Exact Analytical Three and Two-Dimensional Solutions for Systems with Rectangular Fin

BUIKIS ANDRIS, BUIKE MARGARITA
Institute of Mathematics and Computer Science, Latvian University,
Raina bulv.29, Riga, LV1459
LATVIA
buikis@latnet.lv; mbuike@lanet.lv; http://www.lza.lv/scientists/buikis.htm

Abstract: - In this paper we develop mathematical models for three an two dimensional stationary hyperbolic heat equations with inner source power and we construct their analytical solutions. We solved three-dimensional problem for two contacted rectangles with inner heat sources and full non-homogeneous boundary conditions. The application for such mathematical model can be very different. Exact solutions is in the form of the Fredholm integral equation on the continuous plane between both rectangles. We use Green function for the both rectangles.

Key-Words: - Elliptic equation, Non-homogeny boundary condition, Non-canonical domain, Green function, Exact solution.

1 Introduction

Real processes take place in natural or technical systems with complicated structure. Very often such systems consist of separate layers with different thickness and different physical properties. It means that on the surfaces between two adjacent layers we have jump in coefficients of differential equations mathematically describing correspondent physical process. A great number of different engineering branches are concerned with rapid heat energy transitions. In the construction of various types of efficient heat transfer equipment to the so-called prime surface are supplemented with additional surfaces, e.g., a rectangular fin. Such heat transfer equipment is related to refrigerators, radiators, engines and microelectronics, etc. The traditional mathematical description of heat flow between a source and a sink very often is bounded by the so-called Murray-Gardner's hypotheses [1] - [3]. Usually its mathematical modeling is realized by one dimensional steady-state assumptions [3].

We investigate such type of systems with fins more than 20 years. In our previous papers in the years 80-ties and later we elaborate conservative averaging method [4] – [11]. This method is applicable for very different mathematical models of all types of differential equations: parabolic [12] – [22], [24], elliptic [22] and hyperbolic types [23], [27] – [36]. We investigate the parabolic type equations for underground fluid movements in the multilayered systems [12] – [15].

We have constructed two and three dimensional analytical approximate [12] – [15] and exact [21] solutions.

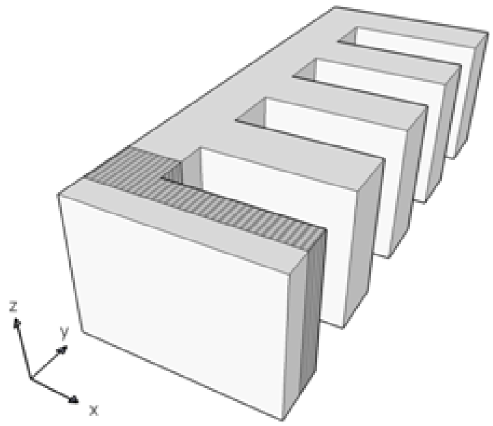
In this paper we obtain exact analytical three and two dimensional solutions by the original method of Green function method for non-canonical domain. Here we look for two connected rectangles. We can use Green function method for two rectangles, use conjugations conditions between both rectangles. It is possible to use this method is possible use for more canonical domains. For example we use this method for system with two fins [17], [22]. We develop our investigation of [38] in additional solution of three dimensional problem.

2 Statement of the 3-D problem

It is widely known that Green function method allows solving the boundary problem for the non-homogeneous equation and non-homogeneous boundary conditions. We ignore the 1-D problem statement and use 3-D statement with energy equality. Important is that Green function method is applicable for canonical domain. System with fin consists for the wall: $\{x \in [0,\delta], y \in [0,b], z \in [0,c]\}$ and the fin: $\{x \in [\delta,a], y \in [0,b_1], z \in [0,c]\}$ in the non-dimensional arguments. This 3-D formulation can be used in very different applications. The main equation for the wall and boundary conditions we assume in non-homogeneous form:

$$\frac{\partial^2 V_0(x,y,z)}{\partial x^2}+\frac{\partial^2 V_0(x,y,z)}{\partial y^2}+\frac{\partial^2 V_0(x,y,z)}{\partial z^2}=$$

$$
\begin{aligned}
&= -Q_0(x,y,z), \left\{x \in [0,\delta], y \in [0,b], z \in [0,c]\right\},\\
&\left.\left(\frac{\partial V_0}{\partial x} - \beta_0 V_0\right)\right|_{x=0} = -q_{00}(y,z), y \in [0,b], z \in [0,c],\\
&\left.\left(\frac{\partial V_0}{\partial y} - \beta_1 V_0\right)\right|_{y=0} = -q_{01}(x,z), x \in [0,\delta], z \in [0,c],\\
&\left.\left(\frac{\partial V_0}{\partial y} + \beta_2 V_0\right)\right|_{y=b} = q_{02}(x,z), x \in [0,\delta], z \in [0,c], \qquad (1)\\
&\left.\left(\frac{\partial V_0}{\partial z} - \beta_3 V_0\right)\right|_{z=0} = -q_{03}(x,y),\ x \in (0,\delta), y \in (0,b),\\
&\left.\left(\frac{\partial V_0}{\partial z} + \beta_4 V_0\right)\right|_{z=c} = q_{04}(x,y),\ x \in (0,\delta), y \in (0,b).
\end{aligned}
$$



Similarly as in the main equation for the fin and boundary conditions we assume in non-homogeneous form:

$$
\begin{aligned}
&\frac{\partial^2 V(x,y,z)}{\partial x^2} + \frac{\partial^2 V(x,y,z)}{\partial y^2} + \frac{\partial^2 V(x,y,z)}{\partial z^2} =\\
&= -Q(x,y,z), \left\{x \in [\delta,a], y \in [0,b_1], z \in [0,c]\right\},\\
&\frac{\partial V}{\partial x} + \gamma_1 V = q_1(y,z), x = a,\\
&y \in [0,b_1], z \in [0,c],\\
&\frac{\partial V}{\partial y} - \gamma_2 V = -q_2(x,z), y = 0, x \in [\delta,a], \qquad (2)\\
&\frac{\partial V}{\partial y} + \gamma_3 V = q_3(x,z), y = b_1, x \in [\delta,a],\\
&\frac{\partial V}{\partial z} - \gamma_4 V = -q_4(x,y), z = 0,\\
&x \in [\delta,a], y \in [0,b_1],\\
&\frac{\partial V}{\partial z} + \gamma_5 V = q_5(x,y), z = c,\\
&x \in [\delta,a], y \in [0,b_1].
\end{aligned}
$$

In the statements for the wall and the fin we have only 5 boundary conditions. But we have two conjugations conditions between the wall and the fin are in the form:

$$
\begin{aligned}
&V_0\big|_{x=\delta-0} = V\big|_{x=\delta+0},\\
&\alpha_0 \left.\frac{\partial V_0}{\partial x}\right|_{x=\delta-0} = \alpha_1 \left.\frac{\partial V}{\partial x}\right|_{x=\delta+0}.
\end{aligned} \qquad (3)
$$

The main idea is to transform both conjugations conditions as third type boundary conditions (BC). The BC for the wall we formulate similar with to the BC for $x = \delta$:

$$
\left.\left(\frac{\partial V_0}{\partial x} + \beta_0 V_0\right)\right|_{x=\delta-0} = F(y,z). \qquad (4)
$$

As we see BC (4) is not known, it has function of the temperature of the wall and its first derivative. We assume that function $F(y,z)$ is known and we get the solve solution for the wall with the Green function.

3 Solution for the wall and fin with method of the Green function

We know the Green function for the rectangular domain, but we don't have boundary for the $x = \delta$:

$$
\begin{aligned}
&\left.\left(\frac{\partial V_0}{\partial x} + \beta_0 V_0\right)\right|_{x=\delta-0} = F(y,z) =\\
&= \frac{\alpha_1}{\alpha_0}\left.\left(\frac{\partial V}{\partial x} + \beta_6 V\right)\right|_{x=\delta+0}, \beta_6 = \beta_0 \frac{\alpha_0}{\alpha_1}.
\end{aligned} \qquad (5)
$$

Of course, the function $F(y,z)$ is unknown. But now we can write the exact solution of problem (1) by Green function in the form:

$$
\begin{aligned}
&V_0(x,y,z) = \Phi_0(x,y,z) +\\
&\int_0^b d\eta \int_0^c F(\eta,\varsigma) G_0(x,y,z,\delta,\eta,\varsigma) d\varsigma, \qquad (6)\\
&\Phi_0(x,y,z) =\\
&\int_0^\delta d\xi \int_0^b d\eta \int_0^c Q_0(\xi,\eta,\varsigma) G_0(x,y,z,\xi,\eta,\varsigma) d\varsigma +\\
&\int_0^b d\eta \int_0^c q_{00}(\eta,\varsigma) G_0(x,y,z,0,\eta,\varsigma) d\varsigma +\\
&\int_0^\delta d\xi \int_0^c q_{01}(\xi,\varsigma) G_0(x,y,z,\xi,0,\eta) d\varsigma +
\end{aligned}
$$

$$+\int_0^\delta d\xi\int_0^c q_{02}(\xi,\varsigma)G_0(x,y,z,\xi,b,\varsigma)d\varsigma \quad (7)$$
$$+\int_0^\delta d\xi\int_0^b q_{03}(\xi,\eta)G_0(x,y,z,\xi,\eta,0)d\eta$$
$$+\int_0^\delta d\xi\int_0^b q_{04}(\xi,\eta)G_0(x,y,z,\xi,\eta,c)d\eta.$$

The Green function has such form [37]:

$$G_0(x,y,z,\xi,\eta,\varsigma)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{s=1}^{\infty}\frac{\varphi_{n0}(x)\varphi_{n0}(\xi)}{\|\varphi_{n0}\|^2}\times$$
$$\frac{\phi_{m0}(y)\phi_{m0}(\eta)\chi_{s0}(z)\chi_{s0}(\varsigma)}{\|\phi_{m0}\|^2\|\varsigma_{s0}\|^2\left(\mu_{n0}^2+\lambda_{m0}^2+\nu_{s0}^2\right)}, \quad (8)$$

$$\varphi_{n0}(x)=\cos(\mu_{n0}x)+\frac{\beta_0}{\mu_n}\sin(\mu_{n0}x),$$
$$\|\varphi_{n0}\|^2=\frac{\beta_0}{\mu_{n0}^2}+\frac{a}{2}\left(1+\frac{\beta_0^2}{\mu_{n0}^2}\right),$$
$$\phi_{n0}(y)=\cos(\lambda_{m0}y)+\frac{\beta_1}{\lambda_{m0}}\sin(\lambda_{m0}y), \quad (9)$$
$$\|\phi_{n0}\|^2=\frac{\beta_2}{2\lambda_{m0}^2}\frac{\lambda_{m0}^2+\beta_1^2}{\lambda_{m0}^2+\beta_2^2}+\frac{\beta_1}{2\lambda_{m0}^2}+\frac{b}{2}\left(1+\frac{\beta_1^2}{\lambda_{m0}^2}\right),$$
$$\chi_{s0}(z)=\cos(\nu_{s0}z)+\frac{\beta_3}{\nu_{s0}}\sin(\nu_{s0}z),$$
$$\|\chi_{s0}\|^2=\frac{\beta_4}{2\nu_{s0}^2}\frac{\nu_{s0}^2+\beta_3^2}{\nu_{s0}^2+\beta_4^2}+\frac{\beta_3}{2\nu_{s0}^2}+\frac{c}{2}\left(1+\frac{\beta_3^2}{\nu_{s0}^2}\right).$$

The eigenvalues $\lambda_{m0},\mu_{n0},\upsilon_{s0}$ are positive roots of the transcendental equations:

$$\frac{2\beta_0}{\mu^2-\beta_0^2}=\frac{tg(\mu a)}{\mu},\ \frac{\beta_1+\beta_2}{\lambda^2-\beta_1\beta_2}=\frac{tg(\lambda b)}{\lambda},$$
$$\frac{\beta_3+\beta_4}{\nu^2-\beta_3\beta_4}=\frac{tg(\nu c)}{\nu}. \quad (10)$$

For the fin we need the combination $F_0(y,z)$ from the conjugation conditions (3):

$$\left.\left(\frac{\partial V}{\partial x}-\gamma_1 V\right)\right|_{x=\delta+0}=F_0(y,z)=$$
$$\frac{\alpha_0}{\alpha_1}\left.\left(\frac{\partial V_0}{\partial x}-\gamma_6 V_0\right)\right|_{x=\delta-0},\ \gamma_6=\gamma_1\frac{\alpha_1}{\alpha_0}, \quad (11)$$
$$y\in(0,b),\ z\in(0,c).$$

We transform the left side boundary condition from equation (4):

$$F_0(y,z)=\frac{\alpha_0}{\alpha_1}\Upsilon_0(x,y,z)\Big|_{x=\delta}-$$
$$\frac{\alpha_0}{\alpha_1}\int_0^b d\eta\int_0^c F(\eta,\varsigma)\Gamma_0(\delta,y,z,\xi,\eta,\varsigma)\Big|_{\xi=\delta}d\varsigma,$$
$$\Upsilon_0(x,y,z)=\frac{\partial\Phi_0(x,y,z)}{\partial x}-\beta_6\Phi_0(x,y,z), \quad (12)$$
$$\Gamma_0(x,y,z,\xi,\eta,\varsigma)=$$
$$\left.\left[\frac{\partial G_0(x,y,z,\xi,\eta,\varsigma)}{\partial x}-\beta_6 G_0(x,y,z,\xi,\eta,\varsigma)\right]\right|_{\xi=\delta}.$$

Now we can go to the fin. Exact solution of the fin we can write in the following form:

$$V(x,y,z)=\Phi(x,y,z)-$$
$$\frac{\alpha_1}{\alpha_0}\int_0^{b_1}d\eta\int_0^c F_0(\eta,\varsigma)G(x,y,z,\delta,\eta,\varsigma)d\varsigma,$$
$$\Phi(x,y,z)=$$
$$\int_\delta^a d\xi\int_0^{b_1}d\eta\int_0^c Q(\xi,\eta,\varsigma)G(x,y,z,\xi,\eta,\varsigma)d\varsigma$$
$$+\int_0^{b_1}d\eta\int_0^c q_1(\eta,\varsigma)G(x,y,z,a,\eta,\varsigma)\,d\varsigma \quad (13)$$
$$+\int_\delta^a d\xi\int_0^c q_2(\xi,\varsigma)G(x,y,z,\xi,0,\varsigma)d\varsigma$$
$$+\int_\delta^a d\xi\int_0^c q_3(\xi,\varsigma)G(x,y,z,\xi,b_1,\varsigma)d\eta$$
$$+\int_\delta^a d\xi\int_0^{b_1} q_4(\xi,\eta)G(x,y,z,\xi,\eta,0)d\eta$$
$$+\int_\delta^a d\xi\int_0^{b_1} q_5(\xi,\eta)G(x,y,z,\xi,\eta,c)d\eta.$$

The Green function has such form [37]:

$$G(x,y,z,\xi,\eta,\varsigma)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{s=1}^{\infty}\frac{\varphi_n(x)\varphi_n(\xi)}{\|\varphi_n\|^2}$$
$$\times\frac{\phi_m(y)\phi_m(\eta)\chi_s(z)\chi_s(\varsigma)}{\|\phi_m\|^2\|\varsigma_s\|^2\left(\mu_n^2+\lambda_m^2+\nu_s^2\right)},\ \varphi_n(x)= \quad (14)$$
$$=\cos(\mu_n(x-\delta))+\frac{\gamma_1}{\mu_n}\sin(\mu_n(x-\delta)),$$
$$\|\varphi_n\|^2=\frac{\gamma_1}{\mu_n^2}+\frac{a-\delta}{2}\left(1+\frac{\gamma_1^2}{\mu_n^2}\right),$$

$$\phi_n(y) = \cos(\lambda_m y) + \frac{\gamma_2}{\lambda_m}\sin(\lambda_m y),$$

$$\|\phi_n\|^2 = \frac{\beta_3}{2\lambda_m^2}\frac{\lambda_m^2+\beta_2^2}{\lambda_m^2+\beta_3^2} + \frac{\beta_3}{2\lambda_m^2} + \frac{b}{2}\left(1+\frac{\beta_2^2}{\lambda_m^2}\right),$$

$$\chi_s(z) = \cos(\nu_s z) + \frac{\gamma_4}{\nu_s}\sin(\nu_s z),$$

$$\|\chi_s\|^2 = \frac{\beta_5}{2\nu_s}\frac{\nu_s^2+\beta_4^2}{\nu_s^2+\beta_5^2} + \frac{\beta_4}{2\nu_s^2} + \frac{c}{2}\left(1+\frac{\beta_4^2}{\nu_s^2}\right).$$

The eigenvalues $\lambda_m, \mu_n, \upsilon_s$ are positive roots of the transcendental equations:

$$\frac{2\beta_1}{\mu^2-\beta_1^2} = \frac{tg\left[\mu(a-\delta)\right]}{\mu}, \frac{\beta_2+\beta_3}{\lambda^2-\beta_2\beta_3} = \frac{tg(\lambda b_1)}{\lambda},$$

$$\frac{\beta_4+\beta_5}{\chi^2-\beta_4\beta_5} = \frac{tg(\chi c)}{\chi}.$$

We transform the equation (13) with respect to $F(y,z)$:

$$F(y,z) = \frac{\alpha_1}{\alpha_0}\left[\frac{\partial\Phi(x,y,z)}{\partial x} + \beta_6\Phi(x,y,z)\right]\Bigg|_{x=\delta} -$$

$$\frac{\alpha_1}{\alpha_0}\int_\delta^a d\xi\int_0^{b_1} d\eta\int_0^c F_0(\xi,\eta,\varsigma)\Gamma(\delta,y,z,\xi,\eta,\varsigma)\Big|_{\xi=\delta} d\varsigma,$$

$$\Gamma(x,y,z,\delta,\eta,\varsigma) = \qquad (15)$$

$$\left[\frac{\partial}{\partial x}G(x,y,z,\xi,\eta,\varsigma) + \beta_6 G(x,y,z,\xi,\eta,\varsigma)\right]\Bigg|_{\xi=\delta}.$$

4 Exact integral solution for the wall and the fin

In this section we describe the connection of the wall and the fin. We designate:

$$\Gamma(x,y,z,\xi,\eta,\varsigma) = \frac{\partial G(x,y,z,\xi,\eta,\varsigma)}{\partial x} + \beta_6 G(x,y,z,\xi,\eta,\varsigma). F_0(y,z) = \frac{\alpha_0}{\alpha_1}\Upsilon_0(x,y,z)\Big|_{x=\delta} -$$

$$\frac{\alpha_0}{\alpha_1}\int_0^b d\eta\int_0^c F(\eta,\varsigma)\Gamma_0(x,y,z,\xi,\eta,\varsigma)\Big|_{\xi=\delta} d\varsigma,\ F(x,y,z) = \frac{\alpha_1}{\alpha_0}\left[\frac{\partial\Phi(x,y,z)}{\partial x} + \beta_6\Phi(x,y,z)\right]\Bigg|_{\xi=\delta} -$$

$$\frac{\alpha_1}{\alpha_0}\int_0^{b_1} d\eta\int_0^c F_0(\eta,\varsigma)\Gamma(\delta,y,z,\xi,\eta,\varsigma)\Big|_{\xi=\delta} d\varsigma,\ F(\xi,\eta,\varsigma) =$$

$$\frac{\alpha_1}{\alpha_0}\left[\frac{\partial\Phi(x,y,z)}{\partial x} + \beta_6\Phi(x,y,z)\right]\Bigg|_{x=\delta} - \frac{\alpha_1}{\alpha_0}\int_0^{b_1} d\eta_1\int_0^c F_0(\eta_1,\varsigma_1)\Gamma(\delta,y,z,\xi,\eta_1,\varsigma_1)\Big|_{\xi=\delta} d\varsigma_1,$$

$$F_0(\eta_1,\varsigma_1) = \frac{\alpha_0}{\alpha_1}\left[\Upsilon_0(x,y,z)\Big|_{x=\delta} + \frac{\alpha_0}{\alpha_1}\int_0^b d\eta_1\int_0^c F(\xi_1,\eta_1,\varsigma_1)\Gamma_0(\delta,y,z,\xi,\eta_1,\varsigma_1)\Big|_{\xi=\delta} d\varsigma_1\right].$$

The formulas (11) and (15) give such representation for $F_0(y,z), F(y,z)$:

$$F(y,z) = \frac{\alpha_1}{\alpha_0}\left[\frac{\partial\Phi(x,y,z)}{\partial x} + \beta_6\Phi(x,y,z)\right]\Bigg|_{x=\delta} -$$

$$\frac{\alpha_1}{\alpha_0}\int_\delta^a d\xi\int_0^{b_1} d\eta\int_0^c F_0(\eta,\varsigma)\Gamma(\delta,y,z,\xi,\eta,\varsigma)\Big|_{\xi=\delta} d\varsigma. (17)$$

Now we give representation for the $F(\eta,\varsigma)$:

$$F(\eta,\varsigma) = \frac{\alpha_1}{\alpha_0}\left[\frac{\partial\Phi(x,y,z)}{\partial x} + \beta_6\Phi(x,y,z)\right]\Bigg|_{x=\delta} -$$

$$\frac{\alpha_1}{\alpha_0}\int_0^{b_1} d\eta_1\int_0^c F_0(\eta_1,\varsigma_1)\Gamma(\delta,y,z,\xi,\eta_1,\varsigma_1)\Big|_{\xi=\delta} d\varsigma_1,$$

$$F_0(\eta_1,\varsigma_1) = \frac{\alpha_1}{\alpha_0}\Big[\Upsilon_0(x,y,z)\Big|_{x=\delta} -$$

$$\frac{\alpha_1}{\alpha_0}\int_0^b d\eta_1\int_0^c F(\eta_1,\varsigma_1)\Gamma_0(\delta,y,z,\xi,\eta_1,\varsigma_1)\Big|_{\xi=\delta} d\varsigma_1\Big].$$

It can be written in short form:

$$F(\eta,\varsigma) = \frac{\alpha_1}{\alpha_0}\left[\frac{\partial\Phi(x,y,z)}{\partial x} + \beta_6\Phi(x,y,z)\right]\Bigg|_{x=\delta} -$$

$$\frac{\alpha_1}{\alpha_0}\int_0^{b_1} d\eta_1\int_0^c \bar{F}(\eta_1,\varsigma_1)\Gamma(\delta,y,z,\xi,\eta_1,\varsigma_1)\Big|_{\xi=\delta} d\varsigma_1,$$

$$\bar{F}(\eta_1,\varsigma_1) =$$

$$\int_0^b d\eta_2\int_0^c F(\eta_2,\varsigma_2)\Gamma_0(\delta,y,z,\xi,\eta_2,\varsigma_2)\Big|_{\xi=\delta} d\varsigma_2.$$

(18)

First equation of (18) is Fredholm second type integral equation:

$$F(\eta,\varsigma) = \frac{\alpha_1}{\alpha_0}\left[\frac{\partial\Phi(x,y,z)}{\partial x} + \beta_6\Phi(x,y,z)\right]\Bigg|_{x=\delta} -$$

$$\frac{\alpha_1}{\alpha_0}\int_0^{b_1} d\eta_1\int_0^c d\varsigma_2\int_0^b d\eta_2\int_0^c F(\eta_2,\varsigma_2)\times$$

$$\Gamma_0(\delta,y,z,\xi,\eta_2,\varsigma_2)\Big|_{\xi=\delta}\Gamma(\delta,y,z,\xi,\eta_1,\varsigma_1)\Big|_{\xi=\delta} d\varsigma_1.$$

To solve this integral equation we can solve integral equation (16) and equation (17). Than is easy to solve stationary equations (6), (13).

5 Statement of the 2-D problem

The main equation for the wall and boundary conditions we assume in non-homogeneous form:

$$\frac{\partial^2 V_0(x,y)}{\partial x^2}+\frac{\partial^2 V_0(x,y)}{\partial y^2}=-Q_0(x,y),$$
$$\{x\in[0,\delta],y\in[0,b]\},$$
$$\left.\left(\frac{\partial V_0}{\partial x}-\beta_0 V_0\right)\right|_{x=0}=-q_{00}(y),y\in[0,b],$$
$$\left.\left(\frac{\partial V_0}{\partial y}-\beta_3 V_0\right)\right|_{y=0}=-q_{03}(x),x\in[0,\delta], \tag{19}$$
$$\left.\left(\frac{\partial V_0}{\partial y}+\beta_4 V_0\right)\right|_{y=b}=q_{04}(x),x\in[0,\delta].$$

Similarly as in the main equation for the fin, and boundary conditions we assume in non-homogeneous form:

$$\frac{\partial^2 V(x,y)}{\partial x^2}+\frac{\partial^2 V(x,y)}{\partial y^2}==-Q(x,y),$$
$$\{x\in[\delta,a],y\in[0,b_1]\},$$
$$\frac{\partial V}{\partial x}-\beta_1 V=-q_1(y),x=0,y\in[0,b_1], \tag{20}$$
$$\frac{\partial V}{\partial y}-\beta_2 V=-q_2(x),y=0,x\in[\delta,a],$$
$$\frac{\partial V}{\partial y}+\beta_5 V=q_3(x),y=b_1,x\in[\delta,a].$$

We have two conjugations conditions between the wall and the fin are in the form:

$$V_0\big|_{x=\delta-0}=V\big|_{x=\delta+0},$$
$$\alpha_0\left.\frac{\partial V_0}{\partial x}\right|_{x=\delta-0}=\alpha_1\left.\frac{\partial V}{\partial x}\right|_{x=\delta+0}. \tag{21}$$

The BC for the wall we formulate similar with to the BC for :

$$\left.\left(\frac{\partial V_0}{\partial x}+\beta_0 V_0\right)\right|_{x=\delta-0}=F(y)=$$
$$=\frac{\alpha_1}{\alpha_0}\left.\left(\frac{\partial V}{\partial x}+\beta_6 V\right)\right|_{x=\delta+0},\beta_6=\beta_0\frac{\alpha_0}{\alpha_1}. \tag{22}$$

Solution of the fin we can write in the form:

$$V_0(x,y)=\Phi_0(x,y)+\int_0^{b_1}F(\eta)G_0(x,y,\delta,\eta)\,d\eta,$$
$$\Phi(x,y)=\int_0^{\delta}d\xi\int_0^{b}Q_0(\xi,\eta)G_0(x,y,\xi,\eta)\,d\eta$$
$$+\int_0^{b}q_{00}(\eta)G_0(x,y,0,\eta)\,d\eta$$
$$+\int_0^{\delta}q_{03}(\xi)G_0(x,y,\xi,0)\,d\xi+$$
$$+\int_0^{\delta}q_{04}(\xi)G_0(x,y,\xi,b)\,d\xi.$$

The Green function has such form, similar with formula (8):

$$G_0(x,y,\xi,\eta)=$$
$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\varphi_{n0}(x)\varphi_{n0}(\xi)\phi_{m0}(y)\phi_{m0}(\eta)}{\|\varphi_{n0}\|^2\|\phi_{m0}\|^2\left(\mu_{n0}^2+\lambda_{m0}^2\right)},$$
$$\varphi_{n0}(x)=\cos(\mu_{n0}x)+\frac{\beta_0}{\mu_{n0}}\sin(\mu_{n0}x),$$
$$\|\varphi_{n0}\|^2=\frac{\beta_0}{\mu_{n0}^2}+\frac{\delta}{2}\left(1+\frac{\beta_0^2}{\mu_{n0}^2}\right),$$
$$\phi_{n0}(y)=\cos(\lambda_{m0}y)+\frac{\beta_3}{\lambda_{m0}}\sin(\lambda_{m0}y),$$
$$\|\phi_{n0}\|^2=\frac{\beta_4}{2\lambda_m^2}\frac{\lambda_{m0}^2+\beta_3^2}{\lambda_{m0}^2+\beta_4^2}+\frac{\beta_3}{2\lambda_{m0}^2}+\frac{b}{2}\left(1+\frac{\beta_3^2}{\lambda_{m0}^2}\right).$$

The eigenvalues a re positive roots of the transcendental equations:

$$\frac{2\beta_0}{\mu^2-\beta_0^2}=\frac{tg[\mu\delta]}{\mu},\frac{\beta_3+\beta_4}{\lambda^2-\beta_3\beta_4}=\frac{tg(\lambda b)}{\lambda}.$$

The Green function for the fin is:

$$G(x,y,\xi,\eta)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\varphi_n(x)\varphi_n(\xi)\phi_m(y)\phi_m(\eta)}{\|\varphi_n\|^2\|\phi_m\|^2\left(\mu_n^2+\lambda_m^2\right)},$$
$$\varphi_n(x)=$$
$$\cos(\mu_n(x-\delta))+\frac{\beta_1}{\mu_n}\sin(\mu_n(x-\delta)),$$

$$\|\varphi_n\|^2 = \frac{\beta_1}{\mu_n^2} + \frac{a-\delta}{2}\left(1+\frac{\beta_1^2}{\mu_n^2}\right),$$

$$\phi_n(y) = \cos(\lambda_m y) + \frac{\beta_2}{\lambda_m}\sin(\lambda_m y),$$

$$\|\phi_n\|^2 = \frac{\beta_3}{2\lambda_m^2}\frac{\lambda_m^2+\beta_2^2}{\lambda_m^2+\beta_3^2} + \frac{\beta_3}{2\lambda_m^2} + \frac{b}{2}\left(1+\frac{\beta_2^2}{\lambda_m^2}\right).$$

The eigenvalues a re positive roots of the transcendental equations:

$$\frac{\beta_1+\beta_6}{\mu^2-\beta_1\beta_6} = \frac{tg[\mu\delta]}{\mu}, \frac{\beta_2+\beta_5}{\lambda^2-\beta_2\beta_5} = \frac{tg(\lambda b)}{\lambda}.$$

We transform the equation (13) with respect to $F(y)$:

$$F(y) = \frac{\alpha_1}{\alpha_0}\left[\frac{\partial\Phi(x,y)}{\partial x} + \beta_6\Phi(x,y)\right]\Bigg|_{x=\delta} -$$

$$\frac{\alpha_1}{\alpha_0}\int_\delta^a d\xi\int_0^{b_1} F_0(\xi,\eta)\Gamma(\delta,y,\xi,\eta)\Big|_{\xi=\delta} d\eta,$$

$$\Gamma(x,y,\delta,\eta) =$$

$$\left[\frac{\partial}{\partial x}G(x,y,\xi,\eta) + \beta_6 G(x,y,\xi,\eta)\right]\Bigg|_{\xi=\delta}.$$

Solution for the fin:

$$V(x,y) = \Phi(x,y) +$$

$$\frac{\alpha_1}{\alpha_0}\int_0^{b_1} F_0(\eta,\varsigma)G(x,y,\delta,\eta)d\eta,$$

$$\Phi(x,y) = \int_\delta^a d\xi\int_0^{b_1} Q(\xi,\eta)G(x,y,\xi,\eta)d\eta +$$

$$\int_0^{b_1} q_1(\eta)G(x,y,a,\eta)d\eta +$$

$$+\int_\delta^a q_2(\xi)G(x,y,\xi,0)d\xi +$$

$$+\int_\delta^a q_3(\xi)G(x,y,\xi,\eta)d\xi$$

$$+\int_\delta^a q_4(\xi)G(x,y,z,\xi,\eta,c)d\xi.$$

The combination for the fin:

$$F(y) = \frac{\alpha_1}{\alpha_0}\left[\frac{\partial\Phi(x,y)}{\partial x} + \beta_6\Phi(x,y)\right]\Bigg|_{x=\delta}$$

$$-\frac{\alpha_1}{\alpha_0}\int_\delta^a F_0(\eta)d\xi\int_0^{b_1}\Gamma(\delta,y,\xi,\eta)\Big|_{\xi=\delta} d\eta.$$

$$F(\eta) = \frac{\alpha_1}{\alpha_0}\left[\frac{\partial\Phi(x,y)}{\partial x} + \beta_6\Phi(x,y)\right]\Bigg|_{x=\delta} -$$

$$\frac{\alpha_1}{\alpha_0}\int_0^{b_1} F_0(\eta_1)\Gamma(\delta,y,\xi,\eta_1)\Big|_{\xi=\delta} d\eta_1,$$

$$F_0(\eta_1) = \left[\frac{\alpha_1}{\alpha_0}\Upsilon_0(x,y)\Big|_{x=\delta} -\right.$$

$$\left.\frac{\alpha_1}{\alpha_0}\int_0^{b} F(\eta_1)\Gamma_0(\delta,y,\xi,\eta_1)\Big|_{\xi=\delta} d\eta_1\right].$$

The connection for both combination:

$$F(\eta) = \frac{\alpha_1}{\alpha_0}\left[\frac{\partial\Phi(x,y)}{\partial x} + \beta_6\Phi(x,y)\right]\Bigg|_{x=\delta} -$$

$$\alpha_1\alpha_0^{-1}\int_0^{b_1} F_0(\eta_1)\Gamma(\delta,y,z,\xi,\eta_1)\Big|_{\xi=\delta} d\eta_1, F_0(\eta_1) =$$

$$\left[\frac{\alpha_1}{\alpha_0}\Upsilon_0(x,y)\Big|_{x=\delta} + \int_0^{b} F(\eta_1)\Gamma_0(\delta,y,\xi,\eta_1)\Big|_{\xi=\delta} d\eta_1\right].$$

$$F(\eta) = \bar{\Phi}(\delta,y) - \frac{\alpha_1}{\alpha_0}\int_0^{b_1}\bar{F}(\eta_1)\Gamma(\delta,y,\xi,\eta_1)\Big|_{\xi=\delta} d\eta_1,$$

$$\bar{F}(\eta_1) = \int_0^{b} F(\eta_2)\Gamma_0(\delta,y,\xi,\eta_2)\Big|_{\xi=\delta} d\eta_2.\ F_0(y) =$$

$$\left[\frac{\alpha_1}{\alpha_0}\Upsilon_0(x,y)\Big|_{x=\delta} - \frac{\alpha_1}{\alpha_0}\int_0^{b} F(\eta)\Gamma_0(\delta,y,\xi,\eta)\Big|_{\xi=\delta} d\eta\right].$$

6 Conclusions

We solved three-dimensional and two-dimensional problems for two contacted rectangles with inner heat sources and full non-homogeneous boundary conditions. The applications for such mathematical models can be very different. Exact solution is in the form of the Fredholm integral equation on t he continuous plane between both rectangles. generalize Green function method for connected canonical rectangles. We generalize Green function method for connected canonical rectangles.

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