Positive Solution for Higher-order Singular Infinite-point Fractional Differential Equation with p-Laplacian

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Abstract: By means of the method of upper and lower solutions together with the Schauder fixed point theorem, the conditions for the existence of at least one positive solution are established for some higher-order singular infinite-point fractional differential equation with p-Laplacian. The nonlinear term may be singular with respect to both the time and space variables.

Key–Words: Fractional differential equations, p-Laplacian, Singularity, Upper and lower solutions, Positive solution

1 Introduction

We investigate the existence of positive solutions for the following fractional differential equations contaning p-Laplacian operator (PFDE, for short) and infinite-point boundary value conditions

$$\begin{cases} D_{0+}^{\beta}(\varphi_p(D_{0+}^{\alpha}u(t))) + f(t,u(t)) = 0, \\ \qquad\qquad\qquad\qquad 0 < t < 1, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \\ D_{0+}^{\alpha}u(0) = 0,\ u^{(i)}(1) = \sum\limits_{j=1}^{\infty} \alpha_j u(\xi_j), \end{cases} \tag{1}$$

where $D_{0+}^{\alpha}, D_{0+}^{\beta}$ is the standard Riemann-Liouville derivative, $\varphi_p(s) = |s|^{p-2}s, p > 1$, $f \in C((0,1) \times J, J)$, $J = (0,+\infty), R^+ = [0,+\infty), h \in L^1[0,1]$ is nonnegative. $f(t,u)$ may be singular at $t = 0, 1$ and $u = 0$, $i \in [1, n-2]$ is a fixed integer, $n-1 < \alpha \leq n, n \geq 3, 0 < \beta \leq 1$, $\alpha_j \geq 0$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{j-1} < \xi_j < \cdots < 1$ $(j = 1,2,\ldots), \Delta - \sum\limits_{j=1}^{\infty} \alpha_j {\xi_j}^{\alpha-1} > 0, \Delta = (\alpha-1)(\alpha-2)\cdots(\alpha-i)$.

In recent years, many excellent results of fractional differential equations have been widely reported for their numerous applications such as in electrodynamics of complex medium, control, electromagnetic, polymer rheology, and so on, see [1-3] for an extensive collection of such results. In [4-6], by means of fixed point theorem and theory of fixed point index together with eigenvalue with respect to the relevant linear operator, the existence and multiplicity of positive solutions, pseudo-solutions are obtained for m-point boundary value problem of fractional differential equations

$$D_{0+}^{\alpha}u(t) + q(t)f(t,u(t)) = 0,\ \ 0 < t < 1 \tag{A}$$

subject to the following boundary conditions

$$u(0) = \cdots = u^{(n-2)}(0) = 0, u(1) = \sum_{i=1}^{m} \alpha_i u(\xi_i). \tag{B_1}$$

Similar results are extended to more general boundary value problems in [7]. Motivated by [8], by introducing height functions of the nonlinear term on some bounded sets, we considered local existence and multiplicity of positive solutions for BVP (A) with infinite-point boundary value conditions in [9]. On the other hand, there have been some papers dealing with the fractional differential equations involving p-Laplacian operator [10-16]. The purpose of this paper is to study the existence of at least one solution for PFDE (1) by means of the upper and lower solutions and the Schauder fixed point theorem.

Compared to [5-7], this paper admits the following three new features. Firstly, the facts p-Laplacian operator is involved in differential operator and infinite points is contained in boundary value problems make the problem considered more general. Secondly, nonlinear term permits singularities with respect to both the time and space variables.

2 Preliminaries and several lemmas

Let E be the Banach space of continuous functions $u : [0,1] \to R$ equipped with the norm $\|u\| = \max_{0\le t\le 1} |u(t)|$. Here, we list some definitions and useful lemmas from fractional calculus theory.

Definition 1 *([3]) The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $y : (0,\infty) \to R$ is given by*

$$I_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\mathrm{d}s$$

provided the right-hand side is pointwise defined on $(0,\infty)$.

Definition 2 *([3]) The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $y : (0,\infty) \to R$ is given by*

$$D_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \Big(\frac{\mathrm{d}}{\mathrm{d}t}\Big)^n \int_0^t \frac{y(s)}{(t-s)^{\alpha-n+1}} \mathrm{d}s$$

where $n = [\alpha]+1, [\alpha]$ denotes the integer part of the number α, provided that the right-hand side is pointwise defined on $(0,\infty)$.

Now, we consider the linear fractional differential equation

$$\begin{cases} D_{0+}^{\alpha} u(t) + y(t) = 0,\ 0 < t < 1, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \\ u^{(i)}(1) = \sum\limits_{j=1}^{\infty} \alpha_j u(\xi_j). \end{cases} \tag{2}$$

Lemma 3 *([9]) Given $y \in C[0,1]$, then the unique solution of the problem*

$$\begin{cases} D_{0+}^{\alpha} u(t) + y(t) = 0,\ 0 < t < 1, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \\ u^{(i)}(1) = \sum\limits_{j=1}^{\infty} \alpha_j u(\xi_j), \end{cases} \tag{3}$$

can be expressed by

$$u(t) = \int_0^1 G(t,s) y(s) \mathrm{d}s,$$

where

$$G(t,s) = \frac{1}{p(0)\Gamma(\alpha)} \begin{cases} t^{\alpha-1} p(s)(1-s)^{\alpha-1-i} \\ -p(0)(t-s)^{\alpha-1}, \\ 0 \le s \le t \le 1, \\ t^{\alpha-1} p(s)(1-s)^{\alpha-1-i}, \\ 0 \le t \le s \le 1, \end{cases} \tag{4}$$

where $p(s) = \Delta - \sum\limits_{s\le\xi_j} \alpha_j \Big(\frac{\xi_j - s}{1-s}\Big)^{\alpha-1} (1-s)^i$. Obviously, $G(t,s)$ is continuous on $[0,1]\times[0,1]$.

Lemma 4 *[7] The function $G(t,s)$ defined by (4) has the following properties:*
(1) $p(0)\Gamma(\alpha)G(t,s) \ge m_1 s(1-s)^{\alpha-1-i} t^{\alpha-1}, \forall\ t, s \in [0,1]$;
(2) $p(0)\Gamma(\alpha)G(t,s) \le [M_1 + p(0)n]\ (1-s)^{\alpha-1-i} t^{\alpha-1},\ \forall\, t,s \in [0,1]$;
(3) $p(0)\Gamma(\alpha)G(t,s) \le [M_1 + p(0)n] s(1-s)^{\alpha-1-i}$, $\forall\, t,s \in [0,1]$;
(4) $G(t,s) > 0,\ \forall\, t,s \in (0,1)$
where $M_1 = \sup\limits_{0<s\le 1} \frac{p(s)-p(0)}{s}, m_1 = \inf\limits_{0<s\le 1} \frac{p(s)-p(0)}{s}$ are positive numbers.

Proof. The proof of (1) and (3) is almost as the same as that in [7] and (4) is obvious. To get (2), check the proof of Lemma 2.4 in [7]. For $0 < s \le t \le 1$, we get that

$$\begin{aligned} p(0)\Gamma(\alpha)G(t,s) &= p(s)(1-s)^{\alpha-1-i} t^{\alpha-1} \\ &\quad -p(0)(t-s)^{\alpha-1} \\ &= [p(s)-p(0)](1-s)^{\alpha-1-i} t^{\alpha-1} \\ &\quad +p(0)[(1-s)^{\alpha-1-i} t^{\alpha-1} \\ &\quad -(t-s)^{\alpha-1}] \\ &\le M_1 s(1-s)^{\alpha-1-i} t^{\alpha-1} \\ &\quad +p(0)(1-s)^{\alpha-1-i} t^{\alpha-1} \\ &\quad [1-(1-\frac{s}{t})][1+(1-\frac{s}{t}) + \\ &\quad (1-\frac{s}{t})^2 + \cdots + (1-\frac{s}{t})^{n-1}] \\ &\le M_1 s(1-s)^{\alpha-1-i} t^{\alpha-1} \\ &\quad +p(0)(1-s)^{\alpha-1-i} t^{\alpha-2} sn \\ &\le M_1 s(1-s)^{\alpha-1-i} t^{\alpha-1} \\ &\quad +p(0)(1-s)^{\alpha-1-i} t^{\alpha-2} tn \\ &\le [M_1 + p(0)n](1-s)^{\alpha-1-i} t^{\alpha-1}. \end{aligned}$$

For $0 < t \le s \le 1$, we have that

$$\begin{aligned} p(0)\Gamma(\alpha)G(t,s) &= p(s)(1-s)^{\alpha-1-i} t^{\alpha-1} \\ &= [p(s)-p(0)](1-s)^{\alpha-1-i} t^{\alpha-1} \\ &\quad +p(0)(1-s)^{\alpha-1-i} t^{\alpha-1} \\ &\le [M_1 + p(0)n](1-s)^{\alpha-1-i} t^{\alpha-1}. \end{aligned}$$

Let $q > 1$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Then, $\varphi_p^{-1}(s) = \varphi_q(s)$. To study the PFDE (1), we first consider the associated linear PFDE

$$\begin{cases} D_{0+}^{\beta}(\varphi_p(D_{0+}^{\alpha} u(t))) + y(t) = 0,\ \ 0 < t < 1, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \\ D_{0+}^{\alpha} u(0) = 0,\ u^{(i)}(1) = \sum\limits_{j=1}^{\infty} \eta_j u(\xi_j), \end{cases} \tag{5}$$

for $y \in L^1[0,1]$ and $y \ge 0$.

Lemma 5 *The unique solution for the associated linear PFDE (5) can be written by*

$$u(t) = \Big(\frac{1}{\Gamma(\beta)}\Big)^{q-1}\int_0^1 G(t,s) \cdot\varphi_p^{-1}\Big(\int_0^s (s-\tau)^{\beta-1}y(\tau)\mathrm{d}\tau\Big)\mathrm{d}s.$$

Proof. Let $w = D_{0+}^{\alpha}u, v = \varphi_p(w)$. Then, the initial value problem

$$\begin{cases} D_{0+}^{\beta}v(t) + y(t) = 0,\ t \in (0,1), \\ v(0) = 0 \end{cases} \tag{6}$$

has the solution $v(t) = c_1 t^{\beta-1} - I^{\beta}y(t),\ t \in [0,1]$. Noticing that $v(0) = 0, 0 < \beta \le 1$, we have that $c_1 = 0$. As a consequence,

$$v(t) = -I^{\beta}y(t),\ t \in [0,1]. \tag{7}$$

Considering that $D_{0+}^{\alpha}u = w, w = \varphi_p^{-1}(v)$, we have from (7) that

$$\begin{cases} D_{0+}^{\alpha}u(t) = \varphi_p^{-1}(-I^{\beta}(y(t))),\ \ 0 < t < 1, \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \\ u^{(i)}(1) = \sum\limits_{j=1}^{\infty}\eta_j u(\xi_j). \end{cases} \tag{8}$$

By Lemma 3, the solution of (8) can be expressed by

$$u(t) = \Big(\frac{1}{\Gamma(\beta)}\Big)^{q-1}\int_0^1 G(t,s) \cdot\varphi_p^{-1}\Big(\int_0^s (s-\tau)^{\beta-1}y(\tau)\mathrm{d}\tau\Big)\mathrm{d}s.$$

Definition 6 *A continuous function $\Psi(t)$ is called a lower solution of the PFDE (1) if it satisfies*

$$\begin{cases} -D_{0+}^{\beta}(\varphi_p(D_{0+}^{\alpha}\Psi(t))) \le f(t,\Psi(t)),\ \ 0 < t < 1, \\ \Psi(0) \ge 0, \Psi'(0) \ge 0, \cdots, \Psi^{(n-2)}(0) \ge 0, \\ D_{0+}^{\alpha}\Psi(0) \ge 0,\ \ \Psi^{(i)}(1) \ge \sum\limits_{j=1}^{\infty}\alpha_j\Psi(\xi_j). \end{cases}$$

Definition 7 *A continuous function $\Phi(t)$ is called a upper solution of the PFDE (1) if it satisfies*

$$\begin{cases} -D_{0+}^{\beta}(\varphi_p(D_{0+}^{\alpha}\Phi(t))) \ge f(t,\Phi(t)),\ \ 0 < t < 1, \\ \Phi(0) \le 0, \Phi'(0) \le 0, \cdots, \Phi^{(n-2)}(0) \le 0, \\ D_{0+}^{\alpha}\Phi(0) \le 0,\ \ \Phi^{(i)}(1) \le \sum\limits_{j=1}^{\infty}\alpha_j\Phi(\xi_j). \end{cases}$$

Let

$$F = \{u \in C([0,1],R), u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, u^{(i)}(1) = \sum_{j=1}^{\infty}\alpha_j u(\xi_j)\}.$$

Lemma 8 *Let $u \in F$ such that $-D_{0+}^{\alpha}u(t) \ge 0, t \in [0,1]$. Then $u(t) \ge 0, t \in [0,1]$.*

Proof. Let $-D_{0+}^{\alpha}u(t) = h(t)$. Noticing that $u \in F$, by Lemma 3, we know that

$$u(t) = \int_0^1 G(t,s)h(s)\mathrm{d}s.$$

It follows from Lemma 4 and $h(t) \ge 0$ that $u(t) \ge 0, t \in [0,1]$.

Lemma 9 *(Leray-Schauder fixed point theorem) Let T be a continuous and compact mapping of a Banach space E into itself, such that the set*

$$\{x \in E : x = \sigma Tx, \text{for some } 0 \le \sigma \le 1\} \tag{9}$$

is bounded. Then T has a fixed point.

3 Main results

Denote $e(t) = t^{\alpha-1}, \overline{m} = \frac{m_1}{p(0)\Gamma(\alpha)}, \overline{M} = \frac{M_1+p(0)n}{p(0)\Gamma(\alpha)}$. We list below some assumptions used in this paper.

(H_0) $0 < \int_0^1 \varphi_p^{-1}\Big(\int_0^s (s-\tau)^{\beta-1}f(\tau,e(\tau))\mathrm{d}\tau\Big)\mathrm{d}s < +\infty$.

(H_1) $f \in C((0,1)\times J, R^+)$, for any fixed $t \in (0,1)$, $f(t,u)$ is non-increasing in u, for any $c \in (0,1)$, there exist $\lambda > 0$ such that for all $(t,u) \in (0,1] \times J$,

$$f(t,cu) \le c^{-\lambda}f(t,u). \tag{10}$$

From (10), it is easy to see that if $c \in [1,+\infty)$, then

$$f(t,cu) \ge c^{-\lambda}f(t,u). \tag{11}$$

Let

$$P = \{x \in C[0,1] : x(t) \ge 0,\ \ t \in [0,1]\}.$$

Obviously, P is a normal cone in the Banach space E. Now, define a subset D in E as follows

$$D = \{u \in P : \text{there exist two positive numbers } l_u < 1 < L_u \text{ such that } l_u e(t) \le u(t) \le L_u e(t),\ t \in [0,1]\}. \tag{12}$$

Obviously, D is nonempty since $e(t) \in P$. Now define an operator A as follows:

$$(Au)(t) = \Big(\frac{1}{\Gamma(\beta)}\Big)^{q-1}\int_0^1 G(t,s) \cdot\varphi_p^{-1}\Big(\int_0^s (s-\tau)^{\beta-1} f(\tau,u(\tau))\mathrm{d}\tau\Big)\mathrm{d}s,\ \ t \in [0,1]. \tag{13}$$

Theorem 10 *Assume that* (H_0) *and* (H_1)*hold. Then the PFDE (1) has at least one positive solution* $w^* \in D$*, and there exist constants* $0 < k < 1$ *and* $K > 1$ *such that* $ke(t) \leq w^*(t) \leq Ke(t),\ \ t \in [0,1]$.

Proof. Firstly, we show that $A : D \to D$ is well defined.

In fact, for any $u \in D$, there exist two positive numbers $L_u > 1 > l_u$ such that

$$l_u e(t) \leq u(t) \leq L_u e(t),\ \ t \in [0,1]. \tag{14}$$

We have from $(\mathrm{H}_0), (\mathrm{H}_1)$, Lemma 4, (10), (11), (14) that

$$\begin{aligned}
&\Big(\frac{1}{\Gamma(\beta)}\Big)^{q-1}\int_0^1 G(t,s)\\
&\quad \varphi_p^{-1}\Big(\int_0^s (s-\tau)^{\beta-1} f(\tau,u(\tau))\mathrm{d}\tau\Big)\mathrm{d}s\\
\leq\ & \Big(\frac{1}{\Gamma(\beta)}\Big)^{q-1}\overline{M}t^{\alpha-1}\\
&\int_0^1 \varphi_p^{-1}\Big(\int_0^s (s-\tau)^{\beta-1} f(\tau,l_u e(\tau))\mathrm{d}\tau\Big)\mathrm{d}s\\
\leq\ & \Big(\frac{1}{\Gamma(\beta)}\Big)^{q-1} l_u^{-\lambda(q-1)}\overline{M}\\
&\cdot\int_0^1 \varphi_p^{-1}\Big(\int_0^s (s-\tau)^{\beta-1} f(\tau,e(\tau))\mathrm{d}\tau\Big)\mathrm{d}s\\
&\cdot e(t)\\
<\ & +\infty,
\end{aligned}$$

and

$$\begin{aligned}
&\Big(\frac{1}{\Gamma(\beta)}\Big)^{q-1}\int_0^1 G(t,s)\\
&\quad \varphi_p^{-1}\Big(\int_0^s (s-\tau)^{\beta-1} f(\tau,u(\tau))\mathrm{d}\tau\Big)\mathrm{d}s\\
\geq\ & \Big(\frac{1}{\Gamma(\beta)}\Big)^{q-1} L_u^{-\lambda(q-1)}\overline{m}\cdot\int_0^1 s(1-s)^{\alpha-1-i}\\
&\quad \varphi_p^{-1}\Big(\int_0^s (s-\tau)^{\beta-1} f(\tau,e(\tau))\mathrm{d}\tau\Big)\mathrm{d}s\cdot e(t).
\end{aligned}$$

By (H_0), it is clear that

$$\begin{aligned}
&\int_0^1 s(1-s)^{\alpha-1-i}\\
&\quad \varphi_p^{-1}\Big(\int_0^s (s-\tau)^{\beta-1} f(\tau,e(\tau))\mathrm{d}\tau\Big)\mathrm{d}s\\
\leq\ & \int_0^1 \varphi_p^{-1}\Big(\int_0^s (s-\tau)^{\beta-1} f(\tau,e(\tau))\mathrm{d}\tau\Big)\mathrm{d}s\\
<\ & +\infty.
\end{aligned}$$

Thus, we have proved that $A : D \to D$ is well defined.

By Lemma 5, we know that $Au(t)$ satisfy the following equation

$$\begin{cases}
-D_{0+}^{\beta}(\varphi_p(D_{0+}^{\alpha}(Au)(t))) = f(t,u(t)),\\
\qquad\qquad\qquad\qquad 0<t<1,\\
(Au)(0) = (Au)'(0) = \cdots\\
\quad = (Au)^{(n-2)}(0) = 0,\ \ D_{0+}^{\alpha}(Au)(0) = 0,\\
(Au)^{(i)}(1) = \sum\limits_{j=1}^{\infty}\eta_j (Au)(\xi_j).
\end{cases} \tag{15}$$

Now, we are in position to find a pair of upper and lower solutions for PFDE (1). Let

$$\begin{aligned}
u_0(t) \ =\ & \Big(\frac{1}{\Gamma(\beta)}\Big)^{q-1}\int_0^1 G(t,s)\\
&\varphi_p^{-1}\Big(\int_0^s (s-\tau)^{\beta-1} f(\tau,e(\tau))\mathrm{d}\tau\Big)\mathrm{d}s,\\
&\qquad\qquad t\in[0,1].
\end{aligned}$$

By Lemma 4, we get that

$$\begin{aligned}
u_0(t) \ \geq\ & \Big(\frac{1}{\Gamma(\beta)}\Big)^{q-1}\overline{m}\int_0^1 \varphi_p^{-1}\Big(\int_0^s (s-\tau)^{\beta-1}\\
&f(\tau,e(\tau))\mathrm{d}\tau\Big)\mathrm{d}s\cdot e(t),\ \ t\in[0,1].
\end{aligned}$$

As a consequence, there exists a constant $k_0 \geq 1$ such that

$$k_0 u_0(t) \geq e(t), \forall\, t \in [0,1]. \tag{16}$$

It following from $(\mathrm{H}_0), (\mathrm{H}_1)$ and (16) that A is decreasing on u, thus for $k > k_0$, we have

$$\begin{aligned}
&\Big(\frac{1}{\Gamma(\beta)}\Big)^{q-1}\int_0^1 G(t,s)\\
&\quad \varphi_p^{-1}\Big(\textstyle\int_0^s (s-\tau)^{\beta-1} f(\tau,ku_0(\tau))\mathrm{d}\tau\Big)\mathrm{d}s\\
\leq\ & \Big(\frac{1}{\Gamma(\beta)}\Big)^{q-1}\int_0^1 G(t,s)\\
&\quad \varphi_p^{-1}\Big(\int_0^s (s-\tau)^{\beta-1} f(\tau,k_0u_0(\tau))\mathrm{d}\tau\Big)\mathrm{d}s\\
\leq\ & \Big(\frac{1}{\Gamma(\beta)}\Big)^{q-1}\int_0^1 G(t,s)\\
&\quad \varphi_p^{-1}\Big(\textstyle\int_0^s (s-\tau)^{\beta-1} f(\tau,e(\tau))\mathrm{d}\tau\Big)\mathrm{d}s < +\infty,
\end{aligned}$$

and

$$\begin{aligned}
u_0(t) \ \leq\ & \Big(\frac{1}{\Gamma(\beta)}\Big)^{q-1}\overline{M}\int_0^1 \varphi_p^{-1}\Big(\int_0^s (s-\tau)^{\beta-1}\\
&f(\tau,e(\tau))\mathrm{d}\tau\Big)\mathrm{d}s < +\infty.
\end{aligned}$$

Let $\rho = \Big(\frac{1}{\Gamma(\beta)}\Big)^{q-1}\overline{M}\int_0^1 \varphi_p^{-1}\Big(\int_0^s (s-\tau)^{\beta-1} f(\tau,e(\tau))\mathrm{d}\tau\Big)\mathrm{d}s + 1$. Take

$$\begin{aligned}
k^* \ =\ & \max\Big\{k_0, \Big[\Big(\frac{1}{\Gamma(\beta)}\Big)^{q-1}\overline{m}\int_0^1 \varphi_p^{-1}\Big(\int_0^s\\
&(s-\tau)^{\beta-1} f(\tau,\rho)\mathrm{d}\tau\Big)\mathrm{d}s\Big]^{\frac{1}{\lambda(q-1)}}\Big\}
\end{aligned}$$

Then, we have

$$
\begin{aligned}
&+\infty > \Big(\frac{1}{\Gamma(\beta)}\Big)^{q-1}\int_0^1 G(t,s)\\
&\quad \varphi_p^{-1}\Big(\int_0^s (s-\tau)^{\beta-1}\\
&\quad f(\tau,k^*u_0(\tau))\mathrm{d}\tau\Big)\mathrm{d}s\\
&\geq (k^*)^{-\lambda(q-1)}\Big(\frac{1}{\Gamma(\beta)}\Big)^{q-1}\overline{m}t^{\alpha-1}\\
&\quad \int_0^1 \varphi_p^{-1}\Big(\int_0^s (s-\tau)^{\beta-1}\\
&\quad f(\tau,u_0(\tau))\mathrm{d}\tau\Big)\mathrm{d}s\\
&\geq (k^*)^{-\lambda(q-1)}\Big(\frac{1}{\Gamma(\beta)}\Big)^{q-1}\overline{m}t^{\alpha-1}\\
&\quad \int_0^1 \varphi_p^{-1}\Big(\int_0^s (s-\tau)^{\beta-1}f(\tau,\rho)\mathrm{d}\tau\Big)\mathrm{d}s\\
&\geq t^{\alpha-1},\ \forall\, t\in[0,1].
\end{aligned}
\tag{17}
$$

Let

$$
\Phi(t) = k^*u_0(t),\ \ \Psi(t) = (A\Phi)(t). \tag{18}
$$

Then, it follows from (16) and (17) that

$$
\begin{aligned}
\Phi(t) &= k^*\Big(\frac{1}{\Gamma(\beta)}\Big)^{q-1}\int_0^1 G(t,s)\\
&\quad \varphi_p^{-1}\Big(\int_0^s (s-\tau)^{\beta-1}f(\tau,e(\tau))\mathrm{d}\tau\Big)\mathrm{d}s\\
&\geq t^{\alpha-1},
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
\Psi(t) &= \Big(\frac{1}{\Gamma(\beta)}\Big)^{q-1}\int_0^1 G(t,s)\\
&\quad \varphi_p^{-1}\Big(\int_0^s (s-\tau)^{\beta-1}f(\tau,k^*u_0(\tau))\mathrm{d}\tau\Big)\mathrm{d}s\\
&\geq t^{\alpha-1}.
\end{aligned}
\tag{20}
$$

In addition, by (15) and (18), we know that

$$
\begin{aligned}
&\Phi(0) = \Phi'(0) = \cdots = \Phi^{(n-2)}(0) = 0,\\
&D_{0+}^{\alpha}\Phi(0) = 0,\ \ \Phi^{(i)}(1) = \sum_{j=1}^{\infty}\eta_j\Phi(\xi_j),
\end{aligned}
$$

$$
\begin{aligned}
&\Psi(0) = \Phi'(0) = \cdots = \Psi^{(n-2)}(0) = 0,\\
&D_{0+}^{\alpha}\Psi(0) = 0,\ \ \Psi^{(i)}(1) = \sum_{j=1}^{\infty}\eta_j\Psi(\xi_j).
\end{aligned}
$$

By (18),

$$
\begin{aligned}
\Psi(t) &= (A\Phi)(t) = \Big(\frac{1}{\Gamma(\beta)}\Big)^{q-1}\int_0^1\\
&\quad G(t,s)\varphi_p^{-1}\Big(\int_0^s (s-\tau)^{\beta-1}\\
&\quad f(\tau,k^*u_0(\tau))\mathrm{d}\tau\Big)\mathrm{d}s\\
&\leq k^*\Big(\frac{1}{\Gamma(\beta)}\Big)^{q-1}\int_0^1 G(t,s)\\
&\quad \varphi_p^{-1}\Big(\int_0^s (s-\tau)^{\beta-1}\\
&\quad f(\tau,u_0(\tau))\mathrm{d}\tau\Big)\mathrm{d}s\\
&\quad = \Phi(t),\ \ \forall\, t\in[0,1].
\end{aligned}
\tag{21}
$$

Considering the fact that f is non-increasing in u, we get from (18)-(21) that

$$
\begin{aligned}
&D_{0+}^{\beta}(\varphi_p(D_{0+}^{\alpha}\Psi(t))) + f(t,\Psi(t))\\
={}& D_{0+}^{\beta}(\varphi_p(D_{0+}^{\alpha}(A\Phi)(t))) + f(t,\Psi(t))\\
\geq{}& D_{0+}^{\beta}(\varphi_p(D_{0+}^{\alpha}(A\Phi)(t))) + f(t,\Phi(t))\\
={}& -f(t,\Phi(t)) + f(t,\Phi(t)) = 0,
\end{aligned}
\tag{22}
$$

$$
\begin{aligned}
&D_{0+}^{\beta}(\varphi_p(D_{0+}^{\alpha}\Phi(t))) + f(t,\Phi(t))\\
={}& D_{0+}^{\beta}(\varphi_p(D_{0+}^{\alpha}A(t^{\alpha-1}))) + f(t,\Phi(t))\\
={}& -f(t,t^{\alpha-1}) + f(t,\Phi(t))\\
\leq{}& -f(t,t^{\alpha-1}) + f(t,t^{\alpha-1}) = 0.
\end{aligned}
\tag{23}
$$

By (22) and (23), we know that $\Phi,\Psi\in P$ are desired upper and lower solutions of the PFDE (1), respectively.

Define a function F as follows

$$
F(t,u) = \begin{cases} f(t,\Psi(t)), & u<\Psi(t),\\ f(t,u(t)), & \Psi(t)\leq u\leq\Phi(t),\\ f(t,\Phi(t)), & \Phi(t)<u. \end{cases}
\tag{24}
$$

This together with (H_1) shows that $F:(0,1)\times R^+\to R^+$ is continuous.

In the following, we shall show that the fractional boundary value problem

$$
\begin{cases} D_{0+}^{\beta}(\varphi_p(D_{0+}^{\alpha}u(t))) + F(t,u(t)) = 0,\\ \qquad\qquad\qquad\qquad 0<t<1,\\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0,\\ D_{0+}^{\alpha}u(0) = 0,\ \ u^{(i)}(1) = \sum\limits_{j=1}^{\infty}\alpha_j u(\xi_j) \end{cases}
\tag{25}
$$

has a positive solution.

Let

$$
\begin{aligned}
(Tu)(t) &= \Big(\frac{1}{\Gamma(\beta)}\Big)^{q-1}\int_0^1 G(t,s)\\
&\quad \varphi_p^{-1}\Big(\int_0^s (s-\tau)^{\beta-1}\\
&\quad F(\tau,u(\tau))\mathrm{d}\tau\Big)\mathrm{d}s,\ \ t\in[0,1].
\end{aligned}
\tag{26}
$$

Then $T: E \to E$ and a fixed point of the operator T is a solution of the PFDE (25). By (20), (21), the definition of F and the fact that $f(t,u)$ is non-increasing in u, we have that

$$f(t,\Phi(t)) \le F(t,u(t)) \le f(t,\Psi(t)),\ \forall\, x \in E, \tag{27}$$

and

$$f(t,\Phi(t)) \le F(t,u(t)) \le f(t,t^{\alpha-1}),\ \forall\, x \in E. \tag{28}$$

By Lemma 4 and (28), for $u \in E$, we have

$$\begin{aligned}(Tu)(t) &= \Big(\frac{1}{\Gamma(\beta)}\Big)^{q-1}\int_0^1 G(t,s)\\ &\quad \varphi_p^{-1}\Big(\int_0^s (s-\tau)^{\beta-1}\\ &\quad F(\tau,u(\tau))\mathrm{d}\tau\Big)\mathrm{d}s\\ &\le \Big(\frac{1}{\Gamma(\beta)}\Big)^{q-1}\overline{M}t^{\alpha-1}\\ &\quad \int_0^1 \varphi_p^{-1}\Big(\int_0^s (s-\tau)^{\beta-1}\\ &\quad f(\tau,e(\tau))\mathrm{d}\tau\Big)\mathrm{d}s < +\infty,\end{aligned} \tag{29}$$

which means that T is uniformly bounded. Considering the uniform continuity of $G(t,s)$ on $[0,1]\times[0,1]$, it can be easily seen that $T: E \to E$ is completely continuous. In addition, we have from (29) that (9) holds. Thus, Schauder fixed point theorem guarantees that T has at least one fixed point w.

Now, we are in position to show that

$$\Psi(t) \le w(t) \le \Phi(t),\ \ t \in [0,1]. \tag{30}$$

Since w is a fixed point of T, we have by (25) that

$$\begin{aligned}&w(0) = w'(0) = \cdots = w^{(n-2)}(0) = 0,\\ &D_{0+}^{\alpha}w(0) = 0,\ \ w^{(i)}(1) = \sum_{j=1}^{\infty}\alpha_j w(\xi_j),\end{aligned} \tag{31}$$

$$\begin{aligned}&\Phi(0) = \Phi'(0) = \cdots = \Phi^{(n-2)}(0) = 0,\\ &D_{0+}^{\alpha}\Phi(0) = 0,\quad \Phi^{(i)}(1) = \sum_{j=1}^{\infty}\eta_j\Phi(\xi_j).\end{aligned} \tag{32}$$

Let $z(t) = \varphi_p(D_{0+}^{\alpha}\Phi(t)) - \varphi_p(D_{0+}^{\alpha}w(t))$. Then

$$\begin{aligned}D_{0+}^{\beta}z(t) &= D_{0+}^{\beta}(\varphi_p(D_{0+}^{\alpha}\Phi(t)))\\ &\quad -D_{0+}^{\beta}(\varphi_p(D_{0+}^{\alpha}w(t)))\\ &= -f(t,t^{\alpha-1}) + F(t,w(t))\\ &\le 0,\quad t\in[0,1],\\ D_{0+}^{\beta}z(0) &= D_{0+}^{\beta}(\varphi_p(D_{0+}^{\alpha}\Phi(0)))\\ &\quad -D_{0+}^{\beta}(\varphi_p(D_{0+}^{\alpha}w(0))) = 0.\end{aligned}$$

By (6) and (8), we know that

$$z(t) \le 0,$$

which means that

$$\varphi_p(D_{0+}^{\alpha}\Phi(t)) - \varphi_p(D_{0+}^{\alpha}w(t)) \le 0.$$

We get from the fact that φ_p is monotone increasing

$$\varphi_p(D_{0+}^{\alpha}\Phi(t)) \le \varphi_p(D_{0+}^{\alpha}w(t)),$$

i.e.,

$$-\varphi_p(D_{0+}^{\alpha}(\Phi - w))(t) \ge 0.$$

It follows from Lemma 8, (32) and (33) that

$$\Phi(t) - w(t) \ge 0.$$

Thus, we have proved that $w(t) \le \Phi(t)$ on $[0,1]$. Similarly, we can get that $w(t) \ge \Psi(t)$ on $[0,1]$. As a consequence, (30) holds. So, $F(t,w(t)) = f(t,w(t)),\ t\in[0,1]$. Hence, $w(t)$ is a positive solution of the PFDE (1). Noticing that $\Phi,\Psi \in D$, by (30), we can easily know that there exist constants $0<k<1$ and $K>1$ such that

$$ke(t) \le w^*(t) \le Ke(t),\ \ t\in[0,1].$$

4 An example

Example Consider the following singular PFDE

$$\begin{cases} D_{0+}^{\frac{1}{2}}(\varphi_3(D_{0+}^{\frac{7}{2}}u))(t) + \frac{1}{2}t^{-\frac{1}{12}}u^{-\frac{1}{6}} = 0,\\ 0<t<1,\ \ u(0)=u'(0)=u''(0)=0,\\ D_{0+}^{\frac{7}{2}}u(0)=0,\ u'(1)=\sum\limits_{j=1}^{\infty}\frac{2}{j^2}u(\frac{1}{j}).\end{cases} \tag{33}$$

In this situation, $f(t,u) = \frac{1}{2}t^{-\frac{1}{12}}u^{-\frac{1}{6}}, \alpha = \frac{7}{2}, \beta = \frac{1}{2}, p = 3, e(t) = t^{\frac{5}{2}}, \Delta = \frac{5}{2}, \alpha_j = \frac{2}{j^2}, \xi_j = \frac{1}{j}$, $\sum_{j=1}^{\infty}\alpha_j\xi_j^{\alpha-1} \approx 2.109 < \Delta$. By simple computation, we have

$$\begin{aligned}0 &< \int_0^1 \varphi_p^{-1}\Big(\int_0^s (s-\tau)^{\beta-1}f(\tau,e(\tau))\mathrm{d}\tau\Big)\mathrm{d}s\\ &= \int_0^1 \Big(\frac{1}{2}\int_0^s (s-\tau)^{-\frac{1}{2}}\tau^{-\frac{1}{12}}\tau^{-\frac{5}{12}}\mathrm{d}\tau\Big)^{\frac{1}{2}}\mathrm{d}s\\ &= \int_0^1 \Big(\frac{1}{2}\int_0^1 (1-\tau)^{-\frac{1}{2}}\tau^{-\frac{1}{2}}\mathrm{d}\tau\Big)^{\frac{1}{2}}\mathrm{d}s\\ &= \frac{\sqrt{2\pi}}{2} < +\infty.\end{aligned}$$

Therefore, (H_1) holds. It is easy to see that (H_2) is satisfied for $\lambda = \frac{1}{6}$. By Theorem 10, PFDE (33) has at least one positive solution w^* such that there exist constants $0<k<1$ and $K>1$ with $ke(t) \le w^*(t) \le Ke(t),\ \ t\in[0,1]$.

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