Evaluation of singular integrals using a variational sigmoidal transformation

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Abstract: In this paper we propose a so-called variational sigmoidal transformation, containing two additional parameters, which inherits principal features of the traditional sigmoidal transformations. The transformation is used for numerical evaluation of singular integrals appearing inevitably in the numerical techniques for engineering problems, for example, the boundary element method. The principal role of the transformation is to weaken or remove the original singularity of the considered integrand. Purpose of this work is to enhance the accuracy of the numerical evaluation techniques such as the Gauss-Legenre quadrature rule for the weakly integrals and the Euler-Maclaurin formula for the Cauchy-principal value and Hadamard finite part integrals. Based on the asymptotic analysis of the transformed integrands, it is proved that the presented transformation combined with the existing quadrature rules will be effective in improving the approximation errors, thanks to the parameters. Availability of the proposed method is verified by the results of some numerical examples. In numerical fulfillment we explore the approximation method, with respect to various values of the parameters included in the presented transformation, for each case of the singularity of the integrand. It is demonstrated that most numerical results of the proposed method are consistent with those of the theoretical analysis.

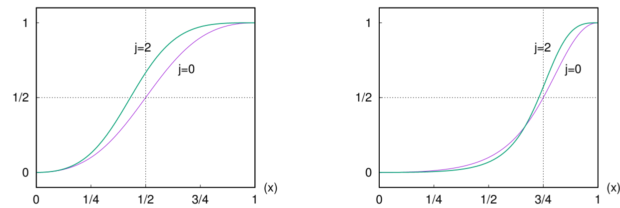
Key–Words: variational sigmoidal transformation, singular integrals, Gauss-Legendre quadrature rule, Euler-Maclaurin formula

1 Introduction

Accurate numerical evaluation of singular integrals including, for example, weakly singular integrals and Cauchy principal value integrals is very important in implementing the boundary element method. Among many approximation methods for the singular integrals [1–28] we are interested in the nonlinear coordinate transformation techniques [5–8, 11–13, 19–27] known to be efficient and easy to use in adaptive approaches.

The order of the singularity in the weakly singular integral can be weakened by nonlinear coordinate transformations having the property of null Jacobian at the singularity. This nonlinear coordinate transformation technique is known to be celebrated because the associated quadrature rule uses the same initial integration points and weights as those used for the regular integrals. For example, the study in the literature [6–8, 11–13, 20, 26, 27] illustrated that a quadrature rule combined with the sigmoidal transformation results in very accurate approximation when the number of integration points is sufficiently large. Explicit estimates of the asymptotic truncation errors of quadrature rules combined with the sigmoidal transformations were derived for weakly singular integrals and stronger singular ones in the literature [13] and [8], respectively. It is noticed that some special polynomial transformation methods [2, 3, 21, 22, 25] and the singularity subtraction method [23] are as powerful as the sigmoidal transformations for weakly singular integrals. However, considering Cauchy-principal value integrals and Hadamard finite part integrals additionally, the sigmoidal transformations combined with the Euler-Maclaurin formula bring about excellent numerical results [7, 8, 17, 27].

In this paper, aiming accurate numerical evaluation of weakly singular integrals as well as stronger singular ones via simple adaptive approaches, we introduce a nonlinear transformation containing two parameters and then consider numerical methods combined with the existing quadrature formulas. The improvement of the approximation errors by the proposed method is explored. To be specific, in the following section we define a so-called variational sigmoidal transformation $g_m^{[j]}(r, x)$ with two additional parameters $0 < r < 1$ and $j \geq 0$ as given in (1). The transformation originates from the simple sigmoidal transformation [20] which is known to be useful in various numerical integration methods. In Section 3 and Section 4 we show that the proposed transformation combined with existing quadrature methods will be effective in improving approximation errors for the singular integrals. Moreover, numerical results of several examples for the weakly singular integrals, Cauchy principal value integrals, and Hadamard finite part integrals show the availability of the presented method with the parameters j and r chosen appropriately.



(a) $r = \frac{1}{2}$ (b) $r = \frac{3}{4}$

Figure 1. Graphs of $g_2^{[j]}(r;x)$, with $j = 0$ and 2, for $r = \frac{1}{2}$ in (a) and $r = \frac{3}{4}$ in (b).

2 A variational sigmoidal transformation

For a real number $0 < r < 1$ and for integers $m > 1$ and $j \geq 0$ we propose a real valued function on the interval $[0, 1]$ as

$$g_m^{[j]}(r;x) = \frac{x^m}{x^m + \left\{\frac{r}{1-r}(1-x)\right\}^{m+j}}, \qquad 0 \leq x \leq 1. \tag{1}$$

Recently the function $g_m^{[0]}$, the special case of $j = 0$, was used for cumulative averaging method for piecewise polynomial interpolations [29]. We, in this work, investigate extended application of the generalized version $g_m^{[j]}$ with an additional parameter j for accurate evaluation of the singular integrals.

We can find some properties of $g_m^{[j]}(r;x)$ as follows.

(i) The special case of $j = 0$ and $r = 1/2$ is

$$g_m^{[0]}\left(\tfrac{1}{2};x\right) = \frac{x^m}{x^m + (1-x)^m} := \gamma_m^{\text{simp}}(x) \tag{2}$$

which is the same with the simple sigmoidal transformation proposed in [20].

(ii) Values of $g_m^{[j]}(r;x)$ at the points $x = 0, r, 1$ are

$$g_m^{[j]}(r;0) = 0, \quad g_m^{[j]}(r;r) = \frac{1}{1+r^j} \tag{3}$$
$$g_m^{[j]}(r;1) = 1,$$

independently of the order m.

(iii) $g_m^{[j]}(r;x)$ is strictly increasing on the interval $[0, 1]$ because the derivative of $g_m^{[j]}(r;x)$ with respect to x satisfies

$$\frac{d}{dx} g_m^{[j]}(r;x) > 0 \tag{4}$$

for all $0 < x < 1$.

(iv) Series expansion of $g_m^{[j]}(r;x)$ near $x = 0$ is

$$g_m^{[j]}(r;x) = \left(\frac{1-r}{r}\right)^{m+j} x^m + O\left(x^{m+1}\right) \tag{5}$$

while that of the simple sigmoidal transformation $\gamma_m^{\text{simp}}(x) = g_m^{[0]}(1/2;x)$ is

$$\gamma_m^{\text{simp}}(x) = x^m + O\left(x^{m+1}\right).$$

It should be noted that the leading coefficient of the series expansion of $g_m^{[j]}$, denoted by

$$C_m^{[j]}(r) = \left(\frac{1-r}{r}\right)^{m+j} \tag{6}$$

decreases to 0 as r approaches to 1.

We call $g_m^{[j]}(r;x)$ a *variational sigmoidal transformation* of order m as it coincides with the traditional sigmoidal transformation when $j = 0$ and $r = \frac{1}{2}$ as mentioned in the property (i). In Figure 1, graphs of $g_2^{[j]}(r;x)$, with $j = 0, 2$, are compared for each parameter $r = \frac{1}{2}$ and $\frac{3}{4}$. The graphs illustrate the aforementioned features of the proposed transformation.

3 Weakly singular integrals

We consider two kinds of weakly singular integrals having end point singularities at $\xi = 1$, for example, such as

$$I_\alpha f := \int_{-1}^{1} (1-\xi)^\alpha f(\xi)\, d\xi, \quad \alpha > -1 \tag{7}$$

and

$$Jf := \int_{-1}^{1} f(\xi) \log(1-\xi)\, d\xi. \tag{8}$$

To evaluate these integrals numerically, referring to the semi-sigmoidal transformation technique [12], we define a modified transformation

$$\widetilde{g}_m^{[j]}(r;x) := 1 - \frac{2}{g_m^{[j]}(r;r)} g_m^{[j]}\left(r; \frac{r(1-x)}{2}\right), \quad (9)$$

$-1 \le x \le 1$, which is a bijective mapping from $[-1,1]$ onto itself. By the change of variables $\xi = \widetilde{g}_m^{[j]}(r;x)$, the weakly singular integrals $I_\alpha f$ and Jf respectively become

$$I_\alpha f = \int_{-1}^{1} \left\{1 - \widetilde{g}_m^{[j]}(r;x)\right\}^\alpha f\left(\widetilde{g}_m^{[j]}(r;x)\right) \times \frac{d}{dx}\widetilde{g}_m^{[j]}(r;x)\,dx \quad (10)$$

and

$$Jf = \int_{-1}^{1} f\left(\widetilde{g}_m^{[j]}(r;x)\right) \log\left(1 - \widetilde{g}_m^{[j]}(r;x)\right) \times \frac{d}{dx}\widetilde{g}_m^{[j]}(r;x)\,dx. \quad (11)$$

From the property (iv) we can see that the asymptotic behaviors of the transformed integrands, near the singular point $x = 1$, are

$$O\left((1-x)^{(1+\alpha)m-1}\right)$$

and

$$O\left((1-x)^{m-1}\log(1-x)\right),$$

respectively. This implies that the regularity of the original integrand becomes higher as the order m of $g_m^{[j]}$ is increasing. Thanks to the work of Johnston and Elliott [13], we can observe that the N-point Gauss-Legendre quadrature rule combined with the transformation $\widetilde{g}_m^{[j]}$ results in the following asymptotic truncation errors for the transformed integrals in (10) and (11).

$$\begin{aligned} E_{\alpha,N}^{[m,j]}f &:= I_\alpha f - I_{\alpha,N}^{[m,j]}f \\ &\sim C_m^{[j]}(r)^{1+\alpha}\left\{(-1)^m 2^{2+(1-2m)(1+\alpha)} m f(1)\cdot\right. \\ &\left.\Gamma\big(2m(1+\alpha)\big)\sin(m\alpha\pi)\right\}/\left(2^{m-1}(2N+1)^{2m}\right)^{1+\alpha} \end{aligned} \quad (12)$$

and

$$\begin{aligned} E_N^{[m,j]}f &:= Jf - J_N^{[m,j]}f \\ &\sim C_m^{[j]}(r)\,\frac{\pi(-1)^m 2^{2+(1-2m)} m^2 f(1)\Gamma(2m)}{2^{m-1}(2N+1)^{2m}} \end{aligned} \quad (13)$$

for sufficiently large N. Therein, $C_m^{[j]}(r)$ is the leading coefficient of the asymptotic behavior of $g_m^{[j]}(r;x)$ near the point $x = 0$ given in (6), that is,

$$C_m^{[j]}(r) = \left(\frac{1-r}{r}\right)^{m+j}$$

and $I_{\alpha,N}^{[m,j]}f$ and $J_N^{[m,j]}f$ denote numerical evaluations using the N-point Gauss-Legendre quadrature rule combined with $\widetilde{g}_m^{[j]}$ for the integrals $I_\alpha f$ and Jf, respectively. The formulas (12) and (13) imply that the errors of the proposed method become better as the additional parameter j is increasing for any $\frac{1}{2} < r < 1$ and m fixed.

Example 1

We take two examples

$$I_\alpha f = \int_{-1}^{1} (1-\xi)^\alpha \left(1+\xi^2\right) d\xi, \quad (14)$$

with $\alpha = -0.7$ and

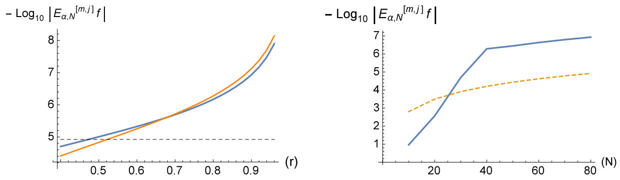
$$Jf = \int_{-1}^{1} \left(1+\xi^2\right)\log(1-\xi)\,d\xi. \quad (15)$$

Exact values of these integrals are $I_\alpha f = 6.5606134847\cdots$ and $Jf = -1.0404964074\cdots$. Numerical errors of the Gauss-Legendre quadrature rule combined with the presented transformation $\widetilde{g}_m^{[j]}(r;x)$ for the integrals $I_{-0.7}f$ and Jf are included in Table 1 and Table 2, respectively, where we chose some values of the parameters $0 \le j \le 4$ and $\frac{1}{2} < r < 1$. In the tables the errors of the quadrature rule combined with the following semi-sigmoidal transformation $\widetilde{\gamma}_m^{\rm simp}(x)$, used in [12], are also included for comparison.

$$\widetilde{\gamma}_m^{\rm simp}(x) = 1 - 4\gamma_m^{\rm simp}\left(\frac{1-x}{4}\right), \quad -1 \le x \le 1 \quad (16)$$

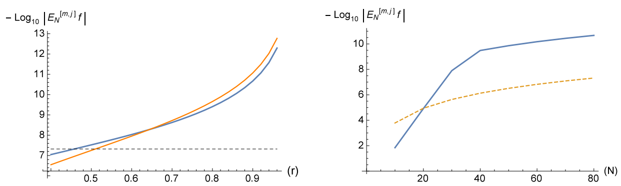
As a result, one can see that the presented transformation $g_m^{[j]}(r;x)$ is effective in improving the errors of the existing method as predicted in (12) and (13) above.

In addition, Figure 2(a) shows the graphs of the approximation errors $E_{\alpha,N}^{[m,j]}f$ associated with the presented transformation $g_m^{[j]}(r;x)$ for the integral $I_\alpha f$ with respect to the parameter $0.4 \le r \le 0.95$ with $j = 4, m = 4$ and $N = 80$ fixed. The thick line shows numerical results of the errors and they are compared with the asymptotic truncation errors in (12) indicated



(a) $0.4 \leq r \leq 0.95 (N = 80)$ (b) $10 \leq N \leq 80 (r = 0.9)$

Figure 2. The errors $-\log_{10}\left|E^{[m,j]}_{\alpha,N} f\right|$, with $m = 4$ and $j = 4$, for the weakly singular integral $I_\alpha f$, $\alpha = -0.7$, in Example 1.



(a) $0.4 \leq r \leq 0.95$ $(N = 80)$ (b) $10 \leq N \leq 80$ $(r = 0.9)$

Figure 3. The errors $-\log_{10}\left|E^{[m,j]}_{N} f\right|$, with $m = 2$ and $j = 2$, for the weakly singular integral Jf in Example 1.

by the thin line. The dotted horizontal line shows the numerical error corresponding to the simple sigmoidal transformation $\gamma_m^{\mathrm{simp}}(x)$. Figure 2(b) illustrates numerical results of approximation errors of the presented method with respect to the number of integration points $10 \leq N \leq 80$ with $j = 4$, $m = 4$ and $r = 0.9$ fixed. The dotted line indicates numerical errors corresponding to $\gamma_m^{\mathrm{simp}}(x)$.

Similarly, for the logarithmic singular integral Jf in (15), Figure 3 illustrates numerical results of the proposed method with $j = 2$, $m = 2$ by the thick lines and those of the existing method combined with the simple sigmoidal transformation $\gamma_m^{\mathrm{simp}}(x)$ by the dotted lines.

From Figure 2 and Figure 3 we can find that the numerical errors of the proposed method are consistent with the theoretical asymptotic errors over the range $0.5 < r < 0.95$, approximately. Furthermore, the superiority of the proposed method over the existing method is maintained for large number of integration points, that is, for $N \geq 30$.

4 Cauchy principal value integrals and hyper singular integrals

In this section, for an interior point $0 < \xi < 1$, we consider the Cauchy principal value integral

$$\begin{aligned} K_1\phi(\xi) &:= \fint_0^1 \frac{\phi(y)}{y-\xi}\,dy \\ &= \lim_{\epsilon\to 0+}\left(\int_0^{\xi-\epsilon} + \int_{\xi+\epsilon}^1\right)\frac{\phi(y)}{y-\xi}\,dy, \end{aligned} \tag{17}$$

where ϕ is Hölder continuous on the interval $[0, 1]$ and the Hadamard finite-part integral

$$\begin{aligned} K_2\phi(\xi) &:= \fint_0^1 \frac{\phi(y)}{(y-\xi)^2}\,dy \\ &= \frac{d}{dx}\fint_0^1 \frac{\phi(y)}{y-\xi}\,dy, \end{aligned} \tag{18}$$

where ϕ' is Hölder continuous on $[0, 1]$.

In general, for the presented transformation $g_m^{[j]}$ in (1), if we change the variable $0 \leq y \leq 1$ by

$$y = g_m^{[j]}(r;t), \quad 0 \leq t \leq 1 \tag{19}$$

then the integrals in (17) and (18) respectively become

$$\begin{aligned} K_1\phi(\xi) &= \fint_0^1 \frac{\Phi_m(t)}{g_m^{[j]}(r;t)-\xi}\,dt \\ K_2\phi(\xi) &= \fint_0^1 \frac{\Phi_m(t)}{(g_m^{[j]}(r;t)-\xi)^2}\,dt, \end{aligned} \tag{20}$$

where

$$\Phi_m(t) = \phi\left(g_m^{[j]}(r;t)\right)\frac{d}{dt}g_m^{[j]}(r;t). \tag{21}$$

Set

$$t_\nu = \frac{1+\nu}{2} \tag{22}$$

for a number $-1 < \nu \leq 1$. Then the literature [7, 8, 17] introduced the formula

$$K_i\phi(\xi) = \left(Q^{[m,j]}_{i,N}\phi\right)(\xi) + \left(E^{[m,j]}_{i,N}\phi\right)(\xi), \tag{23}$$

$i = 1, 2$, where $Q^{[m,j]}_{i,N}\phi$'s, known as the Euler-Maclaurin formula, are the N points quadrature sums for the integrands in (20) using the nodes $t_k = (k + t_\nu)/N$, $k = 0, 1, 2, \cdots, N-1$ (See the literature [7, 8, 17] for complete formulas of $Q^{[m,j]}_{i,N}\phi$'s).

Referring to the literature [17, 28], we can represent the asymptotic truncation errors as given in the following theorem.

Theorem 1 *Suppose a function ϕ is analytic on $(0,1)$ and it has asymptotic behaviors*

$$\phi(y) \sim k_0 y^{\alpha_0}\,, \qquad \phi(y) \sim k_1(1-y)^{\alpha_1}$$

near $y=0$ and $y=1$, respectively, where $\alpha_0,\ \alpha_1 > -1$ and $k_0,\ k_1$ are non-zero constants. Let $C_m^{[j]}(r)$ be the leading coefficient of the transformation $g_m^{[j]}(r;x)$ with $m>1$, $j\geq 0$ and $0<r<1$. Then, for an integer N large enough and for a number t_ν in (22), we have

$$\begin{aligned}&\left|\left(E_{i,N}^{[m,j]}\phi\right)(\xi)\right| \\ &\sim \left|\frac{C_m^{[j]}(r)^{1+\alpha_0}k_0 m\bar{\zeta}\left(-(m(1+\alpha_0)-1),t_\nu\right)}{N^{m(1+\alpha_0)}(-\xi)^i}\right. \\ &\quad \left.+\frac{C_m^{[j]}(r)^{1+\alpha_1}k_1 m\bar{\zeta}\left(-(m(1+\alpha_1)-1),N-t_\nu\right)}{N^{m(1+\alpha_1)}(1-\xi)^i}\right|,\end{aligned} \tag{24}$$

$i=1,2$, *where $0<\xi<1$, and $\bar{\zeta}(s,t)$ denotes a function which is periodic in t with period 1 and coincides with the Riemann– zeta function $\zeta(s,t)$ when $0<t<1$.*

Example 2

We consider two singular integrals,

$$K_1\phi(\xi) = \int_0^1 \sqrt{y(1-y)}\,\frac{U_4(2y-1)}{y-\xi}\,dy \quad \left(= -\frac{\pi}{2}\,T_5(2\xi-1)\right) \tag{25}$$

and

$$K_2\phi(\xi) = \oint_0^1 \sqrt{y(1-y)}\,\frac{U_4(2y-1)}{(y-\xi)^2}\,dy \quad (= -5\pi U_4(2\xi-1)) \tag{26}$$

in which T_n and U_n denote Chebyshev polynomials of degree n of the first and second kind, respectively.

Since

$$\phi(y) \sim U_4(-1)y^{\frac{1}{2}}, \quad \phi(y) \sim U_4(1)(1-y)^{\frac{1}{2}},$$

near $y=0$ and $y=1$, respectively, it follows that $\alpha_0=\alpha_1=1/2$, $k_0=U_4(-1)=5$ and $k_1=U_4(1)=5$. Therefore, in using the presented sigmoidal transformation $g_m^{[j]}(r;x)$, Theorem 1 implies

$$\begin{aligned}\left|\left(E_{i,N}^{[m,j]}\phi\right)(\xi)\right| &\sim \left(\frac{1-r}{r}\right)^{\frac{3}{2}(m+j)} \\ &\times \frac{5m}{N^{\frac{3}{2}m}}\left|\frac{\bar{\zeta}(1-\frac{3}{2}m,t_\nu)}{(-\xi)^i}+\frac{\bar{\zeta}(1-\frac{3}{2}m,N-t_\nu)}{(1-\xi)^i}\right|\end{aligned} \tag{27}$$

for each $i=1,2$. When the simple sigmoidal transformation $\gamma_m^{\mathrm{simp}}(x)$ is employed, it follows that

$$\begin{aligned}\left|\left(E_{i,N}^{[m,j]}\phi\right)(\xi)\right| \sim \frac{5m}{N^{\frac{3}{2}m}}&\left|\frac{\bar{\zeta}(1-\frac{3}{2}m,t_\nu)}{(-\xi)^i}\right. \\ &\left.+\frac{\bar{\zeta}(1-\frac{3}{2}m,N-t_\nu)}{(1-\xi)^i}\right|.\end{aligned} \tag{28}$$

Comparing the formulas (27) and (28), one can observe that the presented variational sigmoidal transformation $g_m^{[j]}(r;x)$ can improve the errors of the simple sigmoidal transformation by the factor

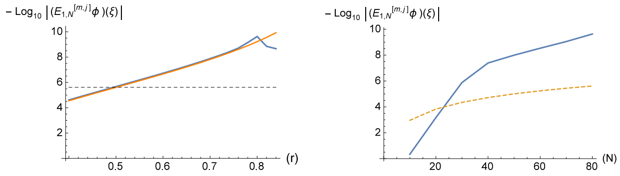
$$R = \left(\frac{1-r}{r}\right)^{\frac{3}{2}(m+j)}. \tag{29}$$

It should be noted that $R<1$ when $\frac{1}{2}<r<1$ and it decreases as the parameter r goes to 1.

Numerical errors of the Euler-Maclaurin formula, with $\nu=0.3$, combined with the presented transformation $g_m^{[j]}(r;x)$ and the simple sigmoidal transformation $\gamma_m^{\mathrm{simp}}(x)$ are given in Table 3 for $K_1\phi\left(\frac{1}{10}\right)$ and in Table 4 for $K_2\phi\left(\frac{1}{10}\right)$. We chose the parameters $j=0,\ 2,\ 4$ and $r=0.6,\ 0.75$ in using $g_m^{[j]}(r;x)$. One can see that the Euler-Maclaurin formulas based on $g_m^{[j]}(r;x)$ highly improve those based on $\gamma_m^{\mathrm{simp}}(x)$ for most of the selected values of the parameters j and r.

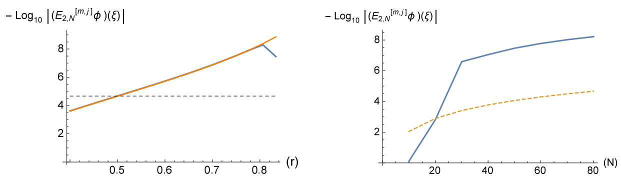
In Figure 4(a) and Figure 5(a) the thick lines show the graphs of the numerical errors $\left(E_{i,N}^{[m,j]}\phi\right)\left(\frac{1}{10}\right)$, associated with the presented transformation $g_m^{[j]}(r;x)$ for the integral $K_i\phi\left(\frac{1}{10}\right)$, $i=1,2$, with respect to the parameter $0.4\leq r\leq 0.85$ with $j=2$, $m=2$ and $N=80$ fixed. The numerical errors are compared with the theoretical asymptotic errors in (24) indicated by the thin line. The dotted horizontal line shows the numerical error corresponding to the simple sigmoidal transformation $\gamma_m^{\mathrm{simp}}(x)$. Figure 4(b) and Figure 5(b) illustrate numerical results of approximation errors corresponding to the presented transformation with respect to the number of integration points $10\leq N\leq 80$ with $r=0.8$ fixed. The dotted lines indicate numerical errors corresponding to the simple sigmoidal transformation $\gamma_m^{\mathrm{simp}}(x)$.

Figure 4 and Figure 5 imply that the numerical errors of the proposed method are consistent with the theoretical asymptotic errors over the range $0.4<r<0.8$, approximately. In addition, like the case of the weakly singular integrals, one can see that the proposed method over the existing method is maintained for $N\geq 30$.



(a) $0.4 \le r \le 0.85$ $(N = 80)$ (b) $10 \le N \le 80$ $(r = 0.8)$

Figure 4. The errors $-\log_{10}\left|\left(E_{1,N}^{[m,j]}\phi\right)(\xi)\right|$ with $m = 2$ and $j = 2$ for the Caucy singular integral $K_1\phi(\xi)$, $\xi = \frac{1}{10}$, in Example 2.



(a) $0.4 \le r \le 0.85(N = 80)$ (b) $10 \le N \le 80(r = 0.8)$

Figure 5. The errors $-\log_{10}\left|\left(E_{2,N}^{[m,j]}\phi\right)(\xi)\right|$ with $m = 2$ and $j = 2$ for the Hadamard finite part integral $K_2\phi(\xi)$, $\xi = \frac{1}{10}$, in Example 2.

5 Conclusion

In this paper, for the purpose of efficient numerical evaluation of singular integrals appearing in implementation of the boundary element method, we introduced a variational sigmoidal transformation containing additional parameters. It was proved that the presented transformation combined with the existing quadrature rules is available in improving the approximation errors. We have observed that the numerical results for some examples of a weakly singular integral, a Cauchy-principal value integral, and a Hadamard finite part integral are consistent with the theoretical results of the asymptotic error analysis.

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Table 1. Numerical results of the errors $\left|E_{\alpha,N}^{[m,j]}f\right|$ corresponding to the presented transformation $\widetilde{g}_m^{[j]}(r;x)$ and the sigmoidal transformation $\widetilde{\gamma}_m^{\rm simp}(x)$ for the weakly singular integral $I_\alpha f$, $\alpha=-0.7$, in Example 1.

			$\widetilde{g}_m^{[0]}(r;x)$		$\widetilde{g}_m^{[2]}(r;x)$		$\widetilde{g}_m^{[4]}(r;x)$	
m	N	$\widetilde{\gamma}_m^{\rm simp}(x)$	$r=0.6$	$r=0.9$	$r=0.6$	$r=0.9$	$r=0.6$	$r=0.9$
4	20	3.2×10^{-4}	2.4×10^{-4}	6.2×10^{-5}	1.7×10^{-4}	$\mathbf{1.0\times10^{-5}}$	1.3×10^{-4}	2.7×10^{-3}
	40	6.2×10^{-5}	4.7×10^{-5}	9.0×10^{-6}	3.3×10^{-5}	2.3×10^{-6}	2.5×10^{-5}	$\mathbf{5.2\times10^{-7}}$
	80	1.2×10^{-5}	9.1×10^{-6}	1.7×10^{-6}	6.4×10^{-6}	4.5×10^{-7}	4.7×10^{-6}	$\mathbf{1.2\times10^{-7}}$
6	20	4.9×10^{-6}	3.3×10^{-6}	4.4×10^{-4}	$\mathbf{2.5\times10^{-6}}$	3.8×10^{-3}	3.7×10^{-6}	1.1×10^{-2}
	40	4.3×10^{-7}	2.9×10^{-7}	2.2×10^{-8}	2.0×10^{-7}	$\mathbf{2.9\times10^{-8}}$	1.5×10^{-7}	1.7×10^{-6}
	80	3.6×10^{-8}	2.4×10^{-8}	2.0×10^{-9}	1.7×10^{-8}	5.2×10^{-10}	1.2×10^{-8}	$\mathbf{1.3\times10^{-10}}$

Table 2. Numerical results of the errors $\left|E_{N}^{[m,j]}f\right|$ corresponding to the presented transformation $\widetilde{g}_m^{[j]}(r;x)$ and the sigmoidal transformation $\widetilde{\gamma}_m^{\rm simp}(x)$ for the weakly singular integral Jf in Example 1.

			$\widetilde{g}_m^{[0]}(r;x)$		$\widetilde{g}_m^{[1]}(r;x)$		$\widetilde{g}_m^{[2]}(r;x)$	
m	N	$\widetilde{\gamma}_m^{\rm simp}(x)$	$r=0.6$	$r=0.9$	$r=0.6$	$r=0.9$	$r=0.6$	$r=0.9$
2	20	1.1×10^{-5}	7.3×10^{-6}	$\mathbf{4.4\times10^{-7}}$	3.9×10^{-6}	2.4×10^{-6}	2.2×10^{-6}	1.2×10^{-5}
	40	7.5×10^{-7}	4.8×10^{-7}	3.0×10^{-8}	2.5×10^{-7}	3.1×10^{-9}	1.4×10^{-7}	$\mathbf{3.3\times10^{-10}}$
	80	4.8×10^{-8}	3.1×10^{-8}	1.9×10^{-9}	1.6×10^{-8}	2.0×10^{-10}	9.2×10^{-9}	$\mathbf{2.1\times10^{-11}}$
3	20	4.0×10^{-8}	2.0×10^{-8}	2.4×10^{-6}	1.1×10^{-8}	2.2×10^{-5}	$\mathbf{6.4\times10^{-9}}$	4.3×10^{-4}
	40	6.7×10^{-10}	3.4×10^{-10}	5.4×10^{-12}	1.8×10^{-10}	$\mathbf{3.7\times10^{-12}}$	1.0×10^{-10}	3.7×10^{-10}
	80	1.1×10^{-11}	5.6×10^{-12}	8.7×10^{-14}	3.0×10^{-12}	9.2×10^{-15}	1.7×10^{-12}	$\mathbf{9.8\times10^{-16}}$

Table 3. Numerical results of the errors $\left|\left(E_{1,N}^{[m,j]}\phi\right)(\xi)\right|$ corresponding to the presented transformation $g_m^{[j]}(r;x)$ and the sigmoidal transformation $\gamma_m^{\rm simp}(x)$ for the Cauchy singular integral $K_1\phi(\xi)$, $\xi=\frac{1}{10}$, in Example 2.

			$g_m^{[0]}(r;x)$		$g_m^{[2]}(r;x)$		$g_m^{[4]}(r;x)$	
m	N	$\gamma_m^{\rm simp}(x)$	$r=0.6$	$r=0.75$	$r=0.6$	$r=0.75$	$r=0.6$	$r=0.75$
2	20	1.5×10^{-4}	9.8×10^{-5}	5.4×10^{-4}	1.0×10^{-5}	6.4×10^{-5}	$\mathbf{3.8\times10^{-6}}$	5.6×10^{-3}
	40	1.9×10^{-5}	1.2×10^{-5}	5.8×10^{-5}	1.4×10^{-6}	1.4×10^{-8}	4.0×10^{-7}	$\mathbf{7.9\times10^{-9}}$
	80	2.4×10^{-6}	1.5×10^{-6}	6.9×10^{-6}	1.9×10^{-7}	2.8×10^{-9}	5.4×10^{-8}	$\mathbf{1.1\times10^{-10}}$
3	20	1.3×10^{-6}	$\mathbf{4.5\times10^{-7}}$	1.9×10^{-5}	1.0×10^{-5}	3.3×10^{-3}	3.0×10^{-4}	4.3×10^{-2}
	40	5.5×10^{-8}	1.7×10^{-8}	1.9×10^{-7}	2.6×10^{-9}	2.3×10^{-9}	$\mathbf{7.2\times10^{-10}}$	1.5×10^{-6}
	80	2.4×10^{-9}	6.7×10^{-10}	6.5×10^{-9}	1.1×10^{-10}	1.5×10^{-12}	3.3×10^{-11}	$\mathbf{2.1\times10^{-14}}$

Table 4. Numerical results of the errors $\left|\left(E_{2,N}^{[m,j]}\phi\right)(\xi)\right|$ corresponding to the presented transformation $g_m^{[j]}(r;x)$ and the sigmoidal transformation $\gamma_m^{\rm simp}(x)$ for the Hadamard finite part integral $K_2\phi(\xi)$, $\xi=\frac{1}{10}$, in Example 2.

			$g_m^{[0]}(r;x)$		$g_m^{[2]}(r;x)$		$g_m^{[4]}(r;x)$	
m	N	$\gamma_m^{\rm simp}(x)$	$r=0.6$	$r=0.75$	$r=0.6$	$r=0.75$	$r=0.6$	$r=0.75$
2	20	1.3×10^{-3}	3.2×10^{-4}	5.4×10^{-4}	1.0×10^{-4}	4.4×10^{-5}	$\mathbf{2.6\times10^{-5}}$	1.9×10^{-3}
	40	1.7×10^{-4}	4.3×10^{-5}	5.7×10^{-5}	1.4×10^{-5}	2.4×10^{-7}	4.0×10^{-6}	$\mathbf{1.0\times10^{-8}}$
	80	2.1×10^{-5}	5.5×10^{-6}	6.8×10^{-6}	1.8×10^{-6}	$\mathbf{2.9\times10^{-8}}$	5.4×10^{-7}	3.7×10^{-8}
3	20	1.2×10^{-5}	$\mathbf{1.6\times10^{-6}}$	4.4×10^{-6}	8.1×10^{-6}	2.8×10^{-4}	3.1×10^{-4}	3.3×10^{-2}
	40	5.4×10^{-7}	7.7×10^{-8}	8.3×10^{-7}	2.6×10^{-8}	$\mathbf{4.0\times10^{-10}}$	1.9×10^{-8}	6.1×10^{-7}
	80	2.4×10^{-8}	3.5×10^{-9}	4.8×10^{-9}	1.1×10^{-9}	9.2×10^{-12}	4.0×10^{-10}	$\mathbf{7.0\times10^{-13}}$