Rapid Decay of Solutions for a Coupled System of Wave Equations with Class of Relaxation Functions in any Space Dimension

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Abstract: We consider a coupled system of viscoelastic wave equations. In weighted spaces, we shall prove a fast decay of energy associated to a coupled system with class of relaxation functions, as $T \to \infty$ in $\mathbb{R}^n$.

Key–Words: Lyapunov function, viscoelastic, density, decay rate, weighted spaces, coupled system

1 Introduction

A coupled system of viscoelastic wave equations in any space dimension is given by

$$\begin{cases} \left(|u'|^{q-2}u'\right)' - \phi(x)\left(\Delta_x u - \int_0^t g_1(t-s)\Delta_x u(s)ds\right) \\ = \alpha v, \\ \\ \left(|v'|^{q-2}v'\right)' - \phi(x)\left(\Delta_x v - \int_0^t g_2(t-s)\Delta_x v(s)ds\right) \\ = \alpha u. \end{cases} \quad (1)$$

Here $x \in \mathbb{R}^n, \alpha \neq 0, t > 0, q \geq 2, n > 2$ and the scalar functions $g_i(s), i = 1, 2$ are assumed to satisfy (A1).
Our problem (1) is supplemented with the next initial data.

$$u(0,x) = u_0(x) \in \mathcal{H}(\mathbb{R}^n),$$
$$u'(0,x) = u_1(x) \in L^q_\rho(\mathbb{R}^n), \quad (2)$$

$$v(0,x) = v_0(x) \in \mathcal{H}(\mathbb{R}^n),$$
$$v'(0,x) = v_1(x) \in L^q_\rho(\mathbb{R}^n). \quad (3)$$

We introduce the weighted spaces in Definition 1, where $(\phi(x))^{-1} = \rho(x)$ satisfies

$$\rho : \mathbb{R}^n \to \mathbb{R}^*_+, \quad \rho(x) \in C^{0,\tilde{\gamma}}(\mathbb{R}^n) \quad (4)$$

with $\tilde{\gamma} \in (0,1)$ and $\rho \in L^s(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, where $s = \frac{2n}{2n-qn+2q}$.
There are many results about the existence by standard Galerkin method, (see [9], [10], [15], [16], [19], [21]), it is well known that, for any initial data $u_0, v_0 \in \mathcal{H}(\mathbb{R}^n)$ and $u_1, v_1 \in L^q_\rho(\mathbb{R}^n)$, the problem (1)-(3) has a unique weak solution

$$(u,v) \in C([0,\infty),\mathcal{H}(\mathbb{R}^n)) \times C([0,\infty),\mathcal{H}(\mathbb{R}^n)),$$

$$(u',v') \in C([0,\infty),L^q_\rho(\mathbb{R}^n)) \times C([0,\infty),L^q_\rho(\mathbb{R}^n)),$$

under hypotheses $(A1)-(A2)$. For completeness, if $u_0, v_0 \in \mathcal{D}(\mathbb{R}^n) \cap \mathcal{H}(\mathbb{R}^n)$ and $u_1, v_1 \in \mathcal{H}(\mathbb{R}^n)$, we state without proof, the regularity result as

$$(u,v) \in C([0,\infty),\mathcal{D}(\mathbb{R}^n)\cap\mathcal{H}(\mathbb{R}^n)) \times C([0,\infty),\mathcal{D}(\mathbb{R}^n)\cap\mathcal{H}(\mathbb{R}^n)),$$

$$(u',v') \in C([0,\infty),\mathcal{H}(\mathbb{R}^n)) \times C([0,\infty),\mathcal{H}(\mathbb{R}^n)).$$

This kind of problem with viscoelasticity was first introduced in [8], where a many qualitative results are obtained (see in this direction [1], [2], [6], [10], [11], [12], [13], [16], [18], [20],[21], [22],[23] [24]).

2 Definitions of Function Spaces and Assumption

We first, state without proof some useful results. Let us make use of the following assumption
(A1) The functions $g_i : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ are of class C^1 satisfying:

$$1 - \overline{g_i} = l_i > 0, \quad g_i(0) = g_{0i} > 0, \quad (5)$$

where $\overline{g_i} = \int_0^\infty g_i(t)dt$.

(A2) There exists a positive function $H \in C^1(\mathbb{R}^+)$ such that

$$g_i'(t) + H(g_i(t)) \leq 0, t \geq 0, \quad H(0) = 0 \quad (6)$$

and H is defined below.
A- With $t_1 > 0$ such that for $i = 1, 2$:

1) $\forall t \geq t_1$: We have

$$\lim_{s\to+\infty} g_i(s) = 0,$$

which gives that

$$\lim_{s\to+\infty} \Big(- g_i'(s)\Big)$$

cannot be positive, so

$$\lim_{s\to+\infty} \Big(- g_i'(s)\Big) = 0.$$

Then $g_i(t_1) > 0$ and

$$\max\{g_1(s), g_2(s), -g_1'(s), -g_2'(s)\}$$

$$< \min\{r, H(r), H_0(r)\},$$

where $H_0(t) = H(D(t))$ provided that D is a positive C^1 function, with $D(0) = 0$, for which H_0 is strictly increasing and strictly convex C^2 function on $(0, r]$ and

$$\int_0^{+\infty} g_i(s)H_0(-g_i'(s))ds < +\infty.$$

2) $\forall t \in [0, t_1]$: As g_i are nonincreasing, $g_i(0) > 0$ and $g_i(t_1) > 0$ then $g_i(t) > 0$ and

$$g_i(0) \geq g_i(t) \geq g_i(t_1) > 0.$$

Then

$$e \leq H(g_1(t)) \leq f$$

$$e' \leq H(g_2(t)) \leq f'$$

for some positive constants e, f, e' and f'. Consequently,

$$g_i'(t) \leq -H(g_i(t)) \leq -kg_i(t), \quad k > 0$$

gives

$$g_i'(t) + kg_i(t) \leq 0, k > 0 \tag{7}$$

B- Let H_0^* be the convex conjugate of H_0 in the sense of Young (see [3], pages 61-64), then

$$H_0^*(s) = s(H_0')^{-1}(s) - H_0[(H_0')^{-1}(s)], \quad s \in (0, H_0'(r))$$

and satisfies the following Young's inequality

$$AB \leq H_0^*(A) + H_0(B), \quad A \in (0, H_0'(r)), B \in (0, r].$$

Definition 1 ([10], [18]) *The function spaces of our problem and their norm is defined as follows:*

$$\mathcal{H}(\mathbb{R}^n) = \Big\{u \in L^{2n/(n-2)}(\mathbb{R}^n) : \nabla_x u \in (L^2(\mathbb{R}^n))^n\Big\}, \tag{8}$$

with the norm $\|u\|_{\mathcal{H}(\mathbb{R}^n)} = \Big(\int_{\mathbb{R}^n} |\nabla u|^2 dx\Big)^{1/2}$.

$$\mathcal{D}(\mathbb{R}^n) = \Big\{u \in L^{2n/(n-2)}(\mathbb{R}^n) : \Delta_x u \in L^2(\mathbb{R}^n)\Big\}. \tag{9}$$

We define the norm

$$\|u\|_{\mathcal{D}(\mathbb{R}^n)} = \Big(\int_{\mathbb{R}^n} |\Delta u|^2 dx\Big)^{1/2},$$

where $\mathcal{D}(\mathbb{R}^n)$ *can be embedded continuously in* $L^{2n/(n-2)}(\mathbb{R}^n)$*, i.e there exists* $k > 0$ *such that*

$$\|u\|_{L^{2n/(n-2)}} \leq k\|u\|_{\mathcal{D}}. \tag{10}$$

and the space $L^2_\rho(\mathbb{R}^n)$ *to be the closure of* $C_0^\infty(\mathbb{R}^n)$ *functions with respect to the inner product*

$$(f, h)_{L^2_\rho(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \rho f h dx. \tag{11}$$

For $1 < p < \infty$*, if* f *is a measurable function on* $\mathbb{R}^n$*, we define*

$$\|f\|_{L^p_\rho(\mathbb{R}^n)} = \Big(\int_{\mathbb{R}^n} \rho|f|^q dx\Big)^{1/p}. \tag{12}$$

The space $L^2_\rho(\mathbb{R}^n)$ is a separable Hilbert space.

Lemma 2 *([7]) Let* $h, w \in C^1(\mathbb{R})$ *be any two functions and* $\theta \in [0, 1]$*, then we have*

$$w'(t)\int_0^t h(t-s)w(s)ds$$
$$= -\frac{1}{2}\frac{d}{dt}\int_0^t h(t-s)|w(t) - w(s)|^2 ds$$
$$+\frac{1}{2}\frac{d}{dt}\left(\int_0^t h(s)ds\right)|w(t)|^2$$
$$+\frac{1}{2}\int_0^t h'(t-s)|w(t) - w(s)|^2 ds - \frac{1}{2}h(t)|w(t)|^2.$$

and

$$\left|\int_0^t h(t-s)(w(t) - w(s))ds\right|^2$$
$$\leq \left(\int_0^t |h(s)|^{2(1-\theta)} ds\right)$$
$$\left(\int_0^t |h(t-s)|^{2\theta}|w(t) - w(s)|^2 ds\right)$$

The modified energy associate to (u,v) at time t is given as

$$E(t)=\frac{(q-1)}{q}\left[\|u'\|^q_{L^q_\rho(\mathbb{R}^n)}+\|v'\|^q_{L^q_\rho(\mathbb{R}^n)}\right]$$
$$+\alpha\int_{\mathbb{R}^n}\rho uv dx+\frac{1}{2}\Big(1-\int_0^t g_1(s)ds\Big)\|\nabla_x u\|_2^2$$
$$+\frac{1}{2}\Big(1-\int_0^t g_2(s)ds\Big)\|\nabla_x v\|_2^2$$
$$+\frac{1}{2}(g_1\circ\nabla_x u)+\frac{1}{2}(g_2\circ\nabla_x v) \tag{13}$$

We easily deduce for $c>0$, that

$$E(t)\geq(1-c|\alpha|\|\rho\|^{-1}_{L^{n/2}})$$
$$\frac{(q-1)}{q}\left[\|u'\|^q_{L^q_\rho(\mathbb{R}^n)}+\|v'\|^q_{L^q_\rho(\mathbb{R}^n)}\right]$$
$$+\frac{1}{2}\Big(1-\int_0^t g_1(s)ds\Big)\|\nabla_x u\|_2^2$$
$$+\frac{1}{2}\Big(1-\int_0^t g_2(s)ds\Big)\|\nabla_x v\|_2^2$$
$$+\frac{1}{2}(g_1\circ\nabla_x u)+\frac{1}{2}(g_2\circ\nabla_x v) \tag{14}$$

for α small enough and by using Lemma 3.
The first derivative of the energy functional for all $t\geq 0$ is given by

$$E'(t)\leq\frac{1}{2}(g_1'\circ\nabla_x u)(t)+\frac{1}{2}(g_2'\circ\nabla_x v)(t), \tag{15}$$

Noting by

$$(g_i\circ\nabla_x h)(t)=\int_0^t g_i(t-\tau)\|\nabla_x h(t)-\nabla_x h(\tau)\|_2^2\,d\tau, \tag{16}$$

for $\psi(t)\in\mathcal{H}(\mathbb{R}^n), t\geq 0, i=1,2$.

3 Main results and Proofs

Lemma 3 *[13] Let ρ satisfy (4), then for any $u\in\mathcal{H}(\mathbb{R}^n)$*

$$\|u\|_{L^p_\rho(\mathbb{R}^n)}\leq\|\rho\|_{L^s(\mathbb{R}^n)}\|\nabla_x u\|_{L^2(\mathbb{R}^n)} \tag{17}$$

with $s=\frac{2n}{2n-pn+2p}, 2\leq p\leq\frac{2n}{n-2}$

Our main result reads as follows.

Theorem 4 *Let $(u_0,v_0)\in(\mathcal{H}(\mathbb{R}^n))^2,(u_1,v_1)\in(L^q_\rho(\mathbb{R}^n))^2$. Then there exists a positive constants a,b,c,d such that the energy of solution of problem (1)-(3) satisfies,*

$$E(t)\leq dH_1^{-1}(bt+c),\quad \textit{for all}\quad t\geq 0,$$

where $(A1)-(A2)$ hold

$$H_1(t)=\int_t^1\frac{1}{sH_0'(as)}ds \tag{18}$$

To prove Theorem 4, let us define

$$L(t)=\xi_1E(t)+\psi_1(t)+\xi_2\psi_2(t) \tag{19}$$

for $\xi_1,\xi_2>1$. Let

$$\psi_1(t)=\int_{\mathbb{R}^n}\rho(x)\left[u|u'|^{q-2}u'+v|v'|^{q-2}v'\right]dx, \tag{20}$$

and

$$\psi_2(t)= \tag{21}$$
$$-\int_{\mathbb{R}^n}\rho(x)|u'|^{q-2}u'\int_0^t g_1(t-s)(u(t)-u(s))dsdx$$
$$-\int_{\mathbb{R}^n}\rho(x)|v'|^{q-2}v'\int_0^t g_2(t-s)(v(t)-v(s))dsdx.$$

Lemma 5 *Suppose that (A1) and (A2) hold. Then, the functional ψ_1 satisfies, along the solution of (1)-(3)*

$$\begin{aligned}\psi_1'(t)&\leq \|u'\|^q_{L^q_\rho(\mathbb{R}^n)}+\|v'\|^q_{L^q_\rho(\mathbb{R}^n)}\\&+(c_\sigma\alpha\|\rho\|^2_{L^{n/2}}-l)\left[\|\nabla_x u\|_2^2+\|\nabla_x v\|_2^2\right]\\&+\frac{(1-l)}{4\sigma}\left[(g_1\circ\nabla_x u)+(g_2\circ\nabla_x v)\right],\end{aligned}$$

where $l=\min\{l_1,l_2\}$.

Proof. From (20), integrating over $\mathbb{R}^n$, we have

$$\psi_1'(t)=\int_{\mathbb{R}^n}\rho(x)|u'|^q dx$$
$$+\int_{\mathbb{R}^n}\rho(x)u\left(|u'|^{q-2}u'\right)'dx$$
$$+\int_{\mathbb{R}^n}\rho(x)|v'|^q dx+\int_{\mathbb{R}^n}\rho(x)v\left(|v'|^{q-2}v'\right)'dx$$
$$=\int_{\mathbb{R}^n}\Big(\rho(x)|u'|^q+u\Delta_x u$$
$$-u\int_0^t g_1(t-s)\Delta_x u(s,x)ds-\alpha\rho(x)uv\Big)dx$$

$$+\int_{\mathbb{R}^n}\Big(\rho(x)|v'|^q+v\Delta_x v$$

$$-v\int_0^t g_2(t-s)\Delta_x v(s,x)ds-\alpha\rho(x)uv\Big)dx$$

$$=\int_{\mathbb{R}^n}\Big(\rho(x)|u'|^q-\nabla_x u\nabla_x u$$

$$+\nabla_x u\int_0^t g_1(t-s)\nabla_x u(s,x)ds-\alpha\rho(x)uv\Big)dx$$

$$+\int_{\mathbb{R}^n}\Big(\rho(x)|v'|^q-\nabla_x v\nabla_x v$$

$$+\nabla_x v\int_0^t g_2(t-s)\nabla_x v(s,x)ds-\alpha\rho(x)uv\Big)dx$$

$$=\int_{\mathbb{R}^n}\Big(\rho(x)|u'|^q-(\nabla_x u)^2$$

$$+\nabla_x u\int_0^t g_1(t-s)(\nabla_x u(s)-\nabla_x u(t))ds\Big)dx$$

$$+\int_{\mathbb{R}^n}\Big(\rho(x)|v'|^q-(\nabla_x v)^2$$

$$+\nabla_x v\int_0^t g_2(t-s)(\nabla_x v(s)-\nabla_x v(t))ds\Big)dx$$

$$+\int_{\mathbb{R}^n}(\nabla_x u)^2\int_0^t g_1(s)dsdx$$

$$+\int_{\mathbb{R}^n}(\nabla_x v)^2\int_0^t g_2(s)dsdx-2\alpha\int_{\mathbb{R}^n}\rho(x)uvdx$$

Thanks to Young's inequality and Lemma 2 and for $\theta=1/2$, we get for small enough positive constant σ

$$\nabla_x u\int_0^t g_1(t-s)(\nabla_x u(s)-\nabla_x u(t))dsdx$$

$$\le\sigma\|\nabla_x u\|_2^2$$

$$+\frac{1}{4\sigma}\int_{\mathbb{R}^n}\left(\int_0^t g_1(t-s)|\nabla_x u(s)-\nabla_x u(t)|ds\right)^2dx$$

$$\le\sigma\|\nabla_x u\|_2^2+\frac{1-l_1}{4\sigma}(g_1\circ\nabla_x u)(t)$$

and

$$\nabla_x v\int_0^t g_2(t-s)(\nabla_x v(s)-\nabla_x v(t))dsdx$$

$$\le\sigma\|\nabla_x v\|_2^2$$

$$+\frac{1}{4\sigma}\int_{\mathbb{R}^n}\left(\int_0^t g_2(t-s)|\nabla_x v(s)-\nabla_x v(t)|ds\right)^2dx$$

$$\le\sigma\|\nabla_x v\|_2^2+\frac{1-l_2}{4\sigma}(g_2\circ\nabla_x v)(t)$$

By (A1) and the fact that $\int_0^t g(s)ds<\int_0^\infty g(s)ds$

$$\psi_1'(t)\le\|u'\|^q_{L^q_\rho(\mathbb{R}^n)}$$

$$+\|v'\|^q_{L^q_\rho(\mathbb{R}^n)}+(\sigma-l_1)\|\nabla_x u\|_2^2$$

$$+(\sigma-l_2)\|\nabla_x v\|_2^2-2\alpha\int_{\mathbb{R}^n}\rho(x)uvdx$$

$$+\frac{1-l_1}{4\sigma}(g_1\circ\nabla_x u)+\frac{1-l_2}{4\sigma}(g_2\circ\nabla_x v)$$

Using Holder's, Young's inequalities and Lemma 3 to get

$$\int_{\mathbb{R}^n}|\rho(x)uv|dx$$

$$=\int_{\mathbb{R}^n}|(\rho(x)^{1/2}u)(\rho(x)^{1/2}v)|dx$$

$$\le\left(\int_{\mathbb{R}^n}\rho(x)|u|^2dx\right)^{1/2}\left(\int_{\mathbb{R}^n}\rho(x)|v|^2dx\right)^{1/2}$$

$$\le\sigma\int_{\mathbb{R}^n}\rho(x)|u|^2dx+\frac{1}{4\sigma}\int_{\mathbb{R}^n}\rho(x)|v|^2dx$$

$$\le\|\rho\|^2_{L^{n/2}}\left[\sigma\int_{\mathbb{R}^n}|\nabla_x u|^2dx+\frac{1}{4\sigma}\int_{\mathbb{R}^n}|\nabla_x v|^2dx\right].$$

Then, we obtain for $l=\min\{l_1,l_2\}, c_\sigma>0$

$$\begin{aligned}\psi_1'(t)&\le\|u'\|^q_{L^q_\rho(\mathbb{R}^n)}+\|v'\|^q_{L^q_\rho(\mathbb{R}^n)}\\&+(c_\sigma\alpha\|\rho\|^2_{L^{n/2}}-l)\left(\|\nabla_x u\|_2^2+\|\nabla_x v\|_2^2\right)\\&+\frac{(1-l)}{4\sigma}\left((g_1\circ\nabla_x u)+(g_2\circ\nabla_x v)\right).\end{aligned}$$

Lemma 6 *Suppose that (A1) and (A2) hold. Then, the functional ψ_2 satisfies, along the solution of (1)-(3), for any $\sigma\in(0,1)$*

$$\begin{aligned}\psi_2'(t)&\le\sigma\left(1+\alpha\|\rho\|^2_{L^{n/2}(\mathbb{R}^n)}\right)\left[\|\nabla_x u\|_2^2+\|\nabla_x v\|_2^2\right]\\&+c_\sigma\left(1+\alpha\|\rho\|^2_{L^{n/2}}\right)\left[(g_1\circ\nabla_x u)+(g_2\circ\nabla_x v)\right]\\&-c_\sigma\|\rho\|^q_{L^s}\left[(g_1'\circ\nabla_x u)^{q/2}+(g_2'\circ\nabla_x v)^{q/2}\right]\\&+\left(\sigma-\int_0^t g(s)ds\right)\left[\|u'\|^{q/2}_{L^q_\rho(\mathbb{R}^n)}+\|v'\|^{q/2}_{L^q_\rho(\mathbb{R}^n)}\right].\end{aligned}$$

where

$$\int_0^t g(s)ds \le \min\left\{\int_0^t g_1(s)ds, \int_0^t g_2(s)ds\right\} \quad (22)$$

Proof. Exploiting Eqs. (1) to get

$$\psi_2'(t) = -\int_{\mathbb{R}^n} \rho(x)\left(|u'|^{q-2}u'\right)' \times$$

$$\int_0^t g_1(t-s)(u(t)-u(s))dsdx$$

$$-\int_{\mathbb{R}^n} \rho(x)|u'|^{q-2}u' \times$$

$$\int_0^t g_1'(t-s)(u(t)-u(s))dsdx - \int_0^t g_1(s)ds\|u'\|^q_{L^q_\rho(\mathbb{R}^n)}$$

$$-\int_{\mathbb{R}^n} \rho(x)\left(|v'|^{q-2}v'\right)' \int_0^t g_2(t-s)(v(t)-v(s))dsdx$$

$$-\int_{\mathbb{R}^n} \rho(x)|v'|^{q-2}v' \int_0^t g_2'(t-s)(v(t)-v(s))dsdx$$

$$-\int_0^t g_2(s)ds\|v'\|^q_{L^q_\rho(\mathbb{R}^n)}$$

$$= \int_{\mathbb{R}^n} \nabla_x u \int_0^t g_1(t-s)(\nabla_x u(t) - \nabla_x u(s))dsdx$$

$$-\int_{\mathbb{R}^n} \left(\int_0^t g_1(t-s)\nabla_x u(s,x)ds\right) \times$$

$$\left(\int_0^t g_1(t-s)(\nabla_x u(t) - \nabla_x u(s))ds\right) dx$$

$$-\int_{\mathbb{R}^n} \rho(x)|u'|^{q-2}u' \int_0^t g_1'(t-s)(u(t)-u(s))dsdx$$

$$-\int_0^t g_1(s)ds\|u'\|^q_{L^q_\rho(\mathbb{R}^n)}$$

$$+\alpha \int_{\mathbb{R}^n} \rho(x)v \int_0^t g_1(t-s)(u(t)-u(s))dsdx$$

$$+\int_{\mathbb{R}^n} \nabla_x v \int_0^t g_2(t-s)(\nabla_x v(t) - \nabla_x v(s))dsdx$$

$$-\int_{\mathbb{R}^n} \left(\int_0^t g_2(t-s)\nabla_x v(s,x)ds\right) \times$$

$$\left(\int_0^t g_2(t-s)(\nabla_x v(t) - \nabla_x v(s))ds\right) dx$$

$$-\int_{\mathbb{R}^n} \rho(x)|v'|^{q-2}v' \int_0^t g_2'(t-s)(v(t)-v(s))dsdx$$

$$-\int_0^t g_2(s)ds\|v'\|^q_{L^q_\rho(\mathbb{R}^n)}$$

$$+\alpha \int_{\mathbb{R}^n} \rho(x)u \int_0^t g_2(t-s)(v(t)-v(s))dsdx,$$

then

$$\psi_2'(t) = \left(1 - \int_0^t g_1(s)ds\right) \times$$

$$\int_{\mathbb{R}^n} \nabla_x u \int_0^t g_1(t-s)(\nabla_x u(t) - \nabla_x u(s))dsdx$$

$$+\int_{\mathbb{R}^n} \left(\int_0^t g_1(t-s)(\nabla_x u(t) - \nabla_x u(s))ds\right)^2 dx$$

$$-\int_{\mathbb{R}^n} \rho(x)|u'|^{q-2}u' \int_0^t g_1'(t-s)(u(t)-u(s))dsdx$$

$$-\int_0^t g_1(s)ds\|u'\|^q_{L^q_\rho(\mathbb{R}^n)} + c(g_1 \circ \nabla_x u)(t)$$

$$+\left(1 - \int_0^t g_2(s)ds\right) \times$$

$$\int_{\mathbb{R}^n} \nabla_x v \int_0^t g_2(t-s)(\nabla_x v(t) - \nabla_x v(s))dsdx$$

$$+\int_{\mathbb{R}^n} \left(\int_0^t g_2(t-s)(\nabla_x v(t) - \nabla_x v(s))ds\right)^2 dx$$

$$-\int_{\mathbb{R}^n} \rho(x)|v'|^{q-2}v' \int_0^t g_2'(t-s)(v(t)-v(s))dsdx$$

$$-\int_0^t g_2(s)ds\|v'\|^q_{L^q_\rho(\mathbb{R}^n)} + c(g_2 \circ \nabla_x v)(t)$$

$$+\alpha \int_{\mathbb{R}^n} \rho(x)\Big(v \int_0^t g_1(t-s)(u(t)-u(s))ds$$

$$+u \int_0^t g_2(t-s)(v(t)-v(s))ds\Big)dx$$

Thanks to Holder's and Young's inequalities and with Lemma 3, we estimate

$$\int_{\mathbb{R}^n} \rho(x)v \int_0^t g_1(t-s)(u(t)-u(s))dsdx$$

$$\le \left(\int_{\mathbb{R}^n} \rho(x)|v|^2 dx\right)^{1/2} \times$$

$$\left(\int_{\mathbb{R}^n} \rho(x)\Big|\int_0^t g_1(t-s)(u(t)-u(s))ds\Big|^2\right)^{1/2}$$

$$\le \sigma\|v\|^2_{L^2_\rho(\mathbb{R}^n)}$$

$$+c_\sigma\Big\|\int_0^t g_1(t-s)(u(t)-u(s))ds\Big\|^2_{L^2_\rho(\mathbb{R}^n)}$$

$$\leq \sigma\|\rho\|^2_{L^{n/2}(\mathbb{R}^n)}\|\nabla_x v\|_2^2 + c_\sigma\|\rho\|^2_{L^{n/2}(\mathbb{R}^n)}(g_1 \circ \nabla_x u)(t).$$

and

$$\int_{\mathbb{R}^n}\rho(x)u\int_0^t g_2(t-s)(v(t)-v(s))dsdx$$

$$\leq \left(\int_{\mathbb{R}^n}\rho(x)|u|^2dx\right)^{1/2}\times$$

$$\left(\int_{\mathbb{R}^n}\rho(x)\Big|\int_0^t g_2(t-s)(v(t)-v(s))ds\Big|^2\right)^{1/2}$$

$$\leq \sigma\|u\|^2_{L^2_\rho(\mathbb{R}^n)}$$

$$+c_\sigma\Big\|\int_0^t g_2(t-s)(v(t)-v(s))ds\Big\|^2_{L^2_\rho(\mathbb{R}^n)}$$

$$\leq \sigma\|\rho\|^2_{L^{n/2}(\mathbb{R}^n)}\|\nabla_x u\|_2^2$$

$$+c_\sigma\|\rho\|^2_{L^{n/2}(\mathbb{R}^n)}(g_2 \circ \nabla_x v)(t).$$

and for the exponents $\frac{q}{q-1}, q$

$$-\int_{\mathbb{R}^n}\rho(x)|u'|^{q-2}u'\int_0^t g_1'(t-s)(u(t)-u(s))dsdx$$

$$\leq \left(\int_{\mathbb{R}^n}\rho(x)|u'|^q dx\right)^{(q-1)/q}\times$$

$$\left(\int_{\mathbb{R}^n}\rho(x)\Big|\int_0^t g_1'(t-s)(u(t)-u(s))ds\Big|^q\right)^{1/q}$$

$$\leq \sigma\|u'\|^q_{L^q_\rho(\mathbb{R}^n)}$$

$$+c_\sigma\Big\|\int_0^t -g_1'(t-s)(u(t)-u(s))ds\Big\|^q_{L^q_\rho(\mathbb{R}^n)}$$

$$\leq \sigma\|u'\|^q_{L^q_\rho(\mathbb{R}^n)} - c_\sigma\|\rho\|^q_{L^s(\mathbb{R}^n)}(g_1' \circ \nabla_x u)^{q/2}(t).$$

and

$$-\int_{\mathbb{R}^n}\rho(x)|v'|^{q-2}v'\int_0^t g_2'(t-s)(v(t)-v(s))dsdx$$

$$\leq \left(\int_{\mathbb{R}^n}\rho(x)|v'|^q dx\right)^{(q-1)/q}\times$$

$$\left(\int_{\mathbb{R}^n}\rho(x)\Big|\int_0^t g_2'(t-s)(v(t)-v(s))ds\Big|^q\right)^{1/q}$$

$$\leq \sigma\|v'\|^q_{L^q_\rho(\mathbb{R}^n)}$$

$$+c_\sigma\Big\|\int_0^t -g_2'(t-s)(v(t)-v(s))ds\Big\|^q_{L^q_\rho(\mathbb{R}^n)}$$

$$\leq \sigma\|v'\|^q_{L^q_\rho(\mathbb{R}^n)} - c_\sigma\|\rho\|^q_{L^s(\mathbb{R}^n)}({g_2}' \circ \nabla_x v)^{q/2}(t).$$

Thanks to Young's and Poincare's inequalities and using Lemma 2 for $\theta = 1/2$, we obtain

$$\psi_2'(t) \leq \sigma\left(1+\alpha\|\rho\|^2_{L^{n/2}(\mathbb{R}^n)}\right)\left(\|\nabla_x u\|_2^2 + \|\nabla_x v\|_2^2\right)$$

$$+c_\sigma\left(1+\alpha\|\rho\|^2_{L^{n/2}(\mathbb{R}^n)}\right)((g_1\circ\nabla_x u)+(g_2\circ\nabla_x v))$$

$$-c_\sigma\|\rho\|^q_{L^s(\mathbb{R}^n)}\left((g_1'\circ\nabla_x u)^{q/2}+(g_2'\circ\nabla_x v)^{q/2}\right)$$

$$+\left(\sigma-\int_0^t g_1(s)ds\right)\|u'\|^{q/2}_{L^q_\rho(\mathbb{R}^n)}$$

$$+\left(\sigma-\int_0^t g_2(s)ds\right)\|v'\|^{q/2}_{L^q_\rho(\mathbb{R}^n)}.$$

For $\xi_1, \xi_2 > 1$, we have

$$\beta_1 L(t) \leq E(t) \leq \beta_2 L(t) \tag{23}$$

which holds for two positive constants β_1 and β_2.

Lemma 7 *For $\xi_1, \xi_2 > 1$, we have*

$$L(t) \sim E(t). \tag{24}$$

Proof. We have by (19)

$$|L(t)-\xi_1 E(t)|$$

$$\leq |\psi_1(t)| + \xi_2|\psi_2(t)|$$

$$\leq \int_{\mathbb{R}^n}\left|\rho(x)u|u'|^{q-2}u'\right|dx + \int_{\mathbb{R}^n}\left|\rho(x)v|v'|^{q-2}v'\right|dx$$

$$+\xi_2\int_{\mathbb{R}^n}\left|\rho(x)|u'|^{q-2}u'\int_0^t g_1(t-s)(u(t)-u(s))ds\right|dx$$

$$+\xi_2\int_{\mathbb{R}^n}\left|\rho(x)|v'|^{q-2}v'\int_0^t g_2(t-s)(v(t)-v(s))ds\right|dx.$$

Using Holder's and Young's inequalities with $\frac{q}{q-1}, q$, since $q \geq 2$, we have

$$\int_{\mathbb{R}^n}\left|\rho(x)u|u'|^{q-2}u'\right|dx$$

$$\leq \left(\int_{\mathbb{R}^n}\rho(x)|u|^q dx\right)^{1/q}\left(\int_{\mathbb{R}^n}\rho(x)|u'|^q dx\right)^{(q-1)/q}$$

$$\leq \frac{1}{q}\left(\int_{\mathbb{R}^n}\rho(x)|u|^q dx\right) + \frac{q-1}{q}\left(\int_{\mathbb{R}^n}\rho(x)|u'|^q dx\right)$$

$$\leq c\|u'\|^q_{L^q_\rho(\mathbb{R}^n)} + c\|\rho\|^q_{L^s(\mathbb{R}^n)}\|\nabla_x u\|_2^q$$

and

$$\int_{\mathbb{R}^n} \left|\rho(x)v|v'|^{q-2}v'\right| dx$$

$$\leq \left(\int_{\mathbb{R}^n} \rho(x)|v|^q dx\right)^{1/q} \left(\int_{\mathbb{R}^n} \rho(x)|v'|^q dx\right)^{(q-1)/q}$$

$$\leq \frac{1}{q}\left(\int_{\mathbb{R}^n} \rho(x)|v|^q dx\right) + \frac{q-1}{q}\left(\int_{\mathbb{R}^n} \rho(x)|v'|^q dx\right)$$

$$\leq c\|v'\|^q_{L^q_\rho(\mathbb{R}^n)} + c\|\rho\|^q_{L^s(\mathbb{R}^n)}\|\nabla_x v\|^q_2$$

and

$$\int_{\mathbb{R}^n} \Big| \left(\rho(x)^{\frac{q-1}{q}}|u'|^{q-2}u'\right) \times$$

$$\left(\rho(x)^{\frac{1}{q}}\int_0^t g_1(t-s)(u(t)-u(s))ds\right)\Big|dx$$

$$\leq \left(\int_{\mathbb{R}^n} \rho(x)|u'|^q dx\right)^{(q-1)/q} \times$$

$$\left(\int_{\mathbb{R}^n} \rho(x)\Big|\int_0^t g_1(t-s)(u(t)-u(s))ds\Big|^q dx\right)^{1/q}$$

$$\leq \frac{q-1}{q}\|u'\|^q_{L^q_\rho(\mathbb{R}^n)}$$

$$+\frac{1}{q}\Big\|\int_0^t g_1(t-s)(u(t)-u(s))ds\Big\|^q_{L^q_\rho(\mathbb{R}^n)}$$

$$\leq \frac{q-1}{q}\|u'\|^q_{L^q_\rho(\mathbb{R}^n)} + \frac{1}{q}\|\rho\|^q_{L^s(\mathbb{R}^n)}(g_1\circ\nabla_x u)^{q/2}(t).$$

and

$$\int_{\mathbb{R}^n} \Big| \left(\rho(x)^{\frac{q-1}{q}}|v'|^{q-2}v'\right) \times$$

$$\left(\rho(x)^{\frac{1}{q}}\int_0^t g_2(t-s)(v(t)-v(s))ds\right)\Big|dx$$

$$\leq \left(\int_{\mathbb{R}^n} \rho(x)|v'|^q dx\right)^{(q-1)/q} \times$$

$$\left(\int_{\mathbb{R}^n} \rho(x)\Big|\int_0^t g_2(t-s)(v(t)-v(s))ds\Big|^q dx\right)^{1/q}$$

$$\leq \frac{q-1}{q}\|v'\|^q_{L^q_\rho(\mathbb{R}^n)}$$

$$+\frac{1}{q}\Big\|\int_0^t g_2(t-s)(v(t)-v(s))ds\Big\|^q_{L^q_\rho(\mathbb{R}^n)}$$

$$\leq \frac{q-1}{q}\|v'\|^q_{L^q_\rho(\mathbb{R}^n)} + \frac{1}{q}\|\rho\|^q_{L^s(\mathbb{R}^n)}(g_2\circ\nabla_x v)^{q/2}(t).$$

Then, since $q \geq 2$, we have

$$|L(t) - \xi_1 E(t)| \leq c(E(t) + E^{q/2}(t))$$

$$\leq c(E(t) + E(t)E^{(q/2)-1}(t))$$

$$\leq c(E(t) + E(t)E^{(q/2)-1}(0))$$

$$\leq cE(t).$$

Therefore, we can choose ξ_1 so that

$$L(t) \sim E(t). \tag{25}$$

Proof of Theorem 4 By (15), Lemma 5 and Lemma 6, we have

$$L'(t) = \xi_1 E'(t) + \psi_1'(t) + \xi_2\psi_2'(t)$$

$$\leq (\frac{\xi_1}{2} - c_\sigma\xi_2\|\rho\|^q_{L^s b}) \times$$

$$\left[(g_1'\circ\nabla_x u)^{q/2} + (g_2'\circ\nabla_x v)^{q/2}\right]$$

$$+M_0\left[(g_2\circ\nabla_x u) + (g_2\circ\nabla_x v)\right]$$

$$-M_1\left[\|u'\|^q_{L^q_\rho} + \|v'\|^q_{L^q_\rho}\right]$$

$$-M_2\left[\|\nabla_x u\|^2_2 + \|\nabla_x v\|^2_2\right]$$

where

$$M_0 = (\frac{4\xi_2 c\left(1+\alpha\|\rho\|^2_{L^{n/2}}\right) + (1-l)}{4\sigma}),$$

$$M_1 = \left(\xi_2\left(\int_0^{t_1} g(s)ds - \sigma\right) - 1\right),$$

$$M_2 = \left(-\xi_2\sigma\left(1+\alpha\|\rho\|^2_{L^{n/2}}\right) + (l - c_\sigma\alpha\|\rho\|^2_{L^{n/2}})\right),$$

and t_1 was given in A.
We now choose σ so small that $\xi_1 > 2c_\sigma\|\rho\|^q_{L^s(\mathbb{R}^n)}\xi_2$. Whence σ is fixed, we can choose ξ_1, ξ_2 large enough so that $M_1, M_2 > 0$, which yield, for all $t \geq t_1$

$$L'(t) \leq M_0\left[(g_1\circ\nabla_x u) + (g_2\circ\nabla_x v)\right] - cE(t) \tag{26}$$

Let us introduce a new functional $F(t) = L(t) + cE(t)$. Then by (26), we get for some positive constant c and $t \geq t_1$

$$F'(t) = L'(t) + cE'(t) \tag{27}$$

$$\leq -cE(t)$$

$$+c\int_{\mathbb{R}^n}\int_{t_1}^{t} g_1(t-s)|\nabla_x u(t)-\nabla_x u(s)|^2 dsdx$$

$$+c\int_{\mathbb{R}^n}\int_{t_1}^{t} g_2(t-s)|\nabla_x v(t)-\nabla_x v(s)|^2 dsdx.$$

By (7) and (15), we have for all $t \geq t_1$

$$\int_{\mathbb{R}^n}\int_{0}^{t_1} g_1(t-s)|\nabla_x u(t)-\nabla_x u(s)|^2 dsdx$$

$$+\int_{\mathbb{R}^n}\int_{0}^{t_1} g_2(t-s)|\nabla_x v(t)-\nabla_x v(s)|^2 dsdx$$

$$\leq -\frac{1}{k}\Big(\int_{\mathbb{R}^n}\int_{0}^{t_1} g_1'(t-s)|\nabla_x u(t)-\nabla_x u(s)|^2 dsdx$$

$$+\int_{\mathbb{R}^n}\int_{0}^{t_1} g_2'(t-s)|\nabla_x v(t)-\nabla_x v(s)|^2 dsdx\Big)$$

$$\leq \quad -cE'(t).$$

At this point, we define

$$I(t)=\int_{t_1}^{t} H_0(-g_1'(s))(g_1\circ\nabla_x u)(t)ds$$

$$+\int_{t_1}^{t} H_0(-g_2'(s))(g_2\circ\nabla_x v)(t)ds. \qquad (28)$$

Since $\int_0^{+\infty} H_0(-g_i'(s))g(s)ds < +\infty, i=1,2,$ from (15) we have

$$I(t)=\int_{t_1}^{t} H_0(-g_1'(s))\times$$

$$\int_{\mathbb{R}^n} g_1(s)|\nabla_x u(t)-\nabla_x u(t-s)|^2 dxds$$

$$+\int_{t_1}^{t} H_0(-g_2'(s))\times$$

$$\int_{\mathbb{R}^n} g_2(s)|\nabla_x v(t)-\nabla_x v(t-s)|^2 dxds$$

$$\leq 2\int_{t_1}^{t} H_0(-g_1'(s))g_1(s)\times$$

$$\int_{\mathbb{R}^n} |\nabla_x u(t)|^2+|\nabla_x u(t-s)|^2 dxds$$

$$+2\int_{t_1}^{t} H_0(-g_2'(s))g_2(s)\times$$

$$\int_{\mathbb{R}^n} |\nabla_x v(t)|^2+|\nabla_x v(t-s)|^2 dxds$$

$$\leq cE(0)\Big[\int_{t_1}^{t} H_0(-g_1'(s))g_1(s)ds$$

$$+\int_{t_1}^{t} H_0(-g_2'(s))g_2(s)ds\Big]. \qquad (29)$$

We have $I(t) < 1$ (see[16], Eq. (3.11)). We now define the functional $\lambda(t)$ related to $I(t)$ as

$$\lambda(t)=-\int_{t_1}^{t} H_0(-g_1'(s))g_1'(s)\times$$

$$\int_{\mathbb{R}^n} g_1(s)|\nabla_x u(t)-\nabla_x u(t-s)|^2 dxds$$

$$-\int_{t_1}^{t} H_0(-g_2'(s))g_2'(s)\times \qquad (30)$$

$$\int_{\mathbb{R}^n} g_2(s)|\nabla_x v(t)-\nabla_x v(t-s)|^2 dxds.$$

By (A1)-(A2), we get

$$H_0(-g_i'(s))g_i(s)\leq H_0(H(g_i(s)))g_i(s)$$

$$=D(g_i(s))g_i(s)\leq k_0.$$

for some positive constant k_0. Then, for all $t\geq t_1$

$$\lambda(t)$$

$$\leq -k_0\int_{t_1}^{t} g_1'(s)\int_{\mathbb{R}^n}|\nabla_x u(t)-\nabla_x u(t-s)|^2 dxds$$

$$-k_0\int_{t_1}^{t} g_2'(s)\int_{\mathbb{R}^n}|\nabla_x v(t)-\nabla_x v(t-s)|^2 dxds$$

$$\leq -k_0\int_{t_1}^{t} g_1'(s)\int_{\mathbb{R}^n}|\nabla_x u(t)|^2+|\nabla_x u(t-s)|^2 dxds$$

$$-k_0\int_{t_1}^{t} g_2'(s)\int_{\mathbb{R}^n}|\nabla_x v(t)|^2+|\nabla_x v(t-s)|^2 dxds$$

$$\leq -cE(0)\left[\int_{t_1}^{t} g_1'(s)ds+\int_{t_1}^{t} g_2'(s)ds\right]$$

$$\leq cE(0)\max\{g_1(t_1),g_2(t_1)\}$$

$$<\min\{r,H(r),H_0(r)\}. \qquad (31)$$

By the definition of H_0, and for $x\in(0,r],\theta\in[0,1]$ we have

$$H_0(\theta x)\leq\theta H_0(x).$$

Using (29), (31)leads to

$$\lambda(t)=I^{-1}(t)\Big\{\int_{t_1}^{t} I(t)H_0[H_0^{-1}(-g_1'(s))]\times$$

$$H_0(-g_1'(s))g_1'(s)$$

$$\int_{\mathbb{R}^n} g_1(s)|\nabla_x u(t)-\nabla_x u(t-s)|^2 dxds$$

$$+\int_{t_1}^t I(t)H_0[H_0^{-1}(-g_2'(s))]H_0(-g_2'(s))g_2'(s)$$

$$\int_{\mathbb{R}^n} g_2(s)|\nabla_x v(t)-\nabla_x v(t-s)|^2 dxds\Big\}$$

$$\geq I^{-1}(t)\Big\{\int_{t_1}^t H_0[I(t)H_0^{-1}(-g_1'(s))]H_0(-g_1'(s))g_1'(s)$$

$$\int_{\mathbb{R}^n} g_1(s)|\nabla_x u(t)-\nabla_x u(t-s)|^2 dxds$$

$$+\int_{t_1}^t H_0[I(t)H_0^{-1}(-g_2'(s))]H_0(-g_2'(s))g_2'(s)$$

$$\int_{\mathbb{R}^n} g_2(s)|\nabla_x v(t)-\nabla_x v(t-s)|^2 dxds\Big\}$$

$$\geq H_0\Big(I^{-1}(t)\int_{t_1}^t I(t)H_0^{-1}(-g_1'(s))H_0(-g_1'(s))g_1'(s)$$

$$\int_{\mathbb{R}^n} g_1(s)|\nabla_x u(t)-\nabla_x u(t-s)|^2 dxds$$

$$+I^{-1}(t)\int_{t_1}^t I(t)H_0^{-1}(-g_2'(s))H_0(-g_2'(s))g_2'(s)$$

$$\int_{\mathbb{R}^n} g_2(s)|\nabla_x v(t)-\nabla_x v(t-s)|^2 dxds\Big)$$

$$\geq H_0\Big(\int_{t_1}^t\int_{\mathbb{R}^n} g_1(s)|\nabla_x u(t)-\nabla_x u(t-s)|^2 dxds$$

$$+\int_{t_1}^t\int_{\mathbb{R}^n} g_2(s)|\nabla_x v(t)-\nabla_x v(t-s)|^2 dxds\Big)$$

which implies

$$\int_{t_1}^t\int_{\mathbb{R}^n} g_1(s)|\nabla_x u(t)-\nabla_x u(t-s)|^2 dxds$$

$$+\int_{t_1}^t\int_{\mathbb{R}^n} g_2(s)|\nabla_x v(t)-\nabla_x v(t-s)|^2 dxds$$

$$\leq H_0^{-1}(\lambda(t)).$$

Then

$$F'(t) \quad\leq\quad -cE(t)+cH_0^{-1}(\lambda(t)), \quad \text{for all} \quad t\geq t_1.$$

Now, we will be following the steps in ([16]) and using the fact that $E'(t)\leq 0, 0<H_0', 0<H_0''$ on $(0,r]$ to define the functional

$$F_1(t)=H_0'\left(a\frac{E(t)}{E(0)}\right)F(t)+cE(t), \quad a<r, 0<c,$$

where $F_1(t)\sim E(t)$ and

$$F_1'(t)=a\frac{E'(t)}{E(0)}H_0''\left(a\frac{E(t)}{E(0)}\right)F(t)$$

$$+H_0'\left(a\frac{E(t)}{E(0)}\right)F'(t)+cE'(t)$$

$$\leq -cE(t)H_0'\left(a\frac{E(t)}{E(0)}\right)$$

$$+cH_0'\left(a\frac{E(t)}{E(0)}\right)H_0^{-1}(\lambda(t))+cE'(t).$$

Let H_0^* be given in A with $A=H_0'\left(a\frac{E(t)}{E(0)}\right), B=H_0^{-1}(\lambda(t))$, we get

$$F_1'(t)\leq -cE(t)H_0'\left(a\frac{E(t)}{E(0)}\right)+cH_0^*\left(H_0'\left(a\frac{E(t)}{E(0)}\right)\right)$$

$$+c\lambda(t)+cE'(t)$$

$$\leq -cE(t)H_0'\left(a\frac{E(t)}{E(0)}\right)+ca\frac{E(t)}{E(0)}H_0'\left(a\frac{E(t)}{E(0)}\right)$$

$$-c'E'(t)+cE'(t).$$

Choosing a, c, c', such that for all $t\geq t_1$ we have

$$\begin{aligned} F_1'(t) &\leq -k\frac{E(t)}{E(0)}H_0'\left(a\frac{E(t)}{E(0)}\right) \\ &= -kH_2\left(\frac{E(t)}{E(0)}\right), \end{aligned}$$

where $H_2(t)=tH_0'(\alpha_0 t)$. Using the strict convexity of H_0 on $(0,r]$, we find that H_2', H_2 are strict positives on $(0,1]$, and then

$$R(t)=\tau\frac{k_1F_1(t)}{E(0)}\sim E(t), \quad \tau\in(0,1) \tag{32}$$

and

$$R'(t)\leq -\tau k_0H_2(R(t)), \quad k_0\in(0,+\infty), t\geq t_1.$$

Integrating and a choosing τ such that,

$$R(t)\leq H_1^{-1}(bt+c), \quad b,c\in(0,+\infty), t\geq t_1,$$

where $H_1(t)=\int_t^1 H_2^{-1}(s)ds$. From (32), for $\alpha_3>0$, we have

$$E(t)\leq dH_1^{-1}(bt+c).$$

Since H_1 is a strictly decreasing $(0, 1]$ and by the properties of H_2, we have

$$\lim_{t\to 0} H_1(t) = +\infty.$$

Therefore

$$E(t) \leq dH_1^{-1}(bt + c), \quad \forall t \geq 0.$$

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