On equivalences between fuzzy dependencies and fuzzy formulas' satisfiability for Yager's fuzzy implication operator

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Abstract: In this paper we consider fuzzy functional and fuzzy multivalued dependencies introduced by Sozat and Yazici. We appropriately relate these dependencies to fuzzy formulas. In particular, we relate any subset of the universal set of attributes to fuzzy conjunction of its attributes. Thus, being in the form of implication between such subsets, we naturally relate a fuzzy dependency to fuzzy implication between corresponding fuzzy conjunctions. In this paper we choose standard min, max as well as Yager's fuzzy implication operator for definitions of fuzzy conjunction, fuzzy disjunction and fuzzy implication, respectively. If any two-element fuzzy relation instance on a given scheme, known to satisfy some set of fuzzy functional and fuzzy multivalued dependencies, satisfies some fuzzy functional or fuzzy multivalued dependency f which is not member of the given set of fuzzy dependencies, then, we prove that satisfiability of the related set of fuzzy formulas yields satisfiability of the fuzzy formula related to f and vice versa. A methodology behind the proofs of our results is mainly based on an application of definitions of the introduced fuzzy logic operators. Our results can be verified for various choices of fuzzy logic operators however.

Key–Words: Fuzzy functional and multivalued dependencies, Fuzzy formulas, Fuzzy relation instances

1 Introduction

In this paper we relate fuzzy dependencies and fuzzy logic theories by joining fuzzy formulas to fuzzy functional and fuzzy multivalued dependencies.

We research the concept of fuzzy relation instance that actively satisfies some fuzzy multivalued dependency. We determine the necessary and sufficient conditions needed to given two-element fuzzy relation instance actively satisfies some fuzzy multivalued dependency. In particular, for Yager's fuzzy implication operator, we prove that a two-element fuzzy relation instance actively satisfies given fuzzy multivalued dependency if and only if:

1) tuples of the instance are conformant on certain, well known set of attributes with degree of conformance greater than or equal to some explicitly known constant,

2) related fuzzy formula is satisfiable in appropriate interpretations.

Finally, for Yager's fuzzy implication operator, we prove that any two-element fuzzy relation instance which satisfies all dependencies from the set $\mathfrak{F}$ satisfies the dependency f if and only if satisfiability of all formulas from the set $\mathfrak{F}^{'}$ implies satisfiability of the formula $f^{'}$. Here, $f \notin \mathfrak{F}$ is a fuzzy functional or a fuzzy multivalued dependency, $\mathfrak{F}$ is a set of fuzzy functional and fuzzy multivalued dependencies, $\mathfrak{F}^{'}$ resp. $f^{'}$ denote the set of fuzzy formulas resp. the fuzzy formula related to $\mathfrak{F}$ resp. f.

2 Preliminaries

As it is usual, we introduce

$$\mathfrak{T}(p \,\&\, q) = \min(\mathfrak{T}(p), \mathfrak{T}(q)),$$
$$\mathfrak{T}(p \,\|\, q) = \max(\mathfrak{T}(p), \mathfrak{T}(q)),$$

where $0 \le \mathfrak{T}(p), \mathfrak{T}(q) \le 1$. Here, $\mathfrak{T}(m)$ is the truth value of m.

An interpretation $\mathcal{I}$ is said to satisfy resp. falsify formula f if $\mathfrak{T}(f) \ge \frac{1}{2}$ resp. $\mathfrak{T}(f) \le \frac{1}{2}$ under $\mathcal{I}$ (see, e.g., [6]).

We introduce the notation following similarity-based fuzzy relational database approach [8] (see also, [2]-[4]).

A similarity relation on $\mathfrak{D}$ is a mapping

$\mathfrak{s} : \mathfrak{D} \times \mathfrak{D} \to [0,1]$ such that (see, [13])

$$\mathfrak{s}(x,x) = 1,$$
$$\mathfrak{s}(x,y) = \mathfrak{s}(y,x),$$
$$\mathfrak{s}(x,z) \geq \max_{y \in \mathfrak{D}} (\min(\mathfrak{s}(x,y), \mathfrak{s}(y,z))),$$

where $\mathfrak{D}$ is a set and $x, y, z \in \mathfrak{D}$.

Let $\mathfrak{R}(\mathfrak{U}) = \mathfrak{R}(\mathcal{B}_1, \mathcal{B}_2, ..., \mathcal{B}_n)$ be a scheme on domains $\mathfrak{D}_1$, $\mathfrak{D}_2$,..., $\mathfrak{D}_n$, where $\mathfrak{U}$ is the set of all attributes $\mathcal{B}_1$, $\mathcal{B}_2$,..., $\mathcal{B}_n$ on $\mathfrak{D}_1$, $\mathfrak{D}_2$,..., $\mathfrak{D}_n$ (we say that $\mathfrak{U}$ is the universal set of attributes). Here, we assume that the domain of $\mathcal{B}_i$ is the finite set $\mathfrak{D}_i$, $i = 1, 2, ..., n$.

A fuzzy relation instance $\mathfrak{r}$ on $\mathfrak{R}(\mathfrak{U})$ is defined as a subset of the cross product of the power sets $2^{\mathfrak{D}_1}$, $2^{\mathfrak{D}_2}$,..., $2^{\mathfrak{D}_n}$ of the domains of the attributes. A member of a fuzzy relation instance corresponding to a horizontal row of the table is called a tuple. More precisely, a tuple is an element t of $\mathfrak{r}$ of the form $(\mathfrak{d}_1, \mathfrak{d}_2, ..., \mathfrak{d}_n)$, where $\mathfrak{d}_i \subseteq \mathfrak{D}_i$, $\mathfrak{d}_i \neq \varnothing$ (see also, [5]). Here, we consider $\mathfrak{d}_i$ as the value of $\mathcal{B}_i$ on t.

Recall that the similarity based database approach allows each domain to be equipped with a similarity relation.

The conformance of attribute $\mathcal{B}$ defined on domain $\mathfrak{D}$ for any two tuples t_1 and t_2 present in relation instance $\mathfrak{r}$ and denoted by $\mathcal{B}^{t_1,t_2}$ is defined by

$$\mathcal{B}^{t_1,t_2} = \min \left\{ \min_{x \in \mathfrak{d}_1} \left\{ \max_{y \in \mathfrak{d}_2} \{\mathfrak{s}(x,y)\} \right\}, \min_{x \in \mathfrak{d}_2} \left\{ \max_{y \in \mathfrak{d}_1} \{\mathfrak{s}(x,y)\} \right\} \right\},$$

where $\mathfrak{d}_i$ denotes the value of attribute $\mathcal{B}$ for tuple t_i, $i = 1, 2$ and $\mathfrak{s} : \mathfrak{D} \times \mathfrak{D} \to [0,1]$ is a similarity relation on $\mathfrak{D}$.

If $\mathcal{B}^{t_1,t_2} \geq q$, where $0 \leq q \leq 1$, then the tuples t_1 and t_2 are said to be conformant on attribute $\mathcal{B}$ with q.

The conformance of attribute set $\mathcal{X}$ for any two tuples t_1 and t_2 present in fuzzy relation instance $\mathfrak{r}$ and denoted by $\mathcal{X}^{t_1,t_2}$ is defined by

$$\mathcal{X}^{t_1,t_2} = \min_{\mathcal{B} \in \mathcal{X}} \left\{ \mathcal{B}^{t_1,t_2} \right\}.$$

Obviously: 1) $\mathcal{X}^{t,t} = 1$ for any t in $\mathfrak{r}$,
2) If $\mathcal{X} \supseteq \mathcal{Y}$, then $\mathcal{Y}^{t_1,t_2} \geq \mathcal{X}^{t_1,t_2}$ for any t_1 and t_2 in $\mathfrak{r}$,
3) If $\mathcal{X} = \{\mathcal{B}_1, \mathcal{B}_2, ..., \mathcal{B}_m\}$ and $\mathcal{B}_k^{t_1,t_2} \geq q$ for all $k \in \{1, 2, ..., m\}$, then $\mathcal{X}^{t_1,t_2} \geq q$ for any t_1 and t_2 in $\mathfrak{r}$.

Let $\mathfrak{r}$ be any fuzzy relation instance on scheme $\mathfrak{R}(\mathcal{B}_1, \mathcal{B}_2, ..., \mathcal{B}_n)$, $\mathfrak{U}$ be the universal set of attributes $\mathcal{B}_1$, $\mathcal{B}_2$,..., $\mathcal{B}_n$ and $\mathcal{X}$, $\mathcal{Y}$ be subsets of $\mathfrak{U}$.

Fuzzy relation instance $\mathfrak{r}$ is said to satisfy the fuzzy functional dependency $\mathcal{X} \xrightarrow{\theta}_F \mathcal{Y}$ if for every pair of tuples t_1 and t_2 in $\mathfrak{r}$, $\mathcal{Y}^{t_1,t_2} \geq \min\left(\theta, \mathcal{X}^{t_1,t_2}\right)$.

Fuzzy relation instance $\mathfrak{r}$ is said to satisfy the fuzzy multivalued dependency $\mathcal{X} \to\xrightarrow{\theta}_F \mathcal{Y}$ if for every pair of tuples t_1 and t_2 in $\mathfrak{r}$, there exists a tuple t_3 in $\mathfrak{r}$ such that:

$$\mathcal{X}^{t_3,t_1} \geq \min\left(\theta, \mathcal{X}^{t_1,t_2}\right),$$
$$\mathcal{Y}^{t_3,t_1} \geq \min\left(\theta, \mathcal{X}^{t_1,t_2}\right), \quad (1)$$
$$\mathcal{Z}^{t_3,t_2} \geq \min\left(\theta, \mathcal{X}^{t_1,t_2}\right),$$

where $\mathcal{Z} = \mathfrak{U} - \mathcal{X}\mathcal{Y}$. Here, $\mathfrak{U} - \mathcal{X}\mathcal{Y}$ means $\mathfrak{U} \setminus (\mathcal{X} \cup \mathcal{Y})$. Moreover, $0 \leq \theta \leq 1$ describes the linguistic strength of the dependency. Namely, some dependencies are precise, some of them are not, some dependencies are more precise than the other ones. Therefore, the linguistic strength of the dependency gives us a method for describing imprecise dependencies as well as precise ones.

Fuzzy relation instance $\mathfrak{r}$ is said to satisfy the fuzzy multivalued dependency $\mathcal{X} \to\xrightarrow{\theta}_F \mathcal{Y}$, $\theta-$actively if $\mathfrak{r}$ satisfies that dependency and if $\mathcal{B}^{t_1,t_2} \geq \theta$ for all $\mathcal{B} \in \mathcal{X}$ and all $t_1, t_2 \in \mathfrak{r}$.

It follows immediately that the instance $\mathfrak{r}$ satisfies the dependency $\mathcal{X} \to\xrightarrow{\theta}_F \mathcal{Y}$, $\theta-$ actively if and only if $\mathfrak{r}$ satisfies $\mathcal{X} \to\xrightarrow{\theta}_F \mathcal{Y}$ and $\mathcal{X}^{t_1,t_2} \geq \theta$ for all t_1, $t_2 \in \mathfrak{r}$.

Let $\mathfrak{r} = \{t_1, t_2\}$ be any two-element fuzzy relation instance on scheme $\mathfrak{R}(\mathcal{B}_1, \mathcal{B}_2, ..., \mathcal{B}_n)$ and $0 \leq \varepsilon \leq 1$.

A mapping $v_\varepsilon^{\mathfrak{r}} : \{\mathcal{B}_1, \mathcal{B}_2, ..., \mathcal{B}_n\} \to [0,1]$ such that

$$v_\varepsilon^{\mathfrak{r}}(\mathcal{B}_k) > \frac{1}{2} \text{ if } \mathcal{B}_k^{t_1,t_2} \geq \varepsilon,$$
$$v_\varepsilon^{\mathfrak{r}}(\mathcal{B}_k) \leq \frac{1}{2} \text{ if } \mathcal{B}_k^{t_1,t_2} < \varepsilon,$$

$k = 1, 2, ..., n$, is called a valuation joined to $\mathfrak{r}$ and ε.

3 Results

Let $\mathcal{X} \xrightarrow{\theta}_F \mathcal{Y}$ ($\mathcal{X} \to\xrightarrow{\theta}_F \mathcal{Y}$) be some fuzzy functional dependency (fuzzy multivalued dependency) on $\mathfrak{U}$, where $\mathfrak{U}$ is the universal set of attributes $\mathcal{B}_1$, $\mathcal{B}_2$,..., $\mathcal{B}_n$ and $\mathfrak{R}(\mathcal{B}_1, \mathcal{B}_2, ..., \mathcal{B}_n)$ is a scheme.

In this paper we associate the fuzzy formula

$$\left(\underset{\mathcal{A} \in \mathcal{X}}{\&} \mathcal{A} \right) \to \left(\underset{\mathcal{B} \in \mathcal{Y}}{\&} \mathcal{B} \right)$$

to $\mathcal{X} \overset{\theta}{\to}_F \mathcal{Y}$ and the fuzzy formula

$$\left(\underset{\mathcal{A}\in\mathcal{X}}{\&} \mathcal{A}\right) \to \left(\left(\underset{\mathcal{B}\in\mathcal{Y}}{\&} \mathcal{B}\right) \parallel \left(\underset{\mathcal{C}\in\mathcal{Z}}{\&} \mathcal{C}\right)\right)$$

to $\mathcal{X} \to\overset{\theta}{\to}_F \mathcal{Y}$, where $\mathcal{Z} = \mathfrak{U} - \mathcal{X}\mathcal{Y}$.

Through the rest of the section, we assume that the fuzzy implication operator is given by

$$\mathfrak{T}(p \to q) = \mathfrak{T}(q)^{\mathfrak{T}(p)}$$

if $\mathfrak{T}(p) \neq 0$ or $\mathfrak{T}(q) \neq 0$, $\mathfrak{T}(p \to q) = 1$ if $\mathfrak{T}(p) = 0$ and $\mathfrak{T}(q) = 0$.

Note that this fuzzy implication operator is known as Yager's operator (see, [11]). It is a typical example of $f-$generated fuzzy implication operator (see, [7], [12]). In general, classes of fuzzy implication operators are very nicely described in [1] and [7].

Theorem 1. *Let $\mathfrak{r} = \{t_1, t_2\}$ be any two-element fuzzy relation instance on scheme $\mathfrak{R}(\mathcal{B}_1, \mathcal{B}_2, ..., \mathcal{B}_n)$, $\mathfrak{U}$ be the universal set of attributes $\mathcal{B}_1$, $\mathcal{B}_2$,..., $\mathcal{B}_n$ and $\mathcal{X}$, $\mathcal{Y}$ be subsets of $\mathfrak{U}$. Let $\mathcal{Z} = \mathfrak{U} - \mathcal{X}\mathcal{Y}$. Then, $\mathfrak{r}$ satisfies the fuzzy multivalued dependency $\mathcal{X} \to\overset{\theta}{\to}_F \mathcal{Y}$, $\theta-$actively if and only if $\mathcal{X}^{t_1,t_2} \geq \theta$ and $v_\theta^{\mathfrak{r}}(\mathcal{K}) > \frac{1}{2}$, where $\mathcal{K}$ denotes the fuzzy formula $(\&_{\mathcal{A}\in\mathcal{X}} \mathcal{A}) \to ((\&_{\mathcal{B}\in\mathcal{Y}} \mathcal{B}) \parallel (\&_{\mathcal{C}\in\mathcal{Z}} \mathcal{C}))$ associated to $\mathcal{X} \to\overset{\theta}{\to}_F \mathcal{Y}$.*

Proof: First, we prove that $\mathfrak{r}$ satisfies $\mathcal{X} \to\overset{\theta}{\to}_F \mathcal{Y}$, $\theta-$actively if and only if $\mathcal{X}^{t_1,t_2} \geq \theta$, $\mathcal{Y}^{t_1,t_2} \geq \theta$ or $\mathcal{X}^{t_1,t_2} \geq \theta$, $\mathcal{Z}^{t_1,t_2} \geq \theta$.

Suppose that the instance $\mathfrak{r}$ satisfies the dependency $\mathcal{X} \to\overset{\theta}{\to}_F \mathcal{Y}$, $\theta-$actively. Now, $\mathcal{X}^{t_1,t_2} \geq \theta$ and there is a tuple $t_3 \in \mathfrak{r}$ such that the conditions given by (1) hold true, i.e., that $\mathcal{X}^{t_3,t_1} \geq \theta$, $\mathcal{Y}^{t_3,t_1} \geq \theta$, $\mathcal{Z}^{t_3,t_2} \geq \theta$. Hence, if $t_3 = t_1$, then $\mathcal{Z}^{t_1,t_2} \geq \theta$. Else, if $t_3 = t_2$, then $\mathcal{Y}^{t_1,t_2} \geq \theta$.

Let $\mathcal{X}^{t_1,t_2} \geq \theta$, $\mathcal{Y}^{t_1,t_2} \geq \theta$. Hence, $\min\left(\theta, \mathcal{X}^{t_1,t_2}\right) = \theta$. Now, there is $t_3 \in \mathfrak{r}$, $t_3 = t_2$ such that $\mathcal{X}^{t_3,t_1} \geq \theta$, $\mathcal{Y}^{t_3,t_1} \geq \theta$, $\mathcal{Z}^{t_3,t_2} = 1 \geq \theta$, i.e., (1) holds true. Analogously, if $\mathcal{X}^{t_1,t_2} \geq \theta$, $\mathcal{Z}^{t_1,t_2} \geq \theta$, then $\min\left(\theta, \mathcal{X}^{t_1,t_2}\right) = \theta$. Moreover, there is $t_3 \in \mathfrak{r}$, $t_3 = t_1$ such that $\mathcal{X}^{t_3,t_1} = 1 \geq \theta$, $\mathcal{Y}^{t_3,t_1} = 1 \geq \theta$, $\mathcal{Z}^{t_3,t_2} \geq \theta$. Therefore, (1) holds true. Now, since $\mathfrak{r}$ satisfies the dependency $\mathcal{X} \to\overset{\theta}{\to}_F \mathcal{Y}$ and $\mathcal{X}^{t_1,t_2} \geq \theta$, it follows that the instance $\mathfrak{r}$ satisfies the dependency $\mathcal{X} \to\overset{\theta}{\to}_F \mathcal{Y}$, $\theta-$actively.

Now, we prove the main assertion.

($\Rightarrow$) Suppose that $\mathfrak{r}$ satisfies $\mathcal{X} \to\overset{\theta}{\to}_F \mathcal{Y}$, $\theta-$actively.

We have, $\mathcal{X}^{t_1,t_2} \geq \theta$, $\mathcal{Y}^{t_1,t_2} \geq \theta$ or $\mathcal{X}^{t_1,t_2} \geq \theta$, $\mathcal{Z}^{t_1,t_2} \geq \theta$.

Suppose that $\mathcal{X}^{t_1,t_2} \geq \theta$, $\mathcal{Y}^{t_1,t_2} \geq \theta$. Now,

$$\min_{\mathcal{A}\in\mathcal{X}} \left\{\mathcal{A}^{t_1,t_2}\right\} = \mathcal{X}^{t_1,t_2} \geq \theta,$$
$$\min_{\mathcal{B}\in\mathcal{Y}} \left\{\mathcal{B}^{t_1,t_2}\right\} = \mathcal{Y}^{t_1,t_2} \geq \theta.$$

Hence, $\mathcal{A}^{t_1,t_2} \geq \theta$ for all $\mathcal{A} \in \mathcal{X}$ and $\mathcal{B}^{t_1,t_2} \geq \theta$ for all $\mathcal{B} \in \mathcal{Y}$. Therefore, $v_\theta^{\mathfrak{r}}(\mathcal{A}) > \frac{1}{2}$ for $\mathcal{A} \in \mathcal{X}$, $v_\theta^{\mathfrak{r}}(\mathcal{B}) > \frac{1}{2}$ for $\mathcal{B} \in \mathcal{Y}$. Now,

$$v_\theta^{\mathfrak{r}}\left(\underset{\mathcal{A}\in\mathcal{X}}{\&} \mathcal{A}\right) = \min\left\{v_\theta^{\mathfrak{r}}(\mathcal{A}) \mid \mathcal{A} \in \mathcal{X}\right\} > \frac{1}{2},$$
$$v_\theta^{\mathfrak{r}}\left(\underset{\mathcal{B}\in\mathcal{Y}}{\&} \mathcal{B}\right) = \min\left\{v_\theta^{\mathfrak{r}}(\mathcal{B}) \mid \mathcal{B} \in \mathcal{Y}\right\} > \frac{1}{2}.$$

We obtain,

$$\begin{aligned}
& v_\theta^{\mathfrak{r}}(\mathcal{K}) \\
&= v_\theta^{\mathfrak{r}}\left(\left(\underset{\mathcal{A}\in\mathcal{X}}{\&} \mathcal{A}\right) \to \left(\left(\underset{\mathcal{B}\in\mathcal{Y}}{\&} \mathcal{B}\right) \parallel \left(\underset{\mathcal{C}\in\mathcal{Z}}{\&} \mathcal{C}\right)\right)\right) \\
&= v_\theta^{\mathfrak{r}}\left(\left(\underset{\mathcal{B}\in\mathcal{Y}}{\&} \mathcal{B}\right) \parallel \left(\underset{\mathcal{C}\in\mathcal{Z}}{\&} \mathcal{C}\right)\right)^{v_\theta^{\mathfrak{r}}(\&_{\mathcal{A}\in\mathcal{X}} \mathcal{A})} \\
&= \max\left(v_\theta^{\mathfrak{r}}\left(\underset{\mathcal{B}\in\mathcal{Y}}{\&} \mathcal{B}\right), v_\theta^{\mathfrak{r}}\left(\underset{\mathcal{C}\in\mathcal{Z}}{\&} \mathcal{C}\right)\right)^{v_\theta^{\mathfrak{r}}(\&_{\mathcal{A}\in\mathcal{X}} \mathcal{A})}.
\end{aligned}$$

Denote $a = v_\theta^{\mathfrak{r}}(\&_{\mathcal{A}\in\mathcal{X}} \mathcal{A})$, $b = \max\left(v_\theta^{\mathfrak{r}}(\&_{\mathcal{B}\in\mathcal{Y}} \mathcal{B}), v_\theta^{\mathfrak{r}}(\&_{\mathcal{C}\in\mathcal{Z}} \mathcal{C})\right)$.

Since $v_\theta^{\mathfrak{r}}(\&_{\mathcal{A}\in\mathcal{X}} \mathcal{A}) > \frac{1}{2}$ and $v_\theta^{\mathfrak{r}}(\&_{\mathcal{B}\in\mathcal{Y}} \mathcal{B}) > \frac{1}{2}$, we have that $a > \frac{1}{2}$, $b > \frac{1}{2}$.

Now, $v_\theta^{\mathfrak{r}}(\mathcal{K}) > \frac{1}{2}$ if and only if $b^a > \frac{1}{2}$.

If $b = 1$, then $b^a > \frac{1}{2}$ holds true and hence $v_\theta^{\mathfrak{r}}(\mathcal{K}) > \frac{1}{2}$.

Let $\frac{1}{2} < b < 1$. Now, $b^a > \frac{1}{2}$ if and only if $a < \log_b \frac{1}{2}$. The last inequality is true since $\log_b \frac{1}{2} > 1$. Therefore, $v_\theta^{\mathfrak{r}}(\mathcal{K}) > \frac{1}{2}$.

Similarly, if $\mathcal{X}^{t_1,t_2} \geq \theta$, $\mathcal{Z}^{t_1,t_2} \geq \theta$, then $v_\theta^{\mathfrak{r}}(\&_{\mathcal{A}\in\mathcal{X}} \mathcal{A}) > \frac{1}{2}$ and $v_\theta^{\mathfrak{r}}(\&_{\mathcal{C}\in\mathcal{Z}} \mathcal{C}) > \frac{1}{2}$. Now, reasoning as in the previous case, we conclude that $a > \frac{1}{2}$, $b > \frac{1}{2}$ and hence $v_\theta^{\mathfrak{r}}(\mathcal{K}) > \frac{1}{2}$.

($\Leftarrow$) Suppose that $\mathcal{X}^{t_1,t_2} \geq \theta$ and $v_\theta^{\mathfrak{r}}(\mathcal{K}) > \frac{1}{2}$. We have $a > \frac{1}{2}$ and then $b^a > \frac{1}{2}$.

If $b = 0$, then $0 > \frac{1}{2}$, i.e., a contradiction. Hence, $0 < b \leq 1$.

If $b = 1$, then $b^a > \frac{1}{2}$ holds true.

Let $0 < b < 1$. We have $b^a > \frac{1}{2}$ if and only if $a < \log_b \frac{1}{2}$. The last inequality is satisfied for $\frac{1}{2} < b < 1$. We conclude, $b > \frac{1}{2}$.

If $b = v_\theta^{\mathfrak{r}}(\&_{\mathcal{B}\in\mathcal{Y}} \mathcal{B})$, then $v_\theta^{\mathfrak{r}}(\mathcal{B}) > \frac{1}{2}$ for all $\mathcal{B} \in \mathcal{Y}$. Hence, $\mathcal{B}^{t_1,t_2} \geq \theta$ for $\mathcal{B} \in \mathcal{Y}$. Now, $\mathcal{Y}^{t_1,t_2} \geq \theta$. Therefore, $\mathcal{X}^{t_1,t_2} \geq \theta$ and $\mathcal{Y}^{t_1,t_2} \geq \theta$ yield the result.

Analogously, if $b = v_{\theta}^{\mathfrak{r}}(\&_{\mathcal{C}\in\mathcal{Z}}\mathcal{C})$, then $\mathcal{Z}^{t_1,t_2} \geq \theta$. Now, $\mathcal{X}^{t_1,t_2} \geq \theta$, $\mathcal{Z}^{t_1,t_2} \geq \theta$ yield the result. This completes the proof. □

Theorem 2. *Let $f \notin \mathfrak{F}$ be a fuzzy functional or a fuzzy multivalued dependency on a set of attributes $\mathfrak{U}$, where $\mathfrak{F}$ is a set of fuzzy functional and fuzzy multivalued dependencies on $\mathfrak{U}$. Let $\mathfrak{F}'$ resp. f' be the set of fuzzy formulas resp. the fuzzy formula related to $\mathfrak{F}$ resp. f. The following two conditions are equivalent:*

(a) Any two-element fuzzy relation instance on scheme $\mathfrak{R}(\mathfrak{U})$ which satisfies all dependencies from the set $\mathfrak{F}$ satisfies also the dependency f.
(b) $v_{\varepsilon}^{\mathfrak{r}}\left(f'\right) > \frac{1}{2}$ for every $v_{\varepsilon}^{\mathfrak{r}}$ such that $v_{\varepsilon}^{\mathfrak{r}}(\mathcal{L}) > \frac{1}{2}$ for all $\mathcal{L} \in \mathfrak{F}'$.

Proof: We denote f by $\mathcal{X} \xrightarrow{\theta_1}_F \mathcal{Y}$ when f is a fuzzy functional dependency and by $\mathcal{X} \rightarrow\xrightarrow{\theta_1}_F \mathcal{Y}$ when f is a fuzzy multivalued dependency. Therefore, $(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A}) \rightarrow (\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B})$ and $(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A}) \rightarrow ((\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B}) \,\|\, (\&_{\mathcal{D}\in\mathcal{Z}}\mathcal{D}))$ will denote f' in the first and the second case, respectively, where $\mathcal{Z} = \mathfrak{U} - \mathcal{X}\mathcal{Y}$.

We may assume that the set $\{p, q\}$ is the domain of each of the attributes in $\mathfrak{U}$.

Fix some $\theta'' \in \left[0, \theta'\right)$, where θ' is the minimum of the strengths of all dependencies that appear in $\mathfrak{F} \cup \{f\}$. Suppose that $\theta' < 1$. Namely, if $\theta' = 1$, then every dependency $f_1 \in \mathfrak{F} \cup \{f\}$ is of the strength 1. This case is not interesting however.

Define $\mathfrak{s}(p,q) = \mathfrak{s}(q,p) = \theta''$ to be a similarity relation on $\{p, q\}$.

(a) ⇒ (b) Suppose that *(b)* is not valid.

Now, there is some $v_{\varepsilon}^{\mathfrak{r}}$ such that $v_{\varepsilon}^{\mathfrak{r}}(\mathcal{L}) > \frac{1}{2}$ for all $\mathcal{L} \in \mathfrak{F}'$ and $v_{\varepsilon}^{\mathfrak{r}}\left(f'\right) \leq \frac{1}{2}$. Here, $v_{\varepsilon}^{\mathfrak{r}}$ is joined to some two-element fuzzy relation instance $\mathfrak{r} = \{t_1, t_2\}$ on $\mathfrak{R}(\mathfrak{U})$ and some ε, $0 \leq \varepsilon \leq 1$.

Define $W = \left\{\mathcal{A} \in \mathfrak{U} \,|\, v_{\varepsilon}^{\mathfrak{r}}(\mathcal{A}) > \frac{1}{2}\right\}$.

Assume that $W = \varnothing$. In this case, $v_{\varepsilon}^{\mathfrak{r}}(\mathcal{A}) \leq \frac{1}{2}$ for all $\mathcal{A} \in \mathfrak{U}$. Hence, $v_{\varepsilon}^{\mathfrak{r}}(\&_{\mathcal{A}\in\mathcal{M}}\mathcal{A}) \leq \frac{1}{2} < 1$ for any $\mathcal{M} \subseteq \mathfrak{U}$.

If $v_{\varepsilon}^{\mathfrak{r}}(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A}) = 0$, $v_{\varepsilon}^{\mathfrak{r}}(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B}) = 0$ resp. $v_{\varepsilon}^{\mathfrak{r}}(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A}) = 0$, $v_{\varepsilon}^{\mathfrak{r}}((\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B}) \,\|\, (\&_{\mathcal{D}\in\mathcal{Z}}\mathcal{D})) = 0$, then $v_{\varepsilon}^{\mathfrak{r}}\left(f'\right) \leq \frac{1}{2}$ yields $1 \leq \frac{1}{2}$, i.e., a contradiction. Hence, $v_{\varepsilon}^{\mathfrak{r}}(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A}) \neq 0$ or $v_{\varepsilon}^{\mathfrak{r}}(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B}) \neq 0$ resp. $v_{\varepsilon}^{\mathfrak{r}}(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A}) \neq 0$ or $\max\left(v_{\varepsilon}^{\mathfrak{r}}(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B}), v_{\varepsilon}^{\mathfrak{r}}(\&_{\mathcal{D}\in\mathcal{Z}}\mathcal{D})\right) \neq 0$.
We may assume that $v_{\varepsilon}^{\mathfrak{r}}(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B}) \neq 0$ resp. $\max\left(v_{\varepsilon}^{\mathfrak{r}}(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B}), v_{\varepsilon}^{\mathfrak{r}}(\&_{\mathcal{D}\in\mathcal{Z}}\mathcal{D})\right) \neq 0$.
Now, $v_{\varepsilon}^{\mathfrak{r}}\left(f'\right) \leq \frac{1}{2}$ implies

$$v_{\varepsilon}^{\mathfrak{r}}\left(\underset{\mathcal{B}\in\mathcal{Y}}{\&}\mathcal{B}\right)^{v_{\varepsilon}^{\mathfrak{r}}(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A})} \leq \frac{1}{2} \tag{2}$$

resp.

$$\max\left(v_{\varepsilon}^{\mathfrak{r}}\left(\underset{\mathcal{B}\in\mathcal{Y}}{\&}\mathcal{B}\right), v_{\varepsilon}^{\mathfrak{r}}\left(\underset{\mathcal{D}\in\mathcal{Z}}{\&}\mathcal{D}\right)\right)^{v_{\varepsilon}^{\mathfrak{r}}(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A})} \leq \frac{1}{2}, \tag{3}$$

i.e.,

$$v_{\varepsilon}^{\mathfrak{r}}\left(\underset{\mathcal{A}\in\mathcal{X}}{\&}\mathcal{A}\right) \geq \log_{v_{\varepsilon}^{\mathfrak{r}}(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B})}\frac{1}{2} \tag{4}$$

resp.

$$v_{\varepsilon}^{\mathfrak{r}}\left(\underset{\mathcal{A}\in\mathcal{X}}{\&}\mathcal{A}\right) \geq \log_{\max(v_{\varepsilon}^{\mathfrak{r}}(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B}), v_{\varepsilon}^{\mathfrak{r}}(\&_{\mathcal{D}\in\mathcal{Z}}\mathcal{D}))}\frac{1}{2}. \tag{5}$$

Therefore, $0 < v_{\varepsilon}^{\mathfrak{r}}(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B}) \leq \frac{1}{2}$ resp. $0 < \max\left(v_{\varepsilon}^{\mathfrak{r}}(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B}), v_{\varepsilon}^{\mathfrak{r}}(\&_{\mathcal{D}\in\mathcal{Z}}\mathcal{D})\right) \leq \frac{1}{2}$ yields $v_{\varepsilon}^{\mathfrak{r}}(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A}) = 1$. This is a contradiction. Hence, $W \neq \varnothing$.

Assume that that $W = \mathfrak{U}$. In this case, $v_{\varepsilon}^{\mathfrak{r}}(\mathcal{A}) > \frac{1}{2}$ for all $\mathcal{A} \in \mathfrak{U}$. Consequently, $v_{\varepsilon}^{\mathfrak{r}}(\&_{\mathcal{A}\in\mathcal{M}}\mathcal{A}) > \frac{1}{2}$ for all $\mathcal{M} \subseteq \mathfrak{U}$.

Now, (2) resp. (3) holds true.

If $v_{\varepsilon}^{\mathfrak{r}}(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B}) = 1$ resp. $\max\left(v_{\varepsilon}^{\mathfrak{r}}(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B}), v_{\varepsilon}^{\mathfrak{r}}(\&_{\mathcal{D}\in\mathcal{Z}}\mathcal{D})\right) = 1$, then $1 \leq \frac{1}{2}$, i.e., a contradiction. Hence, (4) resp. (5) holds true. Therefore, $\frac{1}{2} < v_{\varepsilon}^{\mathfrak{r}}(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B}) < 1$ resp. $\frac{1}{2} < \max\left(v_{\varepsilon}^{\mathfrak{r}}(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B}), v_{\varepsilon}^{\mathfrak{r}}(\&_{\mathcal{D}\in\mathcal{Z}}\mathcal{D})\right) < 1$ yields $v_{\varepsilon}^{\mathfrak{r}}(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A}) > 1$. This is a contradiction. We obtain, $W \neq \mathfrak{U}$.

Define $\mathfrak{r}' = \left\{t', t''\right\}$ by Table 1 below.

$\mathfrak{r}'$ is a two-element fuzzy relation instance on $\mathfrak{R}(\mathfrak{U})$.

We shall prove that this instance satisfies all dependencies from the set $\mathfrak{F}$, but violates the dependency f.

Table 1:

	attributes of W	other attributes
t'	$p, p, \dots, p$	$p, p, \dots, p$
t''	$p, p, \dots, p$	$q, q, \dots, q$

Let $\mathcal{K} \xrightarrow{\theta_2}_F \mathcal{L}$ be any fuzzy functional dependency from the set $\mathfrak{F}$.

Assume that $v_{\varepsilon}^{\mathfrak{r}}\left(\&_{\mathcal{A}\in\mathcal{K}}\mathcal{A}\right)\leq\frac{1}{2}$. Then, there exists $\mathcal{A}_0\in\mathcal{K}$ such that

$$\begin{aligned} v_{\varepsilon}^{\mathfrak{r}}\left(\mathcal{A}_0\right) &= \min\left\{v_{\varepsilon}^{\mathfrak{r}}\left(\mathcal{A}\right)\mid\mathcal{A}\in\mathcal{K}\right\} \\ &= v_{\varepsilon}^{\mathfrak{r}}\left(\underset{\mathcal{A}\in\mathcal{K}}{\&}\mathcal{A}\right)\leq\frac{1}{2}, \end{aligned}$$

i.e., $\mathcal{A}_0\notin W$. We have $\mathcal{A}_0^{t',t''}=\theta''$ and hence

$$\mathcal{K}^{t',t''}=\min_{\mathcal{A}\in\mathcal{K}}\left\{\mathcal{A}^{t',t''}\right\}=\theta''.$$

Since $\mathfrak{s}\left(p,q\right)=\theta''$, we know that $\mathcal{M}^{t',t''}\geq\theta''$ for any set of attributes $\mathcal{M}\subseteq\mathfrak{U}$. Therefore, $\mathcal{L}^{t',t''}\geq\theta''$. We obtain,

$$\mathcal{L}^{t',t''}\geq\theta''=\min\left(\theta_2,\mathcal{K}^{t',t''}\right),$$

i.e., $\mathfrak{r}'$ satisfies $\mathcal{K}\xrightarrow{\theta_2}_F\mathcal{L}$.

Assume that $v_{\varepsilon}^{\mathfrak{r}}\left(\&_{\mathcal{A}\in\mathcal{K}}\mathcal{A}\right)>\frac{1}{2}$. Now,

$$\begin{aligned} & v_{\varepsilon}^{\mathfrak{r}}\left(\underset{\mathcal{B}\in\mathcal{L}}{\&}\mathcal{B}\right)^{v_{\varepsilon}^{\mathfrak{r}}\left(\&_{\mathcal{A}\in\mathcal{K}}\mathcal{A}\right)} \\ &= v_{\varepsilon}^{\mathfrak{r}}\left(\left(\underset{\mathcal{A}\in\mathcal{K}}{\&}\mathcal{A}\right)\to\left(\underset{\mathcal{B}\in\mathcal{L}}{\&}\mathcal{B}\right)\right)>\frac{1}{2}. \end{aligned}$$

The last inequality is satisfied if $v_{\varepsilon}^{\mathfrak{r}}\left(\&_{\mathcal{B}\in\mathcal{L}}\mathcal{B}\right)=1$. If $v_{\varepsilon}^{\mathfrak{r}}\left(\&_{\mathcal{B}\in\mathcal{L}}\mathcal{B}\right)=0$, then $0>\frac{1}{2}$, i.e., a contradiction.

Let $0<v_{\varepsilon}^{\mathfrak{r}}\left(\&_{\mathcal{B}\in\mathcal{L}}\mathcal{B}\right)<1$. We have,

$$v_{\varepsilon}^{\mathfrak{r}}\left(\underset{\mathcal{A}\in\mathcal{K}}{\&}\mathcal{A}\right)<\log_{v_{\varepsilon}^{\mathfrak{r}}\left(\&_{\mathcal{B}\in\mathcal{L}}\mathcal{B}\right)}\frac{1}{2}.$$

Therefore, $v_{\varepsilon}^{\mathfrak{r}}\left(\&_{\mathcal{B}\in\mathcal{L}}\mathcal{B}\right)>\frac{1}{2}$. Now, $v_{\varepsilon}^{\mathfrak{r}}\left(\mathcal{B}\right)>\frac{1}{2}$ for all $\mathcal{B}\in\mathcal{L}$ and then $\mathcal{B}\in W$ for $\mathcal{B}\in\mathcal{L}$. We obtain, $\mathcal{L}\subseteq W$. Hence, $\mathcal{L}^{t',t''}=1$. We have,

$$\mathcal{L}^{t',t''}=1\geq\min\left(\theta_2,\mathcal{K}^{t',t''}\right),$$

i.e., $\mathfrak{r}'$ satisfies the dependency $\mathcal{K}\xrightarrow{\theta_2}_F\mathcal{L}$

Let $\mathcal{K}\to\xrightarrow{\theta_2}_F\mathcal{L}$ be any fuzzy multivalued dependency from the set $\mathfrak{F}$.

Suppose that $v_{\varepsilon}^{\mathfrak{r}}\left(\&_{\mathcal{A}\in\mathcal{K}}\mathcal{A}\right)\leq\frac{1}{2}$. Then, reasoning as in the previous case, we obtain that $\mathcal{K}^{t',t''}=\theta''$. Hence, there is $t'''\in\mathfrak{r}'$, $t'''=t'$ such that

$$\begin{aligned} \mathcal{K}^{t''',t'} &= 1\geq\min\left(\theta_2,\mathcal{K}^{t',t''}\right), \\ \mathcal{L}^{t''',t'} &= 1\geq\min\left(\theta_2,\mathcal{K}^{t',t''}\right), \\ \mathcal{M}^{t''',t''} &\geq\theta''=\min\left(\theta_2,\mathcal{K}^{t',t''}\right), \end{aligned}$$

where $\mathcal{M}=\mathfrak{U}-\mathcal{K}\mathcal{L}$. Therefore, $\mathfrak{r}'$ satisfies $\mathcal{K}\to\xrightarrow{\theta_2}_F\mathcal{L}$.

Let $v_{\varepsilon}^{\mathfrak{r}}\left(\&_{\mathcal{A}\in\mathcal{K}}\mathcal{A}\right)>\frac{1}{2}$. Now,

$$\begin{aligned} & \max\left(v_{\varepsilon}^{\mathfrak{r}}\left(\underset{\mathcal{B}\in\mathcal{L}}{\&}\mathcal{B}\right),v_{\varepsilon}^{\mathfrak{r}}\left(\underset{\mathcal{D}\in\mathcal{M}}{\&}\mathcal{D}\right)\right)^{v_{\varepsilon}^{\mathfrak{r}}\left(\&_{\mathcal{A}\in\mathcal{K}}\mathcal{A}\right)} \\ &= v_{\varepsilon}^{\mathfrak{r}}\left(\left(\underset{\mathcal{B}\in\mathcal{L}}{\&}\mathcal{B}\right)\parallel\left(\underset{\mathcal{D}\in\mathcal{M}}{\&}\mathcal{D}\right)\right)^{v_{\varepsilon}^{\mathfrak{r}}\left(\&_{\mathcal{A}\in\mathcal{K}}\mathcal{A}\right)} \\ &= v_{\varepsilon}^{\mathfrak{r}}\left(\left(\underset{\mathcal{A}\in\mathcal{K}}{\&}\mathcal{A}\right)\to\left(\left(\underset{\mathcal{B}\in\mathcal{L}}{\&}\mathcal{B}\right)\parallel\left(\underset{\mathcal{D}\in\mathcal{M}}{\&}\mathcal{D}\right)\right)\right) \\ &> \frac{1}{2}. \end{aligned}$$

This inequality is satisfied if $\max\left(v_{\varepsilon}^{\mathfrak{r}}\left(\&_{\mathcal{B}\in\mathcal{L}}\mathcal{B}\right),v_{\varepsilon}^{\mathfrak{r}}\left(\&_{\mathcal{D}\in\mathcal{M}}\mathcal{D}\right)\right)=1$.

If $\max\left(v_{\varepsilon}^{\mathfrak{r}}\left(\&_{\mathcal{B}\in\mathcal{L}}\mathcal{B}\right),v_{\varepsilon}^{\mathfrak{r}}\left(\&_{\mathcal{D}\in\mathcal{M}}\mathcal{D}\right)\right)=0$, then $0>\frac{1}{2}$, i.e., a contradiction.

If $0<\max\left(v_{\varepsilon}^{\mathfrak{r}}\left(\&_{\mathcal{B}\in\mathcal{L}}\mathcal{B}\right),v_{\varepsilon}^{\mathfrak{r}}\left(\&_{\mathcal{D}\in\mathcal{M}}\mathcal{D}\right)\right)<1$, then

$$v_{\varepsilon}^{\mathfrak{r}}\left(\underset{\mathcal{A}\in\mathcal{K}}{\&}\mathcal{A}\right)<\log_{\max\left(v_{\varepsilon}^{\mathfrak{r}}\left(\&_{\mathcal{B}\in\mathcal{L}}\mathcal{B}\right),v_{\varepsilon}^{\mathfrak{r}}\left(\&_{\mathcal{D}\in\mathcal{M}}\mathcal{D}\right)\right)}\frac{1}{2}.$$

Therefore, $\max\left(v_{\varepsilon}^{\mathfrak{r}}\left(\&_{\mathcal{B}\in\mathcal{L}}\mathcal{B}\right),v_{\varepsilon}^{\mathfrak{r}}\left(\&_{\mathcal{D}\in\mathcal{M}}\mathcal{D}\right)\right)>\frac{1}{2}$. Hence, $v_{\varepsilon}^{\mathfrak{r}}\left(\&_{\mathcal{B}\in\mathcal{L}}\mathcal{B}\right)>\frac{1}{2}$ or $v_{\varepsilon}^{\mathfrak{r}}\left(\&_{\mathcal{D}\in\mathcal{M}}\mathcal{D}\right)>\frac{1}{2}$.

If $v_{\varepsilon}^{\mathfrak{r}}\left(\&_{\mathcal{B}\in\mathcal{L}}\mathcal{B}\right)>\frac{1}{2}$, then $\mathcal{L}\subseteq W$ and hence $\mathcal{L}^{t',t''}=1$. Similarly, since $v_{\varepsilon}^{\mathfrak{r}}\left(\&_{\mathcal{A}\in\mathcal{K}}\mathcal{A}\right)>\frac{1}{2}$, we conclude that $\mathcal{K}^{t',t''}=1$. Now, there is $t'''\in\mathfrak{r}'$, $t'''=t''$ such that

$$\begin{aligned} \mathcal{K}^{t''',t'} &= 1\geq\min\left(\theta_2,\mathcal{K}^{t',t''}\right), \\ \mathcal{L}^{t''',t'} &= 1\geq\min\left(\theta_2,\mathcal{K}^{t',t''}\right), \\ \mathcal{M}^{t''',t''} &= 1\geq\min\left(\theta_2,\mathcal{K}^{t',t''}\right). \end{aligned} \tag{6}$$

Hence, $\mathfrak{r}'$ satisfies $\mathcal{K}\to\xrightarrow{\theta_2}_F\mathcal{L}$.

If $v_{\varepsilon}^{\mathfrak{r}}\left(\&_{\mathcal{D}\in\mathcal{M}}\mathcal{D}\right)>\frac{1}{2}$, then $\mathcal{M}^{t',t''}=1$. In this case, there is $t'''\in\mathfrak{r}'$, $t'''=t'$ such that (6) holds true. In other words, $\mathfrak{r}'$ satisfies the dependency $\mathcal{K}\to\xrightarrow{\theta_2}_F\mathcal{L}$.

It remains to prove that the instance $\mathfrak{r}'$ violates $\mathcal{X}\xrightarrow{\theta_1}_F\mathcal{Y}$ resp. $\mathcal{X}\to\xrightarrow{\theta_1}_F\mathcal{Y}$.

Let

$$v_{\varepsilon}^{\mathfrak{r}}\left(\left(\underset{\mathcal{A}\in\mathcal{X}}{\&}\mathcal{A}\right)\to\left(\underset{\mathcal{B}\in\mathcal{Y}}{\&}\mathcal{B}\right)\right)=v_{\varepsilon}^{\mathfrak{r}}\left(f'\right)\leq\frac{1}{2}.$$

If $v_{\varepsilon}^{\mathfrak{r}}\left(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A}\right)=0$ and $v_{\varepsilon}^{\mathfrak{r}}\left(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B}\right)=0$, then $1\leq\frac{1}{2}$, i.e., a contradiction. Hence, $v_{\varepsilon}^{\mathfrak{r}}\left(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A}\right)\neq0$ or

$v_\varepsilon^{\mathfrak{r}}\left(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B}\right)\neq 0$. We may assume that $v_\varepsilon^{\mathfrak{r}}\left(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B}\right)\neq 0$. Now,

$$v_\varepsilon^{\mathfrak{r}}\left(\underset{\mathcal{B}\in\mathcal{Y}}{\&}\mathcal{B}\right)^{v_\varepsilon^{\mathfrak{r}}\left(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A}\right)}\leq\frac{1}{2}.$$

If $v_\varepsilon^{\mathfrak{r}}\left(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B}\right)=1$, then $1\leq\frac{1}{2}$, i.e., a contradiction. Therefore, $0<v_\varepsilon^{\mathfrak{r}}\left(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B}\right)<1$. We obtain,

$$v_\varepsilon^{\mathfrak{r}}\left(\underset{\mathcal{A}\in\mathcal{X}}{\&}\mathcal{A}\right)\geq\log_{v_\varepsilon^{\mathfrak{r}}\left(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B}\right)}\frac{1}{2}.$$

If $v_\varepsilon^{\mathfrak{r}}\left(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B}\right)>\frac{1}{2}$, then $v_\varepsilon^{\mathfrak{r}}\left(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A}\right)>1$, i.e., a contradiction. Hence, $0<v_\varepsilon^{\mathfrak{r}}\left(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B}\right)\leq\frac{1}{2}$ and then $v_\varepsilon^{\mathfrak{r}}\left(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A}\right)=1$. Now, as before, we conclude that $\mathcal{Y}^{t',t''}=\theta''$ and $\mathcal{X}^{t',t''}=1$. Therefore,

$$\mathcal{Y}^{t',t''}=\theta''<\theta'\leq\theta_1=\min\left(\theta_1,\mathcal{X}^{t',t''}\right).$$

This means that $\mathfrak{r}'$ violates $\mathcal{X}\xrightarrow{\theta_1}_F\mathcal{Y}$.

Now, let

$$v_\varepsilon^{\mathfrak{r}}\left(\left(\underset{\mathcal{A}\in\mathcal{X}}{\&}\mathcal{A}\right)\rightarrow\left(\left(\underset{\mathcal{B}\in\mathcal{Y}}{\&}\mathcal{B}\right)\parallel\left(\underset{\mathcal{D}\in\mathcal{Z}}{\&}\mathcal{D}\right)\right)\right)$$
$$=v_\varepsilon^{\mathfrak{r}}\left(f'\right)\leq\frac{1}{2}.$$

Reasoning as in the previous case, we conclude that $v_\varepsilon^{\mathfrak{r}}\left(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A}\right)\neq 0$ or $\max\left(v_\varepsilon^{\mathfrak{r}}\left(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B}\right),v_\varepsilon^{\mathfrak{r}}\left(\&_{\mathcal{D}\in\mathcal{Z}}\mathcal{D}\right)\right)\neq 0$.

Assume that $\max\left(v_\varepsilon^{\mathfrak{r}}\left(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B}\right),v_\varepsilon^{\mathfrak{r}}\left(\&_{\mathcal{D}\in\mathcal{Z}}\mathcal{D}\right)\right)\neq 0$. We have,

$$\max\left(v_\varepsilon^{\mathfrak{r}}\left(\underset{\mathcal{B}\in\mathcal{Y}}{\&}\mathcal{B}\right),v_\varepsilon^{\mathfrak{r}}\left(\underset{\mathcal{D}\in\mathcal{Z}}{\&}\mathcal{D}\right)\right)^{v_\varepsilon^{\mathfrak{r}}\left(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A}\right)}\leq\frac{1}{2}.$$

Then, $0<\max\left(v_\varepsilon^{\mathfrak{r}}\left(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B}\right),v_\varepsilon^{\mathfrak{r}}\left(\&_{\mathcal{D}\in\mathcal{Z}}\mathcal{D}\right)\right)\leq\frac{1}{2}$ and $v_\varepsilon^{\mathfrak{r}}\left(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A}\right)=1$, i.e., $v_\varepsilon^{\mathfrak{r}}\left(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B}\right)\leq\frac{1}{2}$, $v_\varepsilon^{\mathfrak{r}}\left(\&_{\mathcal{D}\in\mathcal{Z}}\mathcal{D}\right)\leq\frac{1}{2}$, $v_\varepsilon^{\mathfrak{r}}\left(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A}\right)=1$. We obtain, $\mathcal{Y}^{t',t''}=\theta''$, $\mathcal{Z}^{t',t''}=\theta''$, $\mathcal{X}^{t',t''}=1$.

If $t'''\in\mathfrak{r}'$, $t'''=t'$, then

$$\mathcal{X}^{t''',t'}=1\geq\min\left(\theta_1,\mathcal{X}^{t',t''}\right),$$
$$\mathcal{Y}^{t''',t'}=1\geq\min\left(\theta_1,\mathcal{X}^{t',t''}\right),$$
$$\mathcal{Z}^{t''',t''}=\theta''<\theta'\leq\theta_1=\min\left(\theta_1,\mathcal{X}^{t',t''}\right).$$

If $t'''\in\mathfrak{r}'$, $t'''=t''$, then

$$\mathcal{X}^{t''',t'}=1\geq\min\left(\theta_1,\mathcal{X}^{t',t''}\right),$$
$$\mathcal{Y}^{t''',t'}=\theta''<\theta'\leq\theta_1=\min\left(\theta_1,\mathcal{X}^{t',t''}\right),$$
$$\mathcal{Z}^{t''',t''}=1\geq\min\left(\theta_1,\mathcal{X}^{t',t''}\right).$$

In other words, the instance $\mathfrak{r}'$ violates $\mathcal{X}\rightarrow\xrightarrow{\theta_1}_F\mathcal{Y}$.

(b) ⇒ (a) Suppose that *(a)* is not valid.

Now, there is a two-element fuzzy relation instance $\mathfrak{r}'=\left\{t',t''\right\}$ on scheme $\mathfrak{R}(\mathfrak{U})$, such that $\mathfrak{r}'$ satisfies all dependencies in $\mathfrak{F}$ and $\mathfrak{r}'$ does not satisfy f. Therefore, $\mathfrak{r}'$ does not satisfy $\mathcal{X}\xrightarrow{\theta_1}_F\mathcal{Y}$ resp. $\mathcal{X}\rightarrow\xrightarrow{\theta_1}_F\mathcal{Y}$.

Define $W=\left\{\mathcal{A}\in\mathfrak{U}\,|\,\mathcal{A}^{t',t''}=1\right\}$.

Assume that $W=\varnothing$. Now, $\mathcal{A}^{t',t''}=\theta''$ for all $\mathcal{A}\in\mathfrak{U}$. Therefore, $\mathcal{M}^{t',t''}=\theta''$ for all $\mathcal{M}\subseteq\mathfrak{U}$.

In the case when $\mathfrak{r}'$ does not satisfy $\mathcal{X}\xrightarrow{\theta_1}_F\mathcal{Y}$, we obtain

$$\mathcal{Y}^{t',t''}<\min\left(\theta_1,\mathcal{X}^{t',t''}\right),$$

i.e., $\theta''<\min\left(\theta_1,\theta''\right)=\theta''$. This is a contradiction.

Similarly, in the case when $\mathfrak{r}'$ does not satisfy $\mathcal{X}\rightarrow\xrightarrow{\theta_1}_F\mathcal{Y}$, we have that the conditions

$$\begin{aligned}\mathcal{X}^{t',t'}&\geq\min\left(\theta_1,\mathcal{X}^{t',t''}\right),\\ \mathcal{Y}^{t',t'}&\geq\min\left(\theta_1,\mathcal{X}^{t',t''}\right),\\ \mathcal{Z}^{t',t''}&\geq\min\left(\theta_1,\mathcal{X}^{t',t''}\right)\end{aligned}\tag{7}$$

don't hold simultaneously. Since the first and the second condition in (7) hold obviously true, we obtain

$$\theta''=\mathcal{Z}^{t',t''}<\min\left(\theta_1,\mathcal{X}^{t',t''}\right)$$
$$=\min\left(\theta_1,\theta''\right)=\theta'',$$

which is a contradiction. Therefore, $W\neq\varnothing$.

Assume that $W=\mathfrak{U}$. Now, $\mathcal{A}^{t',t''}=1$ for every $\mathcal{A}\in\mathfrak{U}$. Therefore, $\mathcal{M}^{t',t''}=1$ for every $\mathcal{M}\subseteq\mathfrak{U}$.

In the case when $\mathfrak{r}'$ does not satisfy $\mathcal{X}\xrightarrow{\theta_1}_F\mathcal{Y}$, we have that

$$1=\mathcal{Y}^{t',t''}<\min\left(\theta_1,\mathcal{X}^{t',t''}\right)=\min\left(\theta_1,1\right)=\theta_1.$$

This is a contradiction.

In the case when $\mathfrak{r}'$ does not satisfy $\mathcal{X}\rightarrow\xrightarrow{\theta_1}_F\mathcal{Y}$, the conditions given by (7) don't hold simultaneously. The first and the second condition in (7) are always satisfied, hence

$$1=\mathcal{Z}^{t',t''}<\min\left(\theta_1,\mathcal{X}^{t',t''}\right)=\min\left(\theta_1,1\right)=\theta_1.$$

This is a contradiction. We conclude, $W \neq \mathfrak{U}$.

Now, we define $v_1^{\mathfrak{r}'}$ in the following way. Let

$$\frac{1}{2} < v_1^{\mathfrak{r}'}(\mathcal{A}) \leq 1 \quad \text{if} \quad \mathcal{A} \in W,$$
$$0 \leq v_1^{\mathfrak{r}'}(\mathcal{A}) \leq \frac{1}{2} \quad \text{if} \quad \mathcal{A} \in \mathfrak{U} - W.$$

We shall prove that $v_1^{\mathfrak{r}'}(\mathcal{L}) > \frac{1}{2}$ for every $\mathcal{L} \in \mathfrak{F}'$ and $v_1^{\mathfrak{r}'}\left(f'\right) \leq \frac{1}{2}$.

Suppose that $\mathcal{L} \in \mathfrak{F}'$ is of the form

$$\left(\underset{\mathcal{A}\in\mathcal{K}}{\&} \mathcal{A}\right) \to \left(\underset{\mathcal{B}\in\mathcal{L}}{\&} \mathcal{B}\right).$$

This fuzzy formula corresponds to some fuzzy functional dependency $\mathcal{K} \xrightarrow{\theta_2}_F \mathcal{L}$ from the set $\mathfrak{F}$.

Suppose that $v_1^{\mathfrak{r}'}(\mathcal{L}) \leq \frac{1}{2}$. Then, as earlier, it follows that $v_1^{\mathfrak{r}'}(\&_{\mathcal{A}\in\mathcal{K}} \mathcal{A}) \neq 0$ or $v_1^{\mathfrak{r}'}(\&_{\mathcal{B}\in\mathcal{L}} \mathcal{B}) \neq 0$.

Assume that $v_1^{\mathfrak{r}'}(\&_{\mathcal{B}\in\mathcal{L}} \mathcal{B}) \neq 0$. We have,

$$v_1^{\mathfrak{r}'}\left(\underset{\mathcal{B}\in\mathcal{L}}{\&} \mathcal{B}\right)^{v_1^{\mathfrak{r}'}(\&_{\mathcal{A}\in\mathcal{K}} \mathcal{A})} \leq \frac{1}{2}.$$

Then, $v_1^{\mathfrak{r}'}(\&_{\mathcal{B}\in\mathcal{L}} \mathcal{B}) < 1$. We obtain,

$$v_1^{\mathfrak{r}'}\left(\underset{\mathcal{A}\in\mathcal{K}}{\&} \mathcal{A}\right) \geq \log_{v_1^{\mathfrak{r}'}(\&_{\mathcal{B}\in\mathcal{L}} \mathcal{B})} \frac{1}{2}.$$

Therefore, $0 < v_1^{\mathfrak{r}'}(\&_{\mathcal{B}\in\mathcal{L}} \mathcal{B}) \leq \frac{1}{2}$ and $v_1^{\mathfrak{r}'}(\&_{\mathcal{A}\in\mathcal{K}} \mathcal{A}) = 1$, i.e., $\mathcal{L}^{t',t''} = \theta''$ and $\mathcal{K}^{t',t''} = 1$. We obtain,

$$\begin{aligned} \mathcal{L}^{t',t''} &= \theta'' < \theta' \leq \theta_2 = \min(\theta_2, 1) \\ &= \min\left(\theta_2, \mathcal{K}^{t',t''}\right), \end{aligned}$$

which contradicts the fact that $\mathfrak{r}'$ satisfies $\mathcal{K} \xrightarrow{\theta_2}_F \mathcal{L}$. Therefore, $v_1^{\mathfrak{r}'}(\mathcal{L}) > \frac{1}{2}$.

Suppose that $\mathcal{L} \in \mathfrak{F}'$ is of the form

$$\left(\underset{\mathcal{A}\in\mathcal{K}}{\&} \mathcal{A}\right) \to \left(\left(\underset{\mathcal{B}\in\mathcal{L}}{\&} \mathcal{B}\right) \parallel \left(\underset{\mathcal{D}\in\mathcal{M}}{\&} \mathcal{D}\right)\right),$$

where $\mathcal{M} = \mathfrak{U} - \mathcal{K}\mathcal{L}$. This fuzzy formula corresponds to some fuzzy multivalued dependency $\mathcal{K} \to\xrightarrow{\theta_2}_F \mathcal{L}$ from the set $\mathfrak{F}$.

Assume that $v_1^{\mathfrak{r}'}(\mathcal{L}) \leq \frac{1}{2}$.

As before, we have that $v_1^{\mathfrak{r}'}(\&_{\mathcal{A}\in\mathcal{K}} \mathcal{A}) \neq 0$ or $\max\left(v_1^{\mathfrak{r}'}(\&_{\mathcal{B}\in\mathcal{L}} \mathcal{B}), v_1^{\mathfrak{r}'}(\&_{\mathcal{D}\in\mathcal{M}} \mathcal{D})\right) \neq 0$.

Suppose that $\max\left(v_1^{\mathfrak{r}'}(\&_{\mathcal{B}\in\mathcal{L}} \mathcal{B}), v_1^{\mathfrak{r}'}(\&_{\mathcal{D}\in\mathcal{M}} \mathcal{D})\right) \neq 0$. We have,

$$\begin{aligned} &\max\left(v_1^{\mathfrak{r}'}\left(\underset{\mathcal{B}\in\mathcal{L}}{\&} \mathcal{B}\right), v_1^{\mathfrak{r}'}\left(\underset{\mathcal{D}\in\mathcal{M}}{\&} \mathcal{D}\right)\right)^{v_1^{\mathfrak{r}'}(\&_{\mathcal{A}\in\mathcal{K}} \mathcal{A})} \\ &\leq \frac{1}{2}. \end{aligned}$$

Then, $\max\left(v_1^{\mathfrak{r}'}(\&_{\mathcal{B}\in\mathcal{L}} \mathcal{B}), v_1^{\mathfrak{r}'}(\&_{\mathcal{D}\in\mathcal{M}} \mathcal{D})\right) < 1$. We obtain,

$$v_1^{\mathfrak{r}'}\left(\underset{\mathcal{A}\in\mathcal{K}}{\&} \mathcal{A}\right) \geq \log_{\max\left(v_1^{\mathfrak{r}'}(\&_{\mathcal{B}\in\mathcal{L}} \mathcal{B}), v_1^{\mathfrak{r}'}(\&_{\mathcal{D}\in\mathcal{M}} \mathcal{D})\right)} \frac{1}{2}.$$

Therefore,
$0 < \max\left(v_1^{\mathfrak{r}'}(\&_{\mathcal{B}\in\mathcal{L}} \mathcal{B}), v_1^{\mathfrak{r}'}(\&_{\mathcal{D}\in\mathcal{M}} \mathcal{D})\right) \leq \frac{1}{2}$ and $v_1^{\mathfrak{r}'}(\&_{\mathcal{A}\in\mathcal{K}} \mathcal{A}) = 1$, i.e., $v_1^{\mathfrak{r}'}(\&_{\mathcal{B}\in\mathcal{L}} \mathcal{B}) \leq \frac{1}{2}$, $v_1^{\mathfrak{r}'}(\&_{\mathcal{D}\in\mathcal{M}} \mathcal{D}) \leq \frac{1}{2}$, $v_1^{\mathfrak{r}'}(\&_{\mathcal{A}\in\mathcal{K}} \mathcal{A}) = 1$. Hence, $\mathcal{L}^{t',t''} = \theta''$, $\mathcal{M}^{t',t''} = \theta''$, $\mathcal{K}^{t',t''} = 1$. In this case, the third condition of the conditions

$$\begin{aligned} \mathcal{K}^{t',t'} &\geq \min\left(\theta_2, \mathcal{K}^{t',t''}\right), \\ \mathcal{L}^{t',t'} &\geq \min\left(\theta_2, \mathcal{K}^{t',t''}\right), \\ \mathcal{M}^{t',t''} &\geq \min\left(\theta_2, \mathcal{K}^{t',t''}\right) \end{aligned}$$

does not hold. Furthermore, the second condition of the conditions

$$\begin{aligned} \mathcal{K}^{t'',t'} &\geq \min\left(\theta_2, \mathcal{K}^{t',t''}\right), \\ \mathcal{L}^{t'',t'} &\geq \min\left(\theta_2, \mathcal{K}^{t',t''}\right), \\ \mathcal{M}^{t'',t''} &\geq \min\left(\theta_2, \mathcal{K}^{t',t''}\right) \end{aligned}$$

does not hold. This contradicts the fact that $\mathfrak{r}'$ satisfies the dependency $\mathcal{K} \to\xrightarrow{\theta_2}_F \mathcal{L}$. Hence, $v_1^{\mathfrak{r}'}(\mathcal{L}) > \frac{1}{2}$.

It remains to prove that $v_1^{\mathfrak{r}'}\left(f'\right) \leq \frac{1}{2}$.

Suppose that the instance $\mathfrak{r}'$ does not satisfy the dependency $\mathcal{X} \xrightarrow{\theta_1}_F \mathcal{Y}$.

Assume that $v_1^{\mathfrak{r}'}\left(f'\right) > \frac{1}{2}$.

If $v_1^{\mathfrak{r}'}(\&_{\mathcal{A}\in\mathcal{X}} \mathcal{A}) \leq \frac{1}{2}$, then $\mathcal{X}^{t',t''} = \theta''$. Hence,

$$\mathcal{Y}^{t',t''} \geq \theta'' = \min\left(\theta_1, \theta''\right) = \min\left(\theta_1, \mathcal{X}^{t',t''}\right).$$

This contradicts the fact that $\mathfrak{r}'$ violates $\mathcal{X} \xrightarrow{\theta_1}_F \mathcal{Y}$.

If $v_1^{\mathfrak{r}'}(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A}) > \frac{1}{2}$, then

$$v_1^{\mathfrak{r}'}\left(\underset{\mathcal{B}\in\mathcal{Y}}{\&}\mathcal{B}\right)^{v_1^{\mathfrak{r}'}(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A})} > \frac{1}{2}.$$

This inequality is satisfied if $v_1^{\mathfrak{r}'}(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B}) = 1$.

If $v_1^{\mathfrak{r}'}(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B}) = 0$, then $0 > \frac{1}{2}$, i.e., a contradiction.

If $0 < v_1^{\mathfrak{r}'}(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B}) < 1$, then

$$v_1^{\mathfrak{r}'}\left(\underset{\mathcal{A}\in\mathcal{X}}{\&}\mathcal{A}\right) < \log_{v_1^{\mathfrak{r}'}(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B})}\frac{1}{2}.$$

Therefore, $v_1^{\mathfrak{r}'}(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B}) > \frac{1}{2}$, i.e., $\mathcal{Y}^{t',t''} = 1$. Now,

$$\mathcal{Y}^{t',t''} \geq \min\left(\theta_1, \mathcal{X}^{t',t''}\right),$$

which is a contradiction. We conclude, $v_1^{\mathfrak{r}'}\left(f'\right) \leq \frac{1}{2}$.

Suppose that $\mathfrak{r}'$ does not satisfy $\mathcal{X} \to\!\xrightarrow{\theta_1}_F \mathcal{Y}$.

Now, the third condition of the conditions given by (7) does not hold, i.e.,

$$\mathcal{Z}^{t',t''} < \min\left(\theta_1, \mathcal{X}^{t',t''}\right). \tag{8}$$

Moreover, the first and the second condition of the conditions

$$\begin{aligned} \mathcal{X}^{t'',t'} &\geq \min\left(\theta_1, \mathcal{X}^{t',t''}\right), \\ \mathcal{Y}^{t'',t'} &\geq \min\left(\theta_1, \mathcal{X}^{t',t''}\right), \\ \mathcal{Z}^{t'',t''} &\geq \min\left(\theta_1, \mathcal{X}^{t',t''}\right) \end{aligned} \tag{9}$$

don't hold simultaneously.

Assume that $v_1^{\mathfrak{r}'}\left(f'\right) > \frac{1}{2}$.

If $v_1^{\mathfrak{r}'}(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A}) \leq \frac{1}{2}$, then $\mathcal{X}^{t',t''} = \theta''$. Hence,

$$\mathcal{Z}^{t',t''} \geq \theta'' = \min\left(\theta_1, \theta''\right) = \min\left(\theta_1, \mathcal{X}^{t',t''}\right),$$

which contradicts (8).

If $v_1^{\mathfrak{r}'}(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A}) > \frac{1}{2}$, then $\mathcal{X}^{t',t''} = 1$ and

$$\max\left(v_1^{\mathfrak{r}'}\left(\underset{\mathcal{B}\in\mathcal{Y}}{\&}\mathcal{B}\right), v_1^{\mathfrak{r}'}\left(\underset{\mathcal{D}\in\mathcal{Z}}{\&}\mathcal{D}\right)\right)^{v_1^{\mathfrak{r}'}(\&_{\mathcal{A}\in\mathcal{X}}\mathcal{A})} > \frac{1}{2}.$$

The last inequality is satisfied if
$\max\left(v_1^{\mathfrak{r}'}(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B}), v_1^{\mathfrak{r}'}(\&_{\mathcal{D}\in\mathcal{Z}}\mathcal{D})\right) = 1$.

If $\max\left(v_1^{\mathfrak{r}'}(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B}), v_1^{\mathfrak{r}'}(\&_{\mathcal{D}\in\mathcal{Z}}\mathcal{D})\right) = 0$, then $0 > \frac{1}{2}$, i.e., a contradiction.

If $0 < \max\left(v_1^{\mathfrak{r}'}(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B}), v_1^{\mathfrak{r}'}(\&_{\mathcal{D}\in\mathcal{Z}}\mathcal{D})\right) < 1$, then

$$v_1^{\mathfrak{r}'}\left(\underset{\mathcal{A}\in\mathcal{X}}{\&}\mathcal{A}\right) < \log_{\max\left(v_1^{\mathfrak{r}'}(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B}), v_1^{\mathfrak{r}'}(\&_{\mathcal{D}\in\mathcal{Z}}\mathcal{D})\right)}\frac{1}{2}.$$

Therefore,
$\max\left(v_1^{\mathfrak{r}'}(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B}), v_1^{\mathfrak{r}'}(\&_{\mathcal{D}\in\mathcal{Z}}\mathcal{D})\right) > \frac{1}{2}$, i.e.,
$v_1^{\mathfrak{r}'}(\&_{\mathcal{B}\in\mathcal{Y}}\mathcal{B}) > \frac{1}{2}$ or $v_1^{\mathfrak{r}'}(\&_{\mathcal{D}\in\mathcal{Z}}\mathcal{D}) > \frac{1}{2}$. Hence, $\mathcal{Y}^{t',t''} = 1$ or $\mathcal{Z}^{t',t''} = 1$.

In the first case, the conditions given by (9) are satisfied simultaneously, while in the second case, the condition (8) does not hold. Hence, a contradiction. We conclude, $v_1^{\mathfrak{r}'}\left(f'\right) \leq \frac{1}{2}$.
This completes the proof. □

4 Conclusion

The results presented in this paper can be similarly verified for many other individual fuzzy implication operators. Such operators may be strong (S), residuated (R), quantum logic (QL) fuzzy implication operators, etc. (see, [9], [10], [7]). One could try to vary fuzzy conjunctions as well as fuzzy disjunctions. In general, it would be interesting to determine the extent of fuzzy logic operators to which our results may be applied.

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