

The Bounded Solutions of a Fourth Order Model Equation

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Abstract: The objective of this paper is to construct bounded solutions of a model equation, which governs two-dimensional steady capillary-gravity waves of an ideal fluid flow with Bond number near $1/3$ and Froude number close to one.

Key-Words: Lyapunov's Center Theorem, Schauder fixed point theorem, Bounded Solution.

1 Introduction

Progressive capillary-gravity waves on an irrotational incompressible inviscid fluid of constant density with surface tension in a two-dimensional channel of finite depth have been studied since nineteenth century. Assume that a coordinate system moving with the wave at a speed is chosen so that in reference to it the wave motion is steady. Let H be the depth of water, g the acceleration of gravity, T the coefficient of surface tension, and ρ the constant density of the fluid. Then there are two nondimensional numbers which are important and defined as $F = c^2/(gH)$, the Froude number, and $\tau = T/(\rho g H^2)$, the Bond number.

When F is not close to 1, the linear theory of water waves is applicable. But when F approaches to 1, the solutions of linearized equations of water waves will grow to infinity (Peters and Stoker [12]). Therefore for F close to 1 nonlinear effect must be taken into account and thus $F = 1$ is a critical value. The first study of a solitary wave on water with surface tension is due to Korteweg and DeVries [10] after whom the K-dV equation with surface tension effect is named. A stationary K-dV equation with Bond number not near $1/3$ can also be formally derived by different approaches. However, if τ is close to 1, the formal derivation of the stationary K-dV equation fails. Thus $\tau = 1/3$ is also a critical value.

It becomes apparent that the problems for F near 1 and for τ near $1/3$ depend on each other and are difficult because they are not only strongly nonlinear, but also very delicate. Since the full nonlinear equations for the water waves are too complicated to study, it is of interest to study model equations. In Hunter and Vanden Broeck's work [8], a fifth order ordinary differential equation

considered as a perturbed stationary K-dV equation was obtained with the assumption that $F = 1 + F_2 \epsilon^2$, $\tau = 1/3 + \tau_1 \epsilon$ and ϵ is a small positive parameter. By integrating the fifth order ordinary differential equation once and set the constant of integration to be zero, then the model equation becomes

$$2F_2 \eta - \frac{3}{2} \eta^2 + \tau_1 \eta_{xx} - \frac{1}{45} \eta_{xxxx} = 0 \quad (1)$$

Equation (1) has been studied extensively by many authors [1-8] and several types of solutions have been found, such as periodic solutions [1, 5, 6, 7], solitary wave solutions [2-8], generalized solitary wave solutions (solitary waves with oscillatory tails at infinity) in the parameter region $\tau_1 < 0$ and $F_2 > 0$ [1,8], etc.

2 Problem Formulation

We add a bump $y = b(x)$ at the bottom of the two-dimensional ideal fluid flow and then derive a forced model equation

$$2F_2 \eta - \frac{3}{2} \eta^2 + \tau_1 \eta_{xx} - \frac{1}{45} \eta_{xxxx} = b \quad (2)$$

We follow Zufiria [18] to construct a Hamiltonian associated to (2).

When $\mathbf{b} = 0$, we rewrite (2) as

$$\eta_{xxxx} - 45\tau_1 \eta_{xx} - 90F_2 \eta + \frac{135}{2} \eta^2 = 0. \quad (3)$$

We multiply $-\eta_x$ to (3) and integrate the resulting equation, then equation (3) has first integral as

$$H = 45F_2 \eta^2 + \frac{1}{2} \eta_{xx}^2 - \eta_{xxx} \eta_x + \frac{45}{2} \tau_1 \eta_x^2 - \frac{45}{2} \eta^3, \quad (4)$$

unit eigenvectors of eigenvalues $-w_1i, w_1i, -w_2i$, and w_2i respectively. We see that \hat{a}_{w_1} must be in the form $(\hat{a}_{w_1}, \hat{a}_{w_2}, 0, 0)$ since the periods tend to $2\pi/w_1$ when we approach the fixed point zero along the family. Therefore, vector $a_{w_1} \in R^4$ lies in the two dimensional eigenspace $S_{w_1} = \{\xi_1, \xi_2\}$ where ξ_2 is the conjugate of ξ_1 . Thus, on interval $(-\infty, x_1)$, we have periodic solutions (13) in short-period family with initial values $z_{w_1}(x_1; \epsilon)$ having the properties that $z_{w_1}(x_1; \epsilon)/\epsilon \rightarrow e^{Ax_1} a_{w_2}$ as $\epsilon \rightarrow 0$ and $a_{w_2} \in S_{w_2} \cap R^4$. On interval (x_2, ∞) , we also obtain periodic solutions (13) in short-period family with initial values $z_{w_2}(x_2; \epsilon)$ having the properties that $z_{w_2}(x_2; \epsilon)/\epsilon \rightarrow e^{Ax_2} a_{w_1}$ as $\epsilon \rightarrow 0$ and $a_{w_1} \in S_{w_1} \cap R^4$.

By the same arguments as above on periodic solutions of short-period family, we have periodic solutions $z_{w_2}(x; \epsilon) = \epsilon e^{Ax} a_{w_2} + O(\epsilon^2)$ in long-period family on interval $(-\infty, x_1)$ with initial values $z_{w_2}(x_1; \epsilon)$ having the properties that $z_{w_2}(x_1; \epsilon)/\epsilon \rightarrow e^{Ax_1} a_{w_2}$ as $\epsilon \rightarrow 0$ and $a_{w_2} \in S_{w_2} \cap R^4$ where $S_{w_2} = \{\xi_3, \xi_4\}$ and ξ_4 is the conjugate of ξ_3 . On interval (x_2, ∞) , we also have periodic solutions $z_{w_1}(x; \epsilon)$ in long-period family with initial values $z_{w_1}(x_2; \epsilon)$ having the properties that $z_{w_1}(x_2; \epsilon)/\epsilon \rightarrow e^{Ax_2} a_{w_1}$ as $\epsilon \rightarrow 0$ and $a_{w_1} \in S_{w_1} \cap R^4$.

As in [17], the bounded solution $\eta_c(x)$ on $[x_1, x_2]$ is obtained by Schauder fixed point theorem and it is required that the initial values at $x = x_1$ and the bump b both must be sufficiently small such that MY and M_b satisfy (98) and (100) in [17]. These requirements could be met by choosing a small bump b and sufficiently small ϵ , say ϵ^* . Now, we write the first component of $z_{w_1}(x; \epsilon^*; x_1)$ or $z_{w_2}(x; \epsilon^*; x_1)$ as $\eta_L(x; \epsilon_L; x_1)$ to be the solution of (9) on $(-\infty, x_1)$. In [16], we showed that the zero solution is stable for Case 3, thus bounded $\eta_R(x; \epsilon_R; x_2)$ on interval (x_2, ∞) can be obtained if $(\eta_c(x_2), \eta_c'(x_2), \eta_c''(x_2), \eta_c'''(x_2))$ is small and this could be done as discussed in [17]. As in [17], we combine $\eta_L(x; \epsilon_L; x_1)$, $\eta_c(x; \epsilon^*; x_1)$, and $\eta_R(x; \epsilon_R; x_2)$ to obtain a solution of equation (9) in Case 3 with w_1/w_2 irrational, which is periodic on interval $(-\infty, x_1)$ and bounded on $[x_1, \infty)$.

3.3 Case 4

In this subsection, we would like to discuss the solutions of equation (9) for the parameters τ_1, F_2 corresponding to **Case 4**. As in previous subsections, we shall show the existence of solutions of equation (9) on three different intervals $(-\infty, x_1), [x_1, x_2]$, and (x_2, ∞) . Then combine these solutions to become a C^4 solution of equation (9).

First we show there exist periodic solutions of equation (9) with $\mathbf{b}(x) = 0$ by a theorem from Meyer [13]. In [13], Meyer discussed the bifurcation occurring in restricted 3-body problem. The Hamiltonian he concerned depends on a parameter $\bar{\mu}$ and has the properties that the eigenvalues of the associated linearized operator are (I) $\pm iw_2, \pm iw_2$ if $\bar{\mu} > 0$ where $w_1, w_2 \in \mathbf{R}, w_1 \neq w_2$, and $w_1 w_2 \neq 0$. (e.g., In **Case 3**) (II) $\pm iw, \pm iw$ if $\bar{\mu} = 0$ where $w \in \mathbf{R}$ and $w \neq 0$, with two two-dimensional Jordan blocks. (e.g., In **Case 4**) (III) $\pm a \pm ib$ if $\bar{\mu} < 0$ where $a, b \in \mathbf{R}$ and $ab \neq 0$. (e.g., In **Case 5**). Meyer transforms the perturbed Hamiltonian to Sokol'skii's normal form

$$H(\bar{\mu}) = w\Gamma_1 + \delta\Gamma_2 + \bar{\mu}(a\Gamma_1 + b\Gamma_3) + \frac{1}{2}(c\Gamma_1^2 + 2d\Gamma_1\Gamma_3 + e\Gamma_3^2) + \dots, \quad (14)$$

where

$$\Gamma_1 = i(z_2 z_4 - z_1 z_3), \quad \Gamma_2 = z_1 z_2, \quad \Gamma_3 = z_3 z_4. \quad (15)$$

With higher-order terms in H are functions of Γ_1, Γ_2 and Γ_3 only. Then he proved the following results.

Theorem 2 Consider a Hamiltonian of the form (14) with $w \neq 0, \delta = \pm 1, b \neq 0, e \neq 0$. Assume $\delta e > 0$. There exist two Lyapunov families of periodic orbits emanating from the origin when $\delta b \bar{\mu}$ is small and positive. These families persist when $\bar{\mu} = 0$ as two distinct families of periodic orbits emanating from the origin. As $\delta b \bar{\mu}$ becomes negative, the two families detach from the origin as a single family and recede from the origin.

Theorem 2 can be used to show a Hamiltonian Holf biurcation near **Case 4** since the Hamiltonian system (5) has the same properties as Meyer [13] discussed if we assume

$$F_2 = -\frac{45}{8} \tau_1^2 + \bar{\mu},$$

with $\tau_1 < 0$ and the parameter $\bar{\mu} \approx 0$. After transforming (5) to Sokol'skii's normal form (14), we have

$$w = \left(-\frac{45}{2}\tau_1\right)^{\frac{1}{2}}, \quad \delta = -1, \quad b = \frac{90}{w^2\delta}, \quad e = -\frac{38475}{32w^8}.$$

Since $\delta e > 0$, we have the same results as **Theorem 2** described. When $\bar{\mu} > 0$ is small and we are in the region **Case 3** but near **Case 4**, there are two Lyapunov periodic families emanating from the origin. These families persist when $\bar{\mu} = 0$ as two distinct families of periodic orbits emanating from the origin. As $\bar{\mu} < 0$ is small and we are in the region **Case 5** but near **Case 4**, the two families detach from the origin as a single family and recede from the origin.

The periodic solutions derived by **Theorem 2** with $\bar{\mu} = 0$ which corresponds to **Case 4** can be used as $\eta_L(x)$ in interval $(-\infty, x_1]$. The existence of $\eta_C(x)$ on $[x_1, x_2]$ can also be proved by the same arguments in [17]. On interval (x_2, ∞) , since the zero solution of equation (9) with $\mathbf{b}(x) = 0$ is almost stable, bounded $\eta_R(x)$ for large x is obtained provided that $(\eta_R(x_2) = \eta_C(x_2), \eta'_R(x_2) = \eta'_C(x_2), \eta''_R(x_2) = \eta''_C(x_2), \eta'''_R(x_2) = \eta'''_C(x_2))$ is small and this could be done as we discussed in section [17].

As in [17], we match $\eta_L(x), \eta_C(x)$, and $\eta_R(x)$ at $x = x_1$ and $x = x_2$ to obtain a solution of equation (9) in **Case 4** which is periodic on interval $(-\infty, x_1)$ and bounded on $[x_1, \infty)$ for large x .

3.4 Case 5

In this subsection, we shall construct bounded solutions of the model equation (9) for **Case 5**.

Our idea is to investigate the solutions of equation (9) on three different intervals $(-\infty, -T_1), [-T_1, T_2]$, and (T_2, ∞) , where T_1 and T_2 are positive constants and will be specified later. On intervals $(-\infty, -T_1)$ and (T_2, ∞) , we try to show that equation (9) with initial values at $x = -T_1$ on $(-\infty, -T_1)$ and initial values at $x = T_2$ on (T_2, ∞) has bounded solutions $\eta_L(x)$ and $\eta_R(x)$, respectively, which decay to zero exponentially at negative and positive infinity by using a theorem from [6]. On $[-T_1, T_2]$, we shall use Schauder fixed point theorem to prove there exist bounded solutions $\eta_C(x)$ of equation (9) subject to initial values $(\eta_L(-T_1), -\eta'_L(-T_1), \eta''_L(-T_1), -\eta'''_L(-T_1))$ at $x = -T_1$. Then we

combine $\eta_C(x)$, $\eta_L(x)$, and $\eta_R(x)$ to obtain a solution of equation (9).

3.4.1 Solutions on $(-\infty, -T_1)$ and (T_2, ∞)

On interval (T_2, ∞) , we rewrite (9) as a system of first order differential equations,

$$\frac{dz}{dx} = Az + g(z), \quad (16)$$

where $z(x) = (\eta(x), \eta'(x), \eta''(x), \eta'''(x))'$,

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 90F_2 & 0 & 45\tau_1 & 0 \end{pmatrix}$$

$$\text{and } g(x, z) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\frac{135}{2}\eta^2 - 45b(x) \end{pmatrix}. \quad (17)$$

In the following, we shall use a theorem from [6] to prove that (16) with some restriction on the initial values at $x = T_2$ has bounded solutions. The theorem is stated as follows :

We consider the asymptotic behavior of the solutions of equation

$$\frac{dz}{dx} = Az + f(x, z), \quad (18)$$

where A is a constant matrix and f is a continuous vector function defined for $x \geq x_0, |z| < c$. Then the underlying vector space X can be uniquely represented as the direct sum of three subspaces X_{-1}, X_0, X_1 invariant under A on which all characteristic roots of A have real parts respectively less than, equal to, greater than μ . We shall denote by P_i the corresponding projection of X onto $X_i (i = -1, 0, 1)$.

Theorem 3 *Suppose that at least one characteristic root of A has real part $\mu < 0$ and*

$$f(x, z) = o(|z|) \text{ for } x \rightarrow \infty, |z| \rightarrow 0 \quad (19)$$

holds.

Then there exist positive constants k, K depending only on A and positive constants T, ρ depending also on f such that if $x^* \geq T$ and if $\xi_{-1} \in X_{-1}, \xi_0 \in X_0$ satisfy

$$|\xi_{-1}| \leq k |\xi_0|, \quad 0 < \xi_0 < \frac{\rho}{2K}, \quad (20)$$

then the equation (18) has at least one solution $z(x)$ for $x \geq x^*$ satisfying

$$P_{-1}z(x^*) = \xi_{-1}, \quad P_0z(x^*) = \xi_0, \quad (21)$$

$|z(x)| \leq \rho$ for $x \geq x^*$ and

$$\mu = \lim_{x \rightarrow \infty} x^{-1} \log |z(x)|. \quad (22)$$

For **Case 5**, there exists at least one eigenvalue with negative real part and $g(x, z)$ in (17) satisfies (19) since $\mathbf{b}(x)$ is compact on $[x_1, x_2]$. Hence, by Theorem 1, there are bounded solutions $z_R(x)$ of equation (16) subject to the initial values $z_R(T_2)$ that satisfy (20) and (21) with $T_2 \geq T$. Then we have $\eta_R(x)$, the first component of $z_R(x)$, as the solution of (9) subject to the initial values $z_R(T_2) = (\eta_R(T_2), \eta'_R(T_2), \eta''_R(T_2), \eta'''_R(T_2))'$ on interval (T_2, ∞) .

For interval $(-\infty, -T_1)$, we let $x = -\hat{x}$ and put it in (9), then equation (9) does not change except that the independent variable is replaced by \hat{x} . Thus, by **Theorem 3** again, there exist bounded solutions $z_L(\hat{x})$ of equation (16) subject to the initial value $z_L(T_1) = (\eta_L(T_1), \eta'_L(T_1), \eta''_L(T_1), \eta'''_L(T_1))$ that satisfy (20) and (21) with $T_1 \geq T$. Hence, by substituting $\hat{x} = -x$, we obtain $\eta_L(x)$, the first component of $z_L(x)$, to be the solution of (9) subject to the initial values $z_L(-T_1) = (\eta_L(-T_1), -\eta'_L(-T_1), \eta''_L(-T_1), -\eta'''_L(-T_1))'$ on interval $(-\infty, -T_1)$.

Next, we shall prove there is a bounded solution $\eta_C(x)$ of (9) subject to initial value $(\eta_L(-T_1), -\eta'_L(-T_1), \eta''_L(-T_1), -\eta'''_L(-T_1))$ at $x = -T_1$ on interval $[-T_1, T_2]$ and the end point value, $(\eta_C(T_2), \eta'_C(T_2), \eta''_C(T_2), \eta'''_C(T_2))$, which also satisfies (20) and (21).

3.4.2 Solutions on $[-T_1, T_2]$

From (9) and posing initial values at $x = -T_1$, we have:

$$\begin{aligned} \eta_{xxx} - 45\tau_1\eta_{xx} - 90F_2\eta &= -45(\mathbf{b}(x) + \frac{3}{2}\eta^2) \equiv f(\eta), \quad x \geq -T_1, \\ \eta(-T_1) &= P, \quad \eta_x(-T_1) = Q, \\ \eta_{xx}(-T_1) &= R, \quad \eta_{xxx}(-T_1) = S, \end{aligned} \quad (23)$$

where

$$P = \eta_L(-T_1), Q = -\eta'_L(-T_1), R = \eta''_L(-T_1), S = -\eta'''_L(-T_1).$$

To analyze the solutions of (23), we transform the ordinary differential equation (23) to an integral equation. First we solve the homogeneous equation of (23) :

$$\begin{aligned} Y_{xxxx} - 45\tau_1Y_{xx} - 90F_2Y &= 0, \quad x \geq -T_1 \\ Y(-T_1) &= P, \quad Y_x(-T_1) = Q, \quad Y_{xx}(-T_1) = R, \\ Y_{xxx}(-T_1) &= S \end{aligned} \quad (24)$$

Next, we use $Y(x)$ in (24) and let $\eta = S + Y$ to convert equation (23) as follows :

$$\begin{aligned} S_{xxxx} - 45\tau_1S_{xx} - 90F_2S &= f, \quad x \geq -T_1 \\ S(-T_1) &= 0, \quad S_x(-T_1) = 0, \quad S_{xx}(-T_1) = 0, \\ S_{xxx}(-T_1) &= 0. \end{aligned} \quad (25)$$

Let the causal Green's function of equation (25) be $G(x, t)$, then we have

$$S(x) = \int_{-T_1}^{\infty} G(x, t) f(\eta(t)) dt \quad (26)$$

Thus we transform the differential equation (23) to the integral equation :

$$\begin{aligned} \eta(x) &= Y(x) + \int_{-T_1}^x G(x-t) \\ &\quad \left\{ -45(\mathbf{b}(t) + \frac{3}{2}\eta^2(t)) \right\} dt = Q(\eta)(x) \end{aligned} \quad (27)$$

To prove the existence of a bounded solution of equation (9) initiating at $x = -T_1$ on the interval $[-T_1, T_2]$, we need to show that the operator defined by the right-hand side of (27) has a fixed point. In

other words, we try to find a function $\hat{\eta}$ such that $Q(\hat{\eta})(x) = \hat{\eta}(x)$ for all $x \in [-T_1, T_2]$. We take the domain of Q to be

$$K = \{\eta \in C([-T_1, T_2]; \mathbf{R}) \mid |\eta(x)| \leq M \text{ for } x \in [-T_1, T_2]\}, \quad (28)$$

where M is some positive real number and should be chosen in such a way that Q maps K into itself.

It is clear that the function $x \mapsto Q(\eta)(x)$ is continuous. In order to prove that Q maps K into itself it remains only to analyze the size of $|Q(\eta)(x)|$. If $\eta \in K$, then we have for all $x \in [-T_1, T_2]$

$$|Q(\eta)(x)| \leq M_Y + M_G M_x (M_b + \frac{3}{2} M^2) \quad (29)$$

where

$$M_Y = \max_{x \in [-T_1, T_2]} |Y(x)|, \quad M_G = \max_{x, t \in [-T_1, T_2]} |G(x, t)|$$

and

$$M_x = x_2 - x_1, \quad M_b = \sup_{x \in [-T_1, T_2]} |\mathbf{b}(x)|.$$

If we assume that the right-hand side of (29) $\leq M$, then we have

$$\frac{3}{2} M_G (M - M^+) (M - M^-) \leq 0$$

where

$$M^\pm = \frac{1 \pm (1 - 6M_G M_x (M_G M_b M_x + M_Y))^{\frac{1}{2}}}{3M_G M_x} \quad (30)$$

and

$$(1 - 6M_G M_x (M_G M_b M_x + M_Y)) \geq 0. \quad (31)$$

The inequality (31) can be satisfied if we choose bump \mathbf{b} and the initial values in (23) such that both M_b and M_Y are sufficiently small. Hence, if we take $M \in [M^-, M^+]$ and inequality (31) is also satisfied, then $|Q(\eta)(x)| \leq M$ for all $-T_1 \leq x \leq T_2$, and Q maps K into itself.

The set K is a bounded, closed, and convex subset of the Banach space $C([-T_1, T_2])$. To apply Schauder's theorem it suffices, therefore, to show that Q is a compact map of K into itself. By the Arzelà-Ascoli Theorem and by what we have already proved, this

amounts to showing that the set $\{Q(\eta) \mid \eta \in K\}$ is equicontinuous. The following simple estimate accomplishes the task. Let $-T_1 \leq \xi \leq x$, then

$$\begin{aligned} |Q(\eta)(x) - Q(\eta)(\xi)| &\leq |Y(x) - Y(\xi)| \\ &+ \left| \int_{\xi}^x G(x-t) f(\eta(t)) dt \right| \\ &+ \left| \int_{-T_1}^{\xi} (G(x-t) - G(\xi-t)) f(\eta(t)) dt \right| \\ &\leq |Y(x) - Y(\xi)| + \\ &\sup_{|\eta| \leq M} |f(\eta)| \left\{ \int_0^{x-\xi} |G(x-\xi-t)| dt \right\} + \\ &\sup_{|\eta| \leq M} |f(\eta)| \left\{ \int_0^{\xi+T_1} |G(x-\xi+t) - G(t)| dt \right\}. \end{aligned}$$

Since the function Y and G are continuous, we conclude that the set $\{Q(\eta) \mid \eta \in K\}$ is equicontinuous on $[-T_1, T_2]$. An application of the Schauder Theorem tells us that there exists a fixed point η_C of Q .

To combine $\eta_L(x), \eta_C(x)$ and $\eta_R(x)$ to be a solution of equation (9), it requires that the end point values, $(\eta_C(T_2), \eta'_C(T_2), \eta''_C(T_2), \eta'''_C(T_2))$ which will be used as the initial values of $z_R(x)$ on (x_2, ∞) , satisfy (20) and (21) in Theorem 3. This needs the right hand side of (29) to be small and this could be done by having M_Y, M_b and M sufficiently small. Observing (30), the positive number M^- could be as small as we want by choosing sufficiently small M_b and M_Y , and thus M could be as small as required. Therefore, we obtain an bounded solution of (9) in **Case 5**.

4 Conclusion

We construct bounded solutions of a model equation, which governs two-dimensional steady capillary-gravity waves of an ideal fluid flow with Bond number near 1/3 and Froude number close to one.

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