On the Numerical Solution of System of Linear Algebraic Equations with Positive Definite Symmetric Ill-Posed Matrices

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Abstract: - In this paper, we present the results of a numerical solution to ill-posed systems of linear algebraic equations (SLAEs) with positive definite symmetric matrices by a regularization method. In the paper it is shown that for the regularization of a computational process by the Tikhonov method it is sufficient to replace matrix A_n of the system by matrix $A_n + \alpha E_n$ where E_n is the unit matrix, and α is some positive numbers (the regularization parameters) that tend to zero.

Key-Words: - Ill-posed systems of linear algebraic equations, Hilbert matrices, regularization parameters.

1 Introduction

In the process of solving various physical problems, it is necessary to solve systems of linear algebraic equations with positive definite symmetric ill-posed matrices. Such SLAEs arise, for example, when a function is approximated by algebraic polynomials using the metric of space $L_2(0,1)$. In this case this approximation generates Hilbert matrices. Such SLAEs also arise during the solution of ordinary differential equations by the Ritz method, which leads to Gram matrices. These matrices of order n are symmetric and positively defined, but with an unlimited increase of n, the smallest eigenvalue tends to zero, which leads to the instability of the decision process. Usually, to obtain a reliable solution, regularization methods are used. A common strategy is to use Tikhonov's stabilizer [1] or its modifications [2-19]. In this paper we consider the features of the numerical solution of systems of linear equations with a positive definite symmetric matrix using the regularizer proposed by prof. Ryabov V.M. In the next sections we show that for the regularization of a computational process by the Tikhonov method it is sufficient to replace matrix A_n of the system by matrix $A_n + \alpha E_n$ where E_n is the unit matrix, and α is some positive numbers (the regularization parameters) that tend to zero.
Thus, we reduce the conditional number of SLAE, which increases the stability.

2 Problem Formulation

Let A be a non-degenerate real square matrix of size $n, \det A \neq 0$. In this case the solution to the system of linear algebraic equations

$$Az = f$$

exists and is unique. Various modifications of the Gauss method for solving systems of SLAE are known, for example the Gauss method with a choice of leading element, etc. Suppose conditional number $cond\,(A) = \|A\| \cdot \|A^{-1}\|$ of matrix A is very large, i.e. the matrix of the system of equations is ill-conditioned. The solution of the ill-conditioned SLAE by the Gauss method does not always give a satisfactory solution. For example, suppose

$$\mathrm{A} = \begin{pmatrix} 0.0000001 & 333 & 555 \\ 33333333 & 1 & 70 \\ 55555555 & 70 & 32 \end{pmatrix}, f = \begin{pmatrix} 888 \\ 33333404 \\ 55555657 \end{pmatrix}.$$

The condition number of matrix A ($cond\,A$) equals $cond\,A = 1314691.460$. Solving this system according to the scheme of single division without permutations (using a program written in C++ with real numbers of double type), we obtain $z = (1.0,\ 1.555556,\ 0.666667)^T$, which is significantly different from the exact solution $(1.0,\ 1.0,\ 1.0)^T$. A similar example was considered in [10]. These examples show that it is necessary to avoid dividing by small elements of absolute value in the process of solving. Using the modified Gauss method - the choice of leading element being the greatest of the absolute values of the elements in a column (Wilkinson's strategy) or the greatest of the absolute values of the elements in the entire matrix of the remaining equations (Jordan's strategy of complete ordering) helps to avoid this situation. The application of the Gauss method with a choice of leading element gives the solution $z = (1.0, \quad 1.0, \quad 1.0)^T$.

If a system of equations is ill-conditioned, for example, in the case of SLAEs with the Hilbert matrix $H_n = (h_{ij})_{i,j=1}^{n}$ of order n with elements $h_{ij} = 1/(i+j-1)$, then it is practically impossible to obtain an acceptable SLAE solution by known methods (direct methods such as the Gauss method, square root method, iterative method etc). Table 1 shows the values of the condition numbers for Hilbert matrices of orders 3 to 20, calculated with the Maple package, *Digits*=50. Table 2 shows solution of SLAEs (1) with Hilbert matrices H_n, obtained by the Gauss method.

Table 1. The conditional numbers of the Hilbert matrices H_n

n	cond(H_n)	n	cond(H_n)
3	748	12	$4.1154\cdot10^{16}$
4	28375	13	$1.3244\cdot10^{18}$
5	$9.4366\cdot10^{5}$	14	$4.5378\cdot10^{19}$
6	$2.9070\cdot10^{7}$	15	$1.5392\cdot10^{21}$
7	$9.8519\cdot10^{8}$	16	$5.0628\cdot10^{22}$
8	$3.3873\cdot10^{10}$	17	$1.6808\cdot10^{24}$
9	$1.0996\cdot10^{12}$	18	$5.7661\cdot10^{25}$
10	$3.5357\cdot10^{13}$	19	$1.9258\cdot10^{27}$
11	$1.2337\cdot10^{15}$	20	$6.2836\cdot10^{28}$

Table 2. Solutions of SLAEs with H_n .

n=10	n=12	n=14	n=20
1.000000	1.000000	1.000000	1.000000
1.000000	1.000003	0.999990	1.000075
0.999998	0.999916	1.000310	0.997232
1.000020	1.001125	0.996220	1.043007
0.999903	0.991859	1.019948	0.657461
1.000267	1.035385	0.981738	2.485524
0.999563	0.902315	0.675197	-2.218021
1.000421	1.175419	2.922896	2.012694
0.999779	0.795768	-4.498100	11.914026
1.000048	1.148663	10.515965	-20.728838
	0.938525	-9.428208	5.732215
	1.011022	8.094763	34.233277
		-1.740964	-42.153623
		1.460245	18.727738
			2.721547
			-16.803379
			49.405743
			-57.516277
			33.288344
			-5.798746

The solutions presented in Table 2 are far from the true solutions. In the following sections we present the regularization method, with the help of which we obtain solutions of initial system (1) with Hilbert matrices.

3 Problem Solution

Various approaches to solving systems of equations with ill-conditioned matrices are known [2-6]. In this paper, to obtain an acceptable solution to SLAEs, we consider the application of the modification of the regularization method proposed by professor of Saint Petersburg State University V.M.Ryabov.

A well-known standard Tikhonov regularization method consists of finding the approximation of a normal solution of system $Az = f$. It is based on finding the element on which the functional

$$M_\alpha(z, A, f) = \|Az - f\|^2 + \alpha \, \|z\|^2$$

reaches the smallest value for fixed positive α.

For obtaining a normal solution, we have to solve the Euler equation

$$(A^*A + \alpha E)z = A^*f.$$

The solution of the Euler equation depends on the conditional number of matrix A^*A. This number can be very large. If matrix A is symmetric and positive definite, we propose to find a normal solution in another way. In this paper we propose to find a normal solution by solving a system of equations for which the conditional number is much smaller.

Let matrix A of the SLAE

$$Az = f \tag{1}$$

be symmetric and positive definite (for example, the Hilbert matrix). In our case, there exists a unique positive definite root of matrix $B = \sqrt{A}$, i.e. positive definite matrix B is such that $B^2 = A$. Let us establish a relation between eigenvalues and eigenvectors of matrices A and B: let μ and x be eigenvalue and eigenvector of matrix B:

$$Bx = \mu x. \tag{2}$$

Multiplying (2) by B we get

$$Ax = \mu Bx, \quad (3)$$

Using (2), we rewrite (3) as $Ax = \mu^2 x$. This equality means that the eigenvectors of matrices A and B are the same, and eigenvalues of matrix A are equal to squares of the eigenvalues of matrix B. Multiplying (1) by B^{-1} we get

$$Bz = B^{-1}f. \quad (4)$$

We write the Euler equation for minimizing the smoothing functional $M_\alpha(z, B, B^{-1}f) = \|Bz - B^{-1}f\|^2 + \alpha\|z\|^2$: it has the form:

$$(B^*B + \alpha E)z_\alpha = B^*(B^{-1}f), \qquad \alpha > 0. \quad (5)$$

In the case of symmetric matrix A, matrix B is selfadjoint, therefore we obtain the equation

$$(A + \alpha E)z_\alpha = f. \quad (6)$$

Formally, the shift in the original system is carried out, but in fact this is the regularization method for equation (5) with the same solution as S LAE (1). Numbers are usually stored in computer memory with some error. We assume that the matrix and the vector are given approximately. Later (see section 5) the convergence of the method will be pr oved. It should be noted that convergence holds only when $\delta \to 0$, with an unlimited increase in the accuracy of the initial information (matrix A and right-hand side f: $\|A - A_\delta\| \le \delta$, $\|f - f_\delta\| \le \delta$), which we can not practically achieve, so the results may be far from the desired solutions. Solving (6), we obtain an approximate solution of system (1).

Remark. If matrix A is symmetric, we don't need to c ompute matrix B. The application of regularization to equation (1) will just increase the conditional number of the resulting system, and that is unprofitable. In the case of a nonsymmetric positive definite matrix, the Euler equation for system (1) has the form (5).

The convergence of the regularization method is stated in [1]. Unlike the standard approach of the regularization procedure this representation also has a reduced conditional number, which is very important. But, in contrast to symmetric matrices it is necessary to know matrix B in the case of nonsymmetric matrices. The last requirement complicates the application of this method in practice.

A series of computational experiments on the application of the regularization method for solving SLAE with the Hilbert matrices of order $n = 2, 3, \dots, 20$ were conducted.

4 Results of the Application of the Regularization Method for SLAEs with Hilbert Matrices H_n

Table 3 shows the results of applying the regularization method for various parameters $\alpha = 1.0, 10^{-1}, 10^{-2}, \dots, 10^{-15}$ for solving perturbed SLAE $(H_n + \alpha E_n)z = f$, where the matrix of the original system $H_n = (h_{ij})_{i,j=1}^n$ is the Hilbert matrix of order n. The solution of the perturbed system was calculated by the Gauss method with a choice of leading element. An exact solution of the original unperturbed system $H_n z = f$ is known: that is an n-dimensional vector of units: $z_t = (1.0, \ 1.0, \ \dots, 1.0)^T$.

The error in the solution was calculated using the Euclidean norm. Calculating the solution of the perturbed SLAE for different values of α, we find the optimal value of the parameter at which the error of the solution has a minimum value. Thus, to solve system of equations (1), it is necessary to solve several systems of equations for different α. The norms of the approximate solutions obtained could be calculated, and, in our case, the solutions whose norm is the smallest corresponds to the optimal α, as it is shown in Tables 3, 4. The last rows in Tables 3, 4 show the error in solutions calculated without regularization. In Tables 3, 4 the smallest error value for a given n corresponding to the optimal value of perturbation parameter α is shown in bold type.

Table 3. Errors in the solution by the Gauss method of a perturbed SLAE with the Hilbert matrices of order n=10, 12 for different values of parameter α.

α	n=10	n=12
10^{-12}	0.1310^{-5}	$0.22 \cdot 10^{-5}$
10^{-11}	**0.14** $\cdot 10^{-6}$	**0.69** $\cdot 10^{-7}$
10^{-10}	$0.16 \cdot 10^{-6}$	$0.19 \cdot 10^{-6}$
10^{-8}	$0.16 \cdot 10^{-3}$	$0.20 \cdot 10^{-3}$
10^{-7}	$0.58 \cdot 10^{-3}$	$0.58 \cdot 10^{-3}$
10^{-6}	$0.17 \cdot 10^{-2}$	$0.19 \cdot 10^{-2}$
10^{-5}	$0.56 \cdot 10^{-2}$	$0.63 \cdot 10^{-2}$
10^{-4}	$0.18 \cdot 10^{-1}$	$0.19 \cdot 10^{-1}$
10^{-3}	$0.56 \cdot 10^{-1}$	$0.61 \cdot 10^{-1}$
10^{-2}	0.1799	0.1985
10^{-1}	0.5788	0.6355
Errors in the solution without regularization		
	$0.71 \cdot 10^{-3}$	0.3306

Table 4. Errors in the solution by the Gauss method of a perturbed SLAE with the Hilbert matrices of of order n=14, 20 for different values of parameter α.

α	n=14	n=20
10^{-12}	$0.19\cdot10^{-4}$	$0.17\cdot10^{-4}$
10^{-11}	$0.27\cdot10^{-6}$	**0.23** $\cdot10^{-6}$
10^{-10}	**0.20** $\cdot10^{-6}$	$0.25\cdot10^{-6}$
10^{-8}	$0.20\cdot10^{-3}$	$0.25\cdot10^{-3}$
10^{-7}	$0.65\cdot10^{-3}$	$0.78\cdot10^{-3}$
10^{-6}	$0.21\cdot10^{-2}$	$0.25\cdot10^{-2}$
10^{-5}	$0.66\cdot10^{-2}$	$0.80\cdot10^{-2}$
10^{-4}	$0.21\cdot10^{-1}$	$0.25\cdot10^{-1}$
10^{-3}	$0.67\cdot10^{-1}$	$0.81\cdot10^{-1}$
10^{-2}	0.2153	0.2581
10^{-1}	0.6872	0.8220
Errors in the solution without regularization		
	17.0703	105.2819

The accuracy of floating-point arithmetic can be characterized by machine-epsilon, the smallest positive floating-point number ε such that 1+ ε>1 (see [20]). We can compute a machine epsilon. For example, in C++ 64-bit doubles give $\varepsilon \approx 2.2\cdot10^{-16}$.

Tikhonov's theorem asserts that, theoretically, as α decreases, the regularized solution improves, but in practical calculations for sufficiently small α (within machine precision in C++), the rounding errors and conditional number of the matrix have a significant effect. It can be seen by examining results presented at the beginning of Tables 3, 4.

5 Nonsymmetric ill-posed matrix

Now we consider a system of linear algebraic equations with positive definite nonsymmetric ill-posed matrix A, $\det(A) \neq 0$,

$$Az = f, \tag{7}$$

thus this system has a unique solution.
In practical problems, it often happens that matrix A and vector of the right-hand side f are given with some error $\delta > 0$:

$$\|A - A_\delta\| \leq \delta,\ \|f - f_\delta\| \leq \delta,$$

and instead of system $Az = f$ we have a system with a perturbed A_δ and f_δ:

$$A_\delta z = f_\delta. \tag{8}$$

It is not known whether system (8) has any solution or not, but many pseudo-solutions $\tilde{z}$ can be constructed, i.e., such vectors $\tilde{z}$, on which discrepancy norm

$$\|A_\delta \tilde{z} - f_\delta\|$$

is minimal. We can choose a normal solution z^0 of system (8) from the set of pseudo-solutions (a normal solution is a pseudo-solution with minimum norm). Normal solution z^0 is unique. Further we take $\delta \in [0,\ \delta_1]$, $\delta_1 > 0$, δ tend to 0 in a way that normal solution (8) will tend to normal solution (7).

We consider the smoothing functional:

$$M_\alpha(\tilde{z}, A_\delta, f_\delta) = \|A_\delta \tilde{z} - f_\delta\|^2 + \alpha\|\tilde{z}\|^2, \tag{9}$$

where $\alpha > 0$.

Let us consider the behavior of the functional M_α, when the vector z is changed to a vector tv, where v is an arbitrary nonzero vector, t is numeric parameter. We consider the function:

$$\Phi(t) = M_\alpha(z + tv, A, f) = \|A(z + tv) - f\|^2 + \alpha\|z + tv\|^2. \tag{10}$$

A necessary condition for an extremum is the vanishing of derivative:

$$\left.\frac{d\Phi(t)}{dt}\right|_{t=0} = 0.$$

Using the scalar product instead of norm in (9), (10), we get:

$$\Phi(t) = (A(z + tv) - f, A(z + tv) - f) + \alpha(z + tv, z + tv).$$

We differentiate $\Phi(t)$ by t and calculate the derivative at the point $t = 0$:

$$\Phi_t'(0) = 2(Av, Az - f) + 2\alpha(v, z) = 0.$$

Thus, we obtain relations:

$$\begin{aligned}(Av, Az - f) + \alpha(v, z) &= (v, A^*Az - A^*f) + (v, \alpha z)\\ &= (v, A^*Az - A^*f + \alpha z) = 0\ \forall v.\end{aligned}$$

The last equality is equivalent to the Euler equation:

$$(A^*A + \alpha E)z = A^*f, \alpha > 0. \tag{11}$$

Matrix $A^*A + \alpha E$ *is* positive definite matrix, so (11) has a unique solution z_α which will be the point of minimum of functional (9).

Theorem. Let the original problem $Az = f$ have a solution, z^0 is a normal solution, z_α is the solution of the Euler equation, $\delta \in [0, \delta_1]$,. Let there be given two non-negative decreasing functions $\beta_1(\delta) \ge 0$, $\beta_2(\delta) \ge 0$, such that $\beta_1(0) = 0$, $\beta_2(0) = 0$, $\delta^2/\beta_1(\delta) \le \beta_2(\delta)$. There exists α_δ such that

$$\delta^2/\beta_1(\delta) \le \alpha_\delta \le \beta_2(\delta), \quad (12)$$

and let z_{α_δ} be a solution of the Euler equation. Then $z_{\alpha_\delta} \to z^0$ when $\delta \to 0$.

Proof. We have the following relations

$$\alpha_\delta \|z_\alpha\|^2 \le M_{\alpha_\delta}(z_{\alpha_\delta}, A_\delta, f_\delta) \le M_{\alpha_\delta}(z^0, A_\delta, f_\delta) = \|A_\delta z^0 - f_\delta\|^2 + \alpha_\delta \|z^0\|^2 =$$
$$= \|(A_\delta z^0 - Az^0) + (Az^0 - f) + (f - f_\delta)\|^2 + \alpha_\delta \|z^0\|^2 \le \delta^2(1 + \|z^0\|^2) + \alpha_\delta \|z^0\|^2 \quad (13)$$

Here we used

$A z^0 - f = 0$, $\|A - A_\delta\| \le \delta$, $\|f - f_\delta\| \le \delta$.

Now we obtain

$$\alpha_\delta \|z_\alpha\|^2 \le \delta^2(1 + \|z^0\|^2) + \alpha_\delta \|z^0\|^2. \quad (14)$$

Using (12), (14) we get the following relations

$$\frac{\delta^2}{\alpha_\delta} \le \beta_1(\delta),$$
$$\|z_{\alpha_\delta}\|^2 \le \beta_1(\delta)(1 + \|z^0\|^2) + \|z^0\|^2 \le \beta_1(\delta_1)(1 + \|z^0\|^2) + \|z^0\|^2 = const, \quad (15)$$

when α_δ is such that (12) is satisfied. From (13) for $\delta \to 0$ we have the relation $M_{\alpha_\delta}(z_{\alpha_\delta}, A_\delta, f_\delta) = \|A_\delta z_{\alpha_\delta} - f_\delta\|^2 + \alpha_\delta \|z_{\alpha_\delta}\|^2 \to 0$, where the second term tends to 0, so $\|A_\delta z_{\alpha_\delta} - f_\delta\| \to 0$ for $\delta \to 0$, Furthermore, it is known, that if a sequence is uniformly bound in R^n then it will be compact (i.e. it contains a convergent subsequence). Now we can take subsequence $\{\alpha_\delta\}$ such that $z_{\alpha_\delta} \to z^*$, and for $\delta \to 0$ we have $\|Az^* - f\| = 0$. From (15) for $\delta = 0$ we receive $\|z^*\| \le \|z^0\|$. Normal solution is unique, therefore, $z^* = z^0$. Thus $z_{\alpha_\delta} \to z^0$ *when* $\delta \to 0$.

Example. Let us replace element $h_{20,1} = 1/20$ of the matrix H_{20} with the element $\tilde{h}_{20,1} = 1/(20.1)$ and replace element $h_{1,20} = 1/20$ of the matrix H_{20} with the element $\tilde{h}_{1,20} = 1/(19.9)$. Now we obtain matrix $\tilde{H}_{20}$ in which all elements except $\tilde{h}_{20,1}$ and $\tilde{h}_{1\,20}$ coincide with the elements of the matrix H_{20}. Matrix $\tilde{H}_{20}$ is not symmetric, $\|\tilde{H}_{20} - H\| \le 0.00236 = \delta$. Multiplying matrix $\tilde{H}_{20}$ by vector $z = z_{20} = (1, \dots, 1)^T$, which consists of ones, we obtain vector f. The solution of system of equations

$$\tilde{H}_{20}\, z = f \quad (16)$$

is presented in the first column of Table 5.

Table 5. Solutions of SLAE (16) without regularization and with regularization

Without regulari-zation	Regularization with α	
	$\alpha = 1.0 \cdot 10^{-11}$	$\alpha = 1.0 \cdot 10^{-12}$
1.00	0.99999	0.99999
1.00	1.00020	1.00022
0.99999	0.99926	0.99919
1.00	1.00024	1.00030
0.99991	1.00078	1.00084
1.00118	1.00056	1.00058
0.98985	0.99996	0.99993
1.06106	0.99940	0.99935
0.73153	0.99910	0.99906
1.87958	0.99914	0.99911
-1.17803	0.99946	0.99945
5.10402	0.99994	0.99995
-4.88391	1.00045	1.00048
7.36947	1.00088	1.00092
-4.11708	1.00111	1.00116
3.95614	1.00108	1.00112
-1.16054	1.00071	1.00074
1.27714	0.99998	0.99999
0.96962	0.99886	0.99885
0.99999	0.99884	0.99873

Calculating in Maple with Digits=50 we obtain that $cond(\tilde{H}_{20}) = 0.168 \cdot 10^{28}$ and eigenvalues $\lambda_i^{\tilde{H}}$ of matrix $\tilde{H}_{20}$ are in interval $[0.3 \cdot 10^{-26}, 1.91]$. To solve the system of equations $\tilde{H}_{20}\, z = f$ by the regularization method, we have to calculate the conjugate matrix $\tilde{H}_{20}^* = \tilde{H}_{20}^T$, where the matrix $\tilde{H}_{20}^T$, is transposed to $\tilde{H}_{20}$. Let λ_i be eigenvalues of $\tilde{H}_{20}^T \tilde{H}_{20}$. It can be calculated that $\lambda_i \in [0.98 \cdot 10^{-51}, 3.63]$, and $cond(\tilde{H}_{20}^T \tilde{H}_{20}) =$

$0.128 \cdot 10^{53}$. Using the regularization method for the nonsymmetric matrix $\widetilde{H}_{20}$ with $\alpha = 1.0 \cdot 10^{-11}$ and $\alpha = 1.0 \cdot 10^{-12}$, we obtain solutions of the SLAE

$$(\widetilde{H}_{20}^T \widetilde{H}_{20} + \alpha E_{20})z = \widetilde{H}_{20}^T f \qquad (17)$$

presented in Table 5. Using the regularization method for the nonsymmetric matrix $\widetilde{H}_{20}$ with $\alpha = 1.0 \cdot 10^{-11}$ and $\alpha = 1.0 \cdot 10^{-12}$, we obtain solutions of the SLAE (17) presented in Table 6. Calculations were made in Maple with *Digits=25*. Here calculations were done with help of Maple function LinearSolve**.**

Table 6. Solutions of SLAE with regularization (17)

Regularization with α	
$\alpha = 1.0 \cdot 10^{-19}$	$\alpha = 1.0 \cdot 10^{-20}$
0.99999999	0.99999999
0.99999995	0.99999997
1.00000086	1.00000056
0.99999559	0.99999593
1.00000814	1.00001490
0.99999756	0.99996780
0.99999792	1.00004035
0.99999240	1.00000495
0.99999483	0.99987202
1.00002277	1.00018443
1.00000180	0.99993786
1.00000485	1.00003619
0.99997911	0.99984255
0.99998618	1.00002733
1.00000875	1.00013538
1.00000757	1.00003839
0.99999990	0.99984772
1.00001528	1.00004360
0.99998635	1.00001002
1.00000016	1.00000006

Table 7 shows errors in the solution of SLAE (16) for different values of parameter α.The error in the solution was calculated using the Euclidean norm. Calculations were done in Maple, Digits=25.

We have

$$\|\widetilde{H}_{20} - H\| \leq 0.00236 = \delta,$$

where δ is small. We can obtain solutions of system (16) using regularization method (6). Tables 8, 9 show the results of calculations. Here we can see that solutions obtained with method (6) have smaller errors.

Table 7. Errors in the solution of SLAE with regularization (17) for different values of α.

α	Errors
$\alpha = 1.0 \cdot 10^{-10}$	$0.82 \cdot 10^{-2}$
$\alpha = 1.0 \cdot 10^{-12}$	$0.17 \cdot 10^{-2}$
$\alpha = 1.0 \cdot 10^{-15}$	$0.45 \cdot 10^{-3}$
$\alpha = 1.0 \cdot 10^{-17}$	$0.10 \cdot 10^{-3}$
$\alpha = 1.0 \cdot 10^{-18}$	$0.81 \cdot 10^{-4}$
$\boldsymbol{\alpha = 1.0 \cdot 10^{-19}}$	$\mathbf{0.44 \cdot 10^{-4}}$
$\alpha = 1.0 \cdot 10^{-20}$	$0.36 \cdot 10^{-3}$
$\alpha = 1.0 \cdot 10^{-21}$	$0.35 \cdot 10^{-2}$

Table 8. Solutions of SLAE (16) with regularization method (6)

Regularization with α	
$\alpha = 1.0 \cdot 10^{-19}$	$\alpha = 1.0 \cdot 10^{-20}$
1.00	1.00
1.00	1.00
1.00	1.00
1.00	1.00
1.00	1.00
0.99999998	0.99999999
1.00000005	1.00000005
0.99999991	0.99999991
0.99999997	0.99999995
1.00000058	1.00000065
0.99999849	0.99999839
1.00000196	1.00000203
0.99999878	0.99999882
0.99999997	0.99999982
1.00000045	1.00000063
0.99999999	0.99999984
0.99999969	0.99999978
1.00000020	1.00000017
0.99999999	0.99999996
1.00	1.00

Table 9. Errors in the solution of SLAE (16) for different values of parameter α in method (6).

α	Errors
$\alpha = 1.0 \cdot 10^{-19}$	$0.29 \cdot 10^{-5}$
$\alpha = 1.0 \cdot 10^{-20}$	$0.30 \cdot 10^{-5}$

6 Conclusion

In this paper we presented the results of a numerical solution to SLAEs with positive definite symmetric (or nonsymmetric but almost symmetrical) ill-conditioned matrices by the regularization method, modified by prof. of St. Petersburg State University Ryabov V.M. It is shown that the solution of SLAEs with Hilbert matrices using the regularization method can be substantially improved.

It can be calculated that $cond(H_{20}) = 0.628 \cdot 10^{28}$, and $cond(H_{20}^T H_{20}) = cond(H_{20}^2) = 0.178 \cdot 10^{58}$. Therefore, for solving SLAE $H_{20}z = f$ we should use method (6) instead of (11). The same for SLAE (16). The application of method (11) in these cases requires a large number of additional arithmetic operations, which can lead to an increase in calculation errors.

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