Tubes of finite II-type in the Euclidean 3-space

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Abstract: In this paper, we consider surfaces in the 3-dimensional Euclidean space E^3 which are of finite II-type, that is, they are of finite type, in the sense of B.-Y. Chen, corresponding to the second fundamental form. We present an important family of surfaces, namely, tubes in E^3. We show that tubes are of infinite II-type.

Surfaces in the Euclidean 3-space, Surfaces of finite Chen-type, Beltrami operator

1 Introduction

As is well known, the theory of surfaces of finite type were introduced by B.-Y. Chen about thirty years ago and it has been a topic of active research by many differential geometers since then. Let M^n be an n-dimensional submanifold of an arbitrary dimensional Euclidean space E^m. Denote by Δ^I the Beltrami-Laplace operator on M^n with respect to the first fundamental form I of M^n. The submanifold M^n is said to be of finite type, if the position vector $\mathbf{x}$ of M^n can be written as a finite sum of nonconstant eigenvectors of the operator Δ^I, that is if

$$\mathbf{x} = \mathbf{x}_0 + \sum_{i=1}^{k} \mathbf{x}_i, \qquad \Delta^I \mathbf{x}_i = \lambda_i \mathbf{x}_i, \quad i = 1, ..., k, \tag{1}$$

where $\mathbf{x}_0$ is a fixed vector and $\mathbf{x}_1, ..., \mathbf{x}_k$ are nonconstant maps such that $\Delta^I \mathbf{x}_i = \lambda_i \mathbf{x}_i, i = 1, ..., k$.

The class of finite type submanifolds in an arbitrary dimensional Euclidean spaces is very large, on the other hand, very little is known about surfaces of finite type in the Euclidean 3-space E^3. In particular, other than minimal surfaces, the circular cylinders and the spheres, no surfaces of finite type corresponding to the first fundamental form in the Euclidean 3-space are known. So in [5] B.-Y. Chen mentions the following problem

Problem 1 *Determine all surfaces of finite Chen I-type in E^3.*

In order to give an answer to the above problem, important families of surfaces were studied by different authors by proving that finite type ruled surfaces [7], finite type quadrics [8], finite type tubes [4], finite type cyclides of Dupin [10] and finite type spiral surfaces [1] are surfaces of the only known examples in E^3. However, for another classical families of surfaces, such as surfaces of revolution, translation surfaces as well as helicoidal surfaces, the classification of its finite type surfaces is not known yet. For a more details, the reader can refer to [6].

Following (1) we say that a surface M is of finite type with respect to the fundamental form II, or briefly of finite II-type if the position vector $\mathbf{x}$ of M can be written as a finite sum of nonconstant eigenvectors of the operator Δ^{II}, that is if

$$\mathbf{x} = \mathbf{x}_0 + \sum_{i=1}^{k} \mathbf{x}_i, \qquad \Delta^{II} \mathbf{x}_i = \lambda_i \mathbf{x}_i, \quad i = 1, ..., k, \tag{2}$$

where $\mathbf{x}_0$ is a fixed vector and $\mathbf{x}_1, ..., \mathbf{x}_k$ are nonconstant maps such that $\Delta^{II} \mathbf{x}_i = \lambda_i \mathbf{x}_i, i = 1, ..., k$. If, in particular, all eigenvalues $\lambda_1, \lambda_2, ..., \lambda_k$ are mutually distinct, then M is said to be of II-type k, otherwise M is said to be of infinite type. When $\lambda_i = 0$ for some i = 1,..., k, then M is said to be of null II-type k.

In general when M is of finite type k, it follows from (2) that there exist a monic polynomial, say $R(x) \neq 0$, such that $R(\Delta^{II})(\mathbf{x} - \mathbf{c}) = f0$. Suppose that $R(x) = x^k + \sigma_1 x^{k-1} + ... + \sigma_{k-1} x + \sigma_k$, then coefficients σ_i are given by

$$\begin{aligned}
\sigma_1 &= -(\lambda_1 + \lambda_2 + ... + \lambda_k), \\
\sigma_2 &= (\lambda_1\lambda_2 + \lambda_1\lambda_3 + ... \\
&\quad +\lambda_1\lambda_k + \lambda_2\lambda_3 + ... \\
&\quad +\lambda_2\lambda_k + ... + \lambda_{k-1}\lambda_k), \\
\sigma_3 &= -(\lambda_1\lambda_2\lambda_3 + ... + \lambda_{k-2}\lambda_{k-1}\lambda_k),
\end{aligned}$$

....

$$\sigma_k = (-1)^k \lambda_1 \lambda_2 ... \lambda_k.$$

Therefore the position vector $\mathbf{x}$ satisfies the following equation, (see [3])

$$(\Delta^{II})^k \mathbf{x} + \sigma_1 (\Delta^{II})^{k-1} \mathbf{x} + ... + \sigma_k (\mathbf{x} - \mathbf{c}) = \mathbf{0}. \quad (3)$$

In this paper we will pay attention to surfaces of finite II-type. Firstly, we will give a formula for $\Delta^{II}\mathbf{x}$. Further, we continue our study by proving finite type surfaces for an important class of surfaces, namely, tubes in E^3.

2 Preliminaries

Let $\mathbf{x} = \mathbf{x}(u^1; u^2)$ be a regular parametric representation of a surface M in the Euclidean 3-space E^3 referred to any system of coordinates u^1, u^2, which does not contain parabolic points, we denote by b_{ij} the components of the second fundamental form $II = b_{ij} du^i du^j$ of S. Let $\varphi(u^1, u^2)$ be a sufficient differentiable function on M. Then the second differential parameter of Beltrami with respect to the second fundamental form of M is defined by [14]

$$\Delta^{II} \varphi := -\frac{1}{\sqrt{|b|}} (\sqrt{|b|} b^{ij} \varphi_{/i})_{/j} \quad (4)$$

where (b^{ij}) denotes the inverse tensor of (b_{ij}) and $b := \det(b_{ij})$. Applying (4) for the position vector $\mathbf{x}$ of M, we find

$$\Delta^{II} \mathbf{x} = -\frac{1}{2K} \nabla^{III} (K, \mathbf{n}) - 2\mathbf{n} \quad (5)$$

where $K, \mathbf{n}$ and ∇^{III} denote the curvature, the unit normal vector field and the first Beltrami-operator with respect to III, see [16].

From (5) we obtain the following results which were proved in [16]:

Theorem 1 *A surface S in E^3 is of II-type 1 if and only if S is part of a sphere.*

Theorem 2 *The Gauss map of a surface M in E^3 is of II-type 1 if and only if M is part of a sphere.*

Up to now, the only known surfaces of finite II-type in E^3 are parts of spheres. So the following question seems to be interesting:

Problem 2 *Other than the spheres, which surfaces in E^3 are of finite II-type?*

Another generalization of the above problem is to study surfaces in E^3 of coordinate finite type, that is, their position vector $\mathbf{x}$ satisfying the relation

$$\Delta^{II} \mathbf{x} = A\mathbf{x}, \quad (6)$$

where $A \in R^{3\times 3}$.

From this point of view, we also pose the following problem

Problem 3 *Classify all surfaces in E^3 with the position vector $\mathbf{x}$ satisfying relation (6).*

This paper provides the first attempt at the study of finite type families of surfaces in E^3 corresponding to the second fundamental form. Our main result is the following

Theorem 3 *All tubes in E^3 are of infinite type corresponding to the second fundamental form.*

Our discussion is local, which means that we show in fact that any open part of a tube is of infinite Chen type.

3 Tubes in E^3

Let $\ell : \mathbf{w} = \mathbf{w}(u)$, $u \epsilon (a, b)$ be a regular unit speed curve of finite length which is topologically imbedded in E^3. The total space $N_{\mathbf{w}}$ of the normal bundle of $\mathbf{w}((a, b))$ in E^3 is naturally diffeomorphic to the direct product $(a, b) \times E^2$ via the translation along $\mathbf{w}$ with respect to the induced normal connection. For a sufficiently small $r > 0$ the tube of radius r about the curve $\mathbf{w}$ is the set:

$$T_r(\mathbf{w}) = \{exp_{\mathbf{w}(u)} \mathbf{w} \mid \mathbf{w} \in N_{\mathbf{w}}, \parallel \mathbf{w} \parallel = r, u \in (a, b)\}.$$

Assume that $\mathbf{t}, \mathbf{h}, \mathbf{b}$ is the Frenet frame and κ the curvature of the unit speed curve $\mathbf{w} = \mathbf{w}(u)$. For a small real number r satisfies $0 < r < min \frac{1}{|\kappa|}$, the tube $T_r(\mathbf{w})$ is a smooth surface in E^3, [15]. Then a parametric representation of the tube $T_r(\mathbf{w})$ is given by

$$F : \mathbf{x}(u, \varphi) = \mathbf{w} + r \cos\varphi \mathbf{h} + r \sin\varphi \mathbf{b}. \quad (7)$$

It is easily verified that the first and the second fundamental forms of F are given by

$$\begin{aligned} I &= (\delta^2 + r^2\tau^2) du^2 + 2r^2 \tau du d\varphi + r^2 d\varphi^2, \\ II &= (-\kappa\delta\cos\varphi + r\tau^2) du^2 + 2r\tau du d\varphi + r d\varphi^2, \end{aligned}$$

where $\delta := (1 - r\kappa\cos\varphi)$ and τ is the torsian of the curve $\mathbf{w}$. The Gauss curvature of F is given by

$$K = -\frac{\kappa\cos\varphi}{r\delta}. \tag{8}$$

Notice that $\kappa \neq 0$ since the Gauss curvature vanishes. The Beltrami operator corresponding to the second fundamental form of F can be expressed as follows

$$\begin{aligned}\Delta^{II} = & -\frac{1}{\kappa\delta\cos\varphi}\Bigg[\frac{\partial^2}{\partial u^2} - 2\tau\frac{\partial^2}{\partial u\partial\varphi} \\ & +\Big(\tau^2 - \frac{\kappa\delta\cos\varphi}{r}\Big)\frac{\partial^2}{\partial\varphi^2} + \frac{(1-2\delta)\beta}{2\kappa\delta\cos\varphi}\frac{\partial}{\partial u} \\ & +\Big(-\tau' + \frac{\tau\beta(2\delta-1)}{2\kappa\delta\cos\varphi} + \\ & \frac{\kappa(2\delta-1)\sin\varphi}{2r}\Big)\frac{\partial}{\partial\varphi}\Bigg],\end{aligned} \tag{9}$$

where $\beta := \kappa'\cos\varphi + \kappa\tau\sin\varphi$ and $' := \frac{d}{du}$.

Before we start of proving our main result, we mention and prove the following special case of tubular surfaces for later use.

3.1 Anchor rings

A tube in the Euclidean 3-space is called an anchor ring if the curve ℓ is a plane circle (or is an open portion of a plane circle). In this case, the torsian τ of $\mathbf{w}$ vanishes identically and the curvature κ of $\mathbf{w}$ is a nonzero constant. Then the position vector $\mathbf{x}$ of the anchor ring can be expressed as

$$\begin{aligned}F : \mathbf{x}(u,\varphi) = \\ \{(a + r\cos u)\cos\varphi, (a + r\cos u)\sin\varphi, r\sin u\},\end{aligned} \tag{10}$$

$$a > r, \;\; a\epsilon R.$$

The first fundamental form is

$$I = r^2du^2 + (a + r\cos u)^2 d\varphi^2,$$

while the second is

$$II = rdu^2 + (a + r\cos u)\cos u d\varphi^2.$$

Hence, the Beltrami operator is given by

$$\begin{aligned}\Delta^{II} = -\frac{1}{r}\frac{\partial^2}{\partial u^2} - \Big[\frac{1}{(a + r\cos u)\cos u}\Big]\frac{\partial^2}{\partial\varphi^2} \\ +\frac{\sin u}{2r}\Big[\frac{1}{\cos u} + \frac{r}{a + r\cos u}\Big]\frac{\partial}{\partial u}.\end{aligned} \tag{11}$$

Let x_3 be the third coordinate function of $\mathbf{x}$. By virtue of (11) one can find

$$\Delta^{II}x_3 = \frac{3}{2}\sin u + \frac{r\cos u\sin u}{2(a + r\cos u)}. \tag{12}$$

which can be rewritten as

$$\begin{aligned}(\Delta^{II})^2x_3 = \frac{1}{(a + r\cos u)^2\cos u}f_1(\cos u, \sin u) \\ -\frac{3r^2\cos^2 u\sin^3 u}{4(a + r\cos u)^3\cos u},\end{aligned} \tag{13}$$

where $f_1(\cos u, \sin u)$ is a polynomial in $\cos u, \sin u$ of degree 5. Moreover, by a direct computation, it can be easy seen that

$$\begin{aligned}(\Delta^{II})^n x_3 = \\ \frac{1}{(a + r\cos u)^{2n-2}\cos u}f_{n-1}(\cos u, \sin u) \\ +\frac{\lambda_n r^n\cos^2 u\sin^{2n-1} u}{2^n(a + r\cos u)^{2n-1}\cos u},\end{aligned} \tag{14}$$

where $f_{n-1}(\cos u, \sin u)$ is a polynomial in $\cos u, \sin u$ of degree $2n+1$ and

$$\lambda_n = (-1)^{n-1}\prod_{j=1}^{n}(2j-3)(4j-5),$$

Now, if F is of finite type, then there exist real numbers, $c_1, c_2, ..., c_n$ such that

$$(\Delta^{II})^n\mathbf{x} + c_1(\Delta^{II})^{n-1}\mathbf{x} + ... + c_{n-1}\Delta^{II}\mathbf{x} + c_n\mathbf{x} = \mathbf{0}. \tag{15}$$

Since $x_3 = r\sin u$ is the third coordinate of $\mathbf{x}$, one gets

$$(\Delta^{II})^n x_3 + c_1(\Delta^{II})^{n-1}x_3 + ... + c_{n-1}\Delta^{II}x_3$$

$$+c_n x_3 = 0. \tag{16}$$

From (12-14) and (16) we obtain that

$$\frac{1}{(a + r\cos u)^{2n-2}\cos u} f_{n-1}(\cos u, \sin u)$$

$$+\frac{\lambda_n r^n \cos^2 u \sin^{2n-1} u}{2^n (a + r\cos u)^{2n-1}\cos u}$$

$$+c_1 \frac{1}{(a + r\cos u)^{2n-4}\cos u} f_{n-2}(\cos u, \sin u)$$

$$+c_1 \frac{\lambda_{n-1} r^{n-1} \cos^2 u \sin^{2n-3} u}{2^{n-1}(a + r\cos u)^{2n-3}\cos u}$$

$$+ ... + \frac{3}{2} c_{n-1} \sin u +$$

$$c_{n-1} \frac{r\cos u \sin u}{2(a + r\cos u)} + c_n r \sin u = 0$$

which can be rewritten as

$$\frac{\lambda_n r^n \cos u \sin^{2n-1} u}{2^n (a + r\cos u)} + G(\cos u, \sin u) = 0, \tag{17}$$

where G is a polynomial of the variables $\cos u, \sin u$ of degree $2n - 1$.

This is impossible for any $n \geq 1$ since $\lambda_n \neq 0$. Consequently, we have the following

Corollary 4 *Every anchor ring in the Euclidean 3-space is of infinite II-type.*

4 Proof of the main theorem

Applying relation (9) on the position vector **x** of (7) gives

$$\Delta^{II}\mathbf{x} = \frac{\beta\delta}{2(\kappa\delta\cos\varphi)^2}\mathbf{t}$$

$$-\left(2\cos\varphi + \frac{\sin^2\varphi}{2\delta\cos\varphi}\right)\mathbf{h}$$

$$-\left(2\sin\varphi - \frac{\sin\varphi}{2\delta}\right)\mathbf{b},$$

which can be rewritten as

$$\Delta^{II}\mathbf{x} = \frac{\beta\delta}{2(\kappa\delta\cos\varphi)^2}\mathbf{t}$$

$$+\frac{1}{\kappa\delta\cos\varphi}\mathbf{Q_1}(\cos\varphi, \sin\varphi), \tag{18}$$

where $\mathbf{Q_1}(x, y)$ is a vector valued polynomial in x, y of degree 3 with functions in u as coefficients. Moreover, by a long computation, we obtain

$$(\Delta^{II})^2\mathbf{x} = \frac{\delta(3\delta - 1)(12\delta - 5)\beta^3}{4(\kappa\delta\cos\varphi)^5}\mathbf{t}$$

$$+\frac{1}{(\kappa\delta\cos\varphi)^4}\mathbf{Q_2}(\cos\varphi, \sin\varphi), \tag{19}$$

where $\mathbf{Q_2}(x, y)$ is a vector valued polynomial in x, y of degree 7 with functions in u as coefficients.

We need the following lemma which can be proved directly by using (9).

Lemma 5 *For any natural numbers m and n we have*

$$\left(\Delta^{II}\frac{\delta g(\delta)\beta^m}{(\kappa\delta\cos\varphi)^n}\right) = -\frac{\delta\widetilde{g}(\delta)\beta^{m+2}}{(\kappa\delta\cos\varphi)^{n+3}}$$

$$+\frac{1}{(\kappa\delta\cos\varphi)^{n+2}} P(\cos\varphi, \sin\varphi),$$

where $g(\delta)$ is a polynomial in δ of degree d, P is a polynomial in x, y of degree $n + 3$ with functions in u as coefficients and $\deg(\widetilde{g}(\delta)) = d + 2$.

Using lemma 5 and relation (9) one finds

$$(\Delta^{II})^\lambda\mathbf{x} = d_\lambda \frac{\beta^{2\lambda-1}}{(\kappa\cos\varphi)^{4\lambda-1}}\mathbf{t}$$

$$+\frac{1}{(\kappa\cos\varphi)^{4\lambda-2}}\mathbf{P}_\lambda(\cos\varphi, \sin\varphi), \tag{20}$$

where

$$d_\lambda = (-1)^{\lambda-1}\prod_{j=1}^{2\lambda-1}(2j - 1).$$

It can be seen that $d_\lambda \neq 0$, for each natural number λ. Moreover, we have

$$(\Delta^{II})^{\lambda+1}\mathbf{x} = d_{\lambda+1}\frac{\beta^{2\lambda+1}}{(\kappa\cos\varphi)^{4\lambda+3}}\mathbf{t}$$

$$+\frac{1}{(\kappa\cos\varphi)^{4\lambda+2}}\mathbf{P}_{\lambda+1}(\cos\varphi,\sin\varphi). \qquad (21)$$

Let F be of finite type. Then there exist real numbers, $c_1, c_2, ..., c_\lambda$ such that

$$(\Delta^{II})^{\lambda+1}\mathbf{x} + c_1(\Delta^{II})^{\lambda}\mathbf{x} + ... + c_\lambda\Delta^{II}\mathbf{x} = \mathbf{0}. \quad (22)$$

Using (18-21), one has

$$d_{\lambda+1}\frac{\beta^{2\lambda+1}}{\kappa\cos\varphi}\mathbf{t} = Q_1\mathbf{t} + Q_2\mathbf{h} + Q_3\mathbf{b}, \qquad (23)$$

where $Q_i, i = 1, 2, 3$, are polynomials in r, s with functions in u as coefficients.

Now, if $\beta \neq 0$. From (23) we find

$$d_{\lambda+1}\frac{\beta^{2\lambda+1}}{\kappa\cos\varphi} = Q_1(\cos\varphi,\sin\varphi) \qquad (24)$$

This is impossible, since Q_1 is polynomial in $\cos\varphi$ and $\sin\varphi$. Assume now $\beta = 0$. Then $\kappa' = 0$ and $\kappa\tau = 0$ so $\kappa = const. \neq 0$ and $\tau = 0$. Therefore the curve ℓ is a circle, and so F is anchor ring. Hence, F is of infinite type according to Corollary (4). This completes our proof.

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