j=i}^{k}\gamma_i^{(j)} K(x_{n+j}, x_{n+i}, y_{n+i}).$$

Note that for using the method of (9), we must suppose that the function $K(x, s, y)$ is defined on the ε-extended of the domain D, which can be written as $D_\varepsilon = \{x_0 \le s \le x + \varepsilon \le X + \varepsilon, |y| \le a\}$.

By this way we have showed that, how one can be determine the values of the coefficients in the methods (8) and (9). And also have shown how the value $v(x_m)$ can be calculated by using the values $v(x_{m-l}) (l = 1,2,..., k)$.

Now let us consider to construction the concrete methods and its application to solving model problem:

Example 1. L et us consider to solving of the following problem:

$$y' = 2\lambda\exp(\lambda x) - \lambda - \lambda^2\int_0^x y(s)ds, \quad y(0) = 1,$$

Exact solution for which is equal to $y(x) = \exp(\lambda x)$.

In assumption the known of the values y_0, y_1, v_0, v_1 to solving the above mentioned problem have applied the following algorithm:

Step 1. Calculate

$$y_{i+2} = y_i + 2hf_{i+1} + 2hv_{i+1};$$
$$v_{i+2} = v_{i+1} + h(5K(x_{i+2}, x_{i+2}, y_{i+2}) + 4K(x_{i+2}, x_{i+1}, y_{i+1}) +$$
$$+ 4K(x_{i+1}, x_{i+1}, y_{i+1}) + K(x_{i+1}, x_i, y_i) - 2K(x_i, x_i, y_i))/12;$$

Step 2. Calculate

$$y_{i+2} = y_{i+1} + h(5f_{i+2} + 8f_{i+1} - f_i)/12;$$

Step 3. Calculate

$$v_{i+2} = v_i + h(K(x_{i+2}, x_{i+2}, y_{i+2}) + 2K(x_{i+2}, x_{i+1}, y_{i+1}) +$$
$$+ 2K(x_{i+1}, x_{i+1}, y_{i+1}) + 2K(x_{i+1}, x_{i+1}, y_{i+1}) + 3K(x_{i+2}, x_i, y_i) -$$
$$- K(x_{i+1}, x_i, y_i) - K(x_i, x_i, y_i) / 3;$$

Remark that this algorithm has the order of accuracy $p = 3$.

Table 1.

Variable x	Error for the step size		
	$h = 0{,}1$	$h = 0{,}05$	$h = 0{,}01$
0.2	5.1E-6	9.09E-7	8.8E-9
0.4	1.6E-5	2.34E-6	1.9E-8
0.6	3.04E-5	3.9E-6	3.2E-8
0.8	4.6E-5	5.86E-6	4.6E-8
1.0	6.3E-5	7.9E-6	6.3E-8

4 Conclusion

Here we investigated the necessary conditions imposed on the coefficients of the methods (8) and (9), and also determined the necessary conditions for the convergence of the proposed method. Here prove the possibility application of the same method to solving the problem (2) and to calculate the values of the function of $v(x)$. By the same token receive some any approximately equivalence between the initial value problem for the ODEs and for the Integro-Differential Equations of Volterra type. For the application of the multistep methods to solving Volterra Integro-Differential Equations, here have define the dependence of its construction from the region of the definition of the function $K(x,s,y)$.This approach allows us to apply a wide arsenal of methods for solving of the integro-differential equations which have used in solve of the ODE . The results of the example show the advantages of this investigation.

5 Acknowledgments

The authors express their thanks to the academicians Telman Aliyev, Abel Maharramov and Ali Abbasov for their suggestion to investigate the computational aspects of our problem. This work was supported by the Science Development Foundation under the President of the Republic of Azerbaijan – Grant № EIF-KETPL-2-20151(25)-56/07/1.

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