A Doubly Nonlinear Parabolic Equations with a σ-Finite Measure as its Initial Value

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Abstract: A class of equations

$$u_t - \text{div}\vec{a}(u, \nabla u) = f(x,t), (x,t) \in S_T = \mathbb{R}^N \times (0,T),$$

is considered. These equations arise in the study of turbulent filtration of gas or liquid through porous media. If the initial value is a σ-finite measure, the existence and and no existence of the solutions of the equation are researched.

Key–Words: Nonlinear parabolic equation, Cauchy problem, Existence,σ-Finite measure.

1 Introduction

Let $S_T = \mathbb{R}^N \times (0,T)$. Consider an equation of the type

$$u_t - \text{div}\vec{a}(u, \nabla u) = f(x,t), (x,t) \in S_T, \quad (1)$$

where $\nabla u = (u_{x_1}, \cdots, u_{x_N})$, $\vec{a} = (a^1, \cdots, a^N)$, $f = f(x,t)$ is a bounded function on S_T, the functions $a^i(u, \vec{\zeta})$ are continuous on $\mathbb{R} \times \mathbb{R}^N$ and for any $u \in \mathbb{R}, \vec{\zeta} \in \mathbb{R}^N$, $\vec{a}$ satisfies

$$\vec{a}(u,\vec{\zeta}) \cdot \vec{\zeta} \geq \nu_0 |u|^{(m-1)(p-1)} |\vec{\zeta}|^p - \Phi_0(u), \quad (2)$$

$$|\vec{a}(u,\vec{\zeta})| \leq \mu_1 |u|^{(m-1)(p-1)} |\vec{\zeta}|^{p-1} + \Phi_0(u), \quad (3)$$

with the constants $p \in (1,2)$, $m > 1$, $\nu_0 > 0$, $\mu_1 > 0$ and $\Phi_i(u) \geq 0$. Equation (1) with the conditions (2)-(3) are the particular cases of the so-called doubly nonlinear parabolic equations (DNPE), which arise particularly in the study of turbulent filtration of gas or liquid through porous media. A typical example is the polytropic filtration equation

$$u_t = \text{div}(| \nabla u^m |^{p-2} \nabla u^m), \quad (4)$$

which includes when $m = 1$, the well-known p-diffusion equation, when $p = 2$, the well-known porous medium equation. Whether $m = 1$ or $p = 2$, if the initial value $u(x,0)$ is suitably regular, the existence of the weak solutions had been profoundly probed in Wu-Zhao [1], Gmira [2], Yuan-Zhao [3], Zhao [4, 10, 15], Zhao-Yuan [5], Li-Xia [11], DiBenedetto [12], Ivanov [13, 30-35], DiBenedetto-Herrero [16], Zhao-Xu [19], Yuan [22], Berins [23], Filo [24], Ishige [25], Aronson-Caffarelli [26], Aronson-Peletier [27], Dahlberg-Kenig [28], Vazquez [30-31], Zhan [6, 22, 36-38] and references therein. Recently, the author [36] had prove the existence of the solution to (4) with

$$u(x,0) = \mu, \quad x \in \mathbb{R}^N, \quad (5)$$

where the initial value μ is a nonnegative σ-finite measure.

In particular, Ivanov [35] had proved that there is an unique strong solution of the initial boundary value problem of equation (1), and the strong solution is Hölder continuous, provided that the following conditions are true.

(0) The functions $a^i(u,\vec{\zeta}), u^{-\alpha}a^i(u,\vec{\zeta})$, are continuous on $\overline{\mathbb{R}_+} \times \mathbb{R}^N$, where $\alpha = \frac{(m-1)(p-1)}{p}$.

(1)(Growth conditions) For any $u \in \mathbb{R}$, $\vec{\zeta} \in \mathbb{R}^N$,

$$\vec{a}(u,\vec{\zeta}) \cdot \vec{\zeta} \geq \nu_0 |u|^{(m-1)(p-1)} |\vec{\zeta}|^p - \mu_0(|u|^{m(p-1)+1} + 1), \quad (6)$$

$$|\vec{a}|(u,\vec{\zeta}) \leq \mu_1 |u|^{(m-1)(p-1)} |\vec{\zeta}|^{p-1} + \mu(u)|u|^{\frac{(m-1)(p-1)}{p}}, \quad (7)$$

where $\mu(u) \geq 1$ is nondecreasing, $\nu_0 > 0$,$\mu_0 > 0$, $\mu_1 > 0$ are constants.

(2)(Monotonicity condition) There exists $\nu_1 > 0$ and a continuous vector function $\vec{b}(u) \in \mathbb{R}^N$ such that for any $u \in \mathbb{R}, \vec{\zeta}_1, \vec{\zeta}_2 \in \mathbb{R}^N$,

$$[\vec{a}(u,\vec{\zeta}_1) - \vec{a}(v,\vec{\zeta}_2)] \cdot (\vec{\zeta}_1 - \vec{\zeta}_2)$$
$$\geq \nu_1 |u|^{(m-1)(p-1)} |\vec{\zeta}_1 - \vec{\zeta}_2|^2 [|\vec{\zeta}_1 - \vec{b}|^p + |\vec{\zeta}_2 - \vec{b}|^p]^{1-\frac{2}{p}}. \quad (8)$$

(3)(Local Lipschitz condition) For any $u, v \in [\varepsilon, M], \varepsilon > 0, M > \varepsilon$, and any $\vec{\zeta} \in \mathbb{R}^N$,

$$|\vec{a}(u,\vec{\zeta}) - \vec{a}(v,\vec{\zeta})| \leq \Lambda |u - v| (|\vec{\zeta}|^{p-1} + 1), \quad (9)$$

where $\Lambda = \Lambda(\varepsilon, M) \geq 0$is constant.

Certainly, we must point out that many papers by Dibenedetto, Gianazza, Vespri etc. had studied the Harnack inequality and the regularity of the equation with the type of (1), for examples, one can refer to their reviewing article [39]. Very recently, Fornaro S., Sosio M. and Vespri V. had published a paper to get the $L^r_{loc} - L^\infty_{loc}$ eatimates of the solutions to the equation with the type of (1).

In the paper, by the above conditions (0)-(3), we shall research the existence of Cauchy problem of equation (1) with the measure initial value (5) when $1 < m < \frac{1}{p-1}$. For simplicity, we assume that when $u \geq 1$,

$$\mu(u) = u^{\frac{p-1}{p}}, \quad (10)$$

while $u \leq 1, \mu(u) \equiv 1$.

We define that

Definition 1 *A measurable nonnegative function u is said to be a weak solution of Cauchy problem of equation (1) with the initial value (5), if u satisfies*

$$u \in C(0,T; L^1_{loc}(\mathbb{R}^N)) \quad (11)$$

$$u^m \in L^p(0,T; W^{1,p}_{loc}(\mathbb{R}^N)) \quad (12)$$

$$\nabla u^m \in L^p_{loc}(S_T), \quad (13)$$

and for any $t \in (0,T)$, denoting $S_t = \mathbb{R}^N \times (0,t)$,

$$\int_{\mathbb{R}^N} u(x,t)\varphi(x,t)dx + \iint_{S_t} (-u\varphi_t + \vec{a}(u,\nabla u)\nabla\varphi)dxdt = \int_{\mathbb{R}^N} \varphi(x,0)d\mu + \iint_{S_t} f(x,t)\varphi dxdt \quad (14)$$

where $\varphi \in C^1(\bar{S}_t)$ and $\varphi = 0$ if $|x|$ is large enough.

Let us introduce a basic iteration lemma. One can refer to [8, page 161, Lemma 3.1; or 1, page 141, Lemma 1.6] for its proof.

Lemma 2 *Let $f(t)$ be a nonnegative bounded function defined in $[r_0, r_1]$. If for $r_0 \leq t < s \leq r_1$,*

$$g(t) \leq \theta g(s) + (A(s-t)^{-\alpha} + B), \quad (15)$$

where A, B, α, θ are nonnegative constants, $0 \leq \theta < 1$. Then for $r_0 \leq \rho < r \leq r_1$,

$$g(\rho) \leq c(A(r-\rho)^{-\alpha} + B), \quad (16)$$

where c is a constant depending on α, θ.

By this lemma, using the standard Moser's iteration method, we will prove the following theorems.

Theorem 3 *Suppose $f(x,t) \leq 0$, and*

$$2 > p > \frac{(m+1)N}{mN+1},\ 1 < m < \frac{1}{p-1}, N \geq 2. \quad (17)$$

Assume that (10), and the conditions (0),(1), (3) are true. Then there is a weak solution u for equation (1) with (5), which satisfies

$$\sup_{0<\tau<t} \int_{B_R} u(x,\tau)dx \leq c\int_{B_{2R}} |d\mu| + cR^{M_1}, \quad (18)$$

where

$$M_1 = \frac{N}{1-m(p-1)}[t^{\frac{1}{p-(m+1)(p-1)}} + t^{\frac{1}{1-m(p-1)}} + t^{\frac{p}{p-m(p-1)}} + t^{\frac{1}{p-(m+1)(p-1)}}].$$

$$\sup_{x\in B_R} |u(x,t)| \leq c(N,p,R_0)[t^{-\frac{N}{k}} \left(\int_{B_{4R}} u_0 dx\right)^{\frac{p}{k}} + R^{M_2}, R \geq R_0, \quad (19)$$

where

$$M_2 = \frac{Np}{k[1-m(p-1)]} t^{-\frac{N}{k}} [t^{\frac{1}{p-(m+1)(p-1)}} + t^{\frac{1}{1-m(p-1)}} + t^{\frac{p}{p-m(p-1)}}],$$

$k = p - N[1-m(p-1)]$, *$R_0 \geq 1$ is a constant.*

Theorem 4 *Suppose $f(x,t) \leq 0$,*

$$1 < p \leq \frac{(m+1)N}{mN+1},\ 1 < m < \frac{1}{p-1}, \quad (20)$$

and

$$\frac{1}{m} < \min\{1 + \frac{2}{m} - p, 1 - m(p-1)\}, \quad (21)$$

then there is not solution of Cauchy problem of equation (1) with the following initial value

$$u(x,0) = \delta(x), \quad (22)$$

where δ is classical Dirac function.

We use some ideas of [15] and [32] in our paper. By the way, if we only consider equation (4), condition (21) is unnecessary.

2 Proof of Theorem 3

In order to prove the theorem, we quote a lemma.

Lemma 4[1] *Let $Q_l(l = 0,1,2,\cdots,)$ be the sequences of the bounded open domains in S_T, $Q_{l+1} \subset Q_l$. If for $\forall q \geq 1$, $v \in L^q(Q_0)$, and there are constants $\alpha_0 \geq 0$, $\lambda, C_0 > 0$, $K > 1$ such that*

$$\int\int_{Q_{l+1}} |v|^{\alpha_0+\lambda K^{l+1}} dxdt$$

$$\leq \left(C_0 C_1^l \int\int_{Q_l} |v|^{\alpha_0+\lambda K^l} dxdt \right)^K,$$

then

$$ess\sup_{Q_\infty} |v|$$

$$\leq \left(C_0^{\frac{K}{K-1}} \overline{C_1} \int\int_{Q_{l_0}} |v|^{\alpha_0+\lambda K^{l_0}} dxdt \right)^{\frac{1}{\lambda K^{l_0}}},$$

where $\overline{C_1} = C_1^{K_1}$, $K_1 = \sum_{l=l_0}^{\infty} lK^{-(l-l_0)}$, and $l_0 \geq 0$ is any nonnegative integer.

The proof of Theorem 3 Let u be the solution of the regularized equation of (1)

$$u_t - \text{div}(\vec{a}(u,\nabla u)) - \varepsilon \triangle u = f(x,t), \qquad (23)$$

with nonnegative initial value

$$u(x,0) = u_0 \in C_0^\infty(\mathbb{R}^N). \qquad (24)$$

Then, $\vec{a}_1(u,\nabla u) = \vec{a}(u,\nabla u) + \varepsilon \nabla u$ satisfies the assumptions of (0)-(3), from [35], we know that the problem (23)-(24) has a unique strong solution u_ε. Let $\varepsilon \to 0$. By a similar discussion as [5], we are able to show that $u_\varepsilon \to u$ is the solution of equation (1) with initial value (24), u is Hölder continuous and satisfies

$$\int_{\mathbb{R}^N} u(x,t)\varphi(x,t)dx$$

$$+ \iint_{S_t} (-u\varphi_t + \vec{a}(u,\nabla u)\nabla\varphi)dxdt$$

$$= \int_{\mathbb{R}^N} \varphi(x,0)u_0(x)dx + \iint_{S_t} f(x,t)\varphi dxdt, \quad (25)$$

Moreover, similar as in [32], it is easily to know that

$$u^m \in L^\infty(0,T; L^1(\mathbb{R}^N)) \cap L^p(0,T; W^{1,p}(\mathbb{R}^N)).$$

Now, let $u_{0n} \in C_0^\infty(\mathbb{R}^{\mathbb{N}})$ be nonnegative such that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} u_{0n}\varphi dx = \int_{R^N} \varphi d\mu.$$

Assume that u_n is the solution of equation (1) with initial value u_{0n}, by the following lemma 6-lemma 10, $\{u_n\}$ is uniformly bounded on every compact set $K \subset S_T$. Hence by [13] or [34-35], $\{u_n\}$ is equicontinuous on every compact set $K \in S_T$. Then there exists a subsequence such that $u_n \to u$ as $n \to \infty$ in $C(K)$, $\nabla u_n^m \rightharpoonup \nabla u^m$ in $L^p_{loc}(0,T; L^p(\mathbb{R}^{\mathbb{N}}))$, and by a standard limiting process (referring to [1],[5], [6],[13], [34]), the properties (18), (19) of u are true, then Theorem 3 is proved..

In what follows, we will give lemma 6-lemma 10 and the proofs to complete the proof of Theorem 3. First of all, due to that u_n is Hölder continuous, $D_1 = \{(x,t) \in (0,t) \times B_{2R} : u_n \geq 1\}$ and $D_2 = \{(x,t) \in (0,t) \times B_{2R} : 0 \leq u_n \leq 1\}$ are well defined, then we have the additive property of the integral domains, for example, we have

$$\int_0^t \int_{B_{2R}} \xi^p u_n^{\alpha m-1}(u_n^{m(p-1)+1} + 1)dxd\tau$$

$$= \iint_{D_1 \bigcup D_2} \xi^p u_n^{\alpha m-1}(u_n^{m(p-1)+1} + 1)dxd\tau,$$

However, if $0 \leq u_n \leq 1$, the following estimates is true clearly, so we only need to deal with the domain D_1 instead of the whole domain $(0,t) \times B_{2R}$. Without loss generality, in the proof of what follows, we may assume that $u_n \geq 1$ in the whole domain $(0,t) \times B_{2R}$ instead of D_2. For simplism, we denote u_n as u if it is clear from the context.

Lemma 6 *Suppose that (6) and (7) are true. For any given $R_0 > 0$, if $R > R_0$, then the nonnegative solution of equation (1) with the initial value (24) satisfies*

$$\sup_{x\in B_R} |u(x,t)|$$

$$\leq c(N,p,R_0)t^{-\frac{N+p}{k}} \left(\int_0^t \int_{B_{2R}} |u(x,t)|\, dxdt \right)^{\frac{p}{k}},$$

where $k = p + N(m(p-1) - 1)$.

Proof As usual, let $B_R(x_0) = \{x : |x - x_0| < R\}$, and if $x_0 = 0$, simply denoted as B_R. Let ξ be the cut function of $B_{2R} \times (0,T)$ and $\alpha > 0$ be a constant to be chosen later. Choosing the testing function in (25) as $\xi^p u^{\alpha m}$, then one can obtain

$$\int_0^t \int_{B_{2R}} \vec{a}(u,\nabla u)\nabla\varphi dxd\tau$$

$$= \int_0^t \int_{B_{2R}} \vec{a}(u,\nabla u) M_3 dxd\tau$$

$$= p\int_0^t\int_{B_{2R}}\vec{a}(u,\nabla u)\xi^{p-1}u^{\alpha\, m}\nabla\xi dxd\tau$$

$$+m\alpha\int_0^t\int_{B_{2R}}\vec{a}(u,\nabla u)\xi^{p}u^{\alpha m-1}\nabla udxd\tau.$$

where$M_3=(p\xi^{p-1}u^{\alpha m}\nabla\xi+m\alpha\xi^{p}u^{\alpha m-1}\nabla u)$.
According to the assumptions (6) and (7),

$$m\alpha\int_0^t\int_{B_{2R}}\vec{a}(u,\nabla u)\xi^{p}u^{\alpha m-1}\nabla udxd\tau$$

$$\geq m\alpha\nu_0\int_0^t\int_{B_{2R}}\xi^{p}u^{\alpha m-1+(m-1)(p-1)}|\nabla u|^{p}dxd\tau$$

$$-m\alpha\mu_0\int_0^t\int_{B_{2R}}\xi^{p}u^{\alpha m-1}(u^{m(p-1)+1}+1)dxd\tau$$

$$\geq m\alpha\nu_0\int_0^t\int_{B_{2R}}\xi^{p}u^{\alpha m-1+(m-1)(p-1)}|\nabla u|^{p}dxd\tau$$

$$-2m\alpha\mu_0\int_0^t\int_{B_{2R}}\xi^{p}u^{\alpha m+1}dxd\tau.$$

$$p\int_0^t\int_{B_{2R}}\vec{a}(u,\nabla u)\xi^{p-1}u^{\alpha m}\nabla\xi dxd\tau$$

$$\leq p\mu_1\int_0^t\int_{B_{2R}}M_4|\nabla\xi||\nabla u|^{p-1}dxd\tau$$

$$+p\int_0^t\int_{B_{2R}}\xi^{p-1}u^{1+\alpha m}|\nabla\xi|dxd\tau$$

$$=\frac{p\mu_1}{m^{p-1}}\int_0^t\int_{B_{2R}}\xi^{p-1}u^{\alpha m}|\nabla\xi||\nabla u^m|^{p-1}dxd\tau$$

$$+p\int_0^t\int_{B_{2R}}\xi^{p-1}u^{1+\alpha m}|\nabla\xi|dxd\tau$$

$$\leq\frac{p\mu_1}{m^{p-1}}\int_0^t\int_{B_{2R}}\xi^{p-1}u^{\alpha m}|\nabla\xi||\nabla u^m|^{p-1}dxd\tau$$

$$+p\int_0^t\int_{B_{2R}}u^{\alpha m+1}|\nabla\xi|dxd\tau.$$

where$M_4=\xi^{p-1}u^{\alpha m+(m-1)(p-1)}$.

$$\int_0^t\int_{B_{2R}}\xi^{p}u^{\alpha m}f(x,t)dxd\tau\leq 0.$$

By the above calculations and (25), we get

$$\frac{1}{\alpha m+1}\int_{B_{2R}}\xi^{p}u^{\alpha m+1}dx+\alpha m^{1-p}\nu_0$$

$$\int_0^t\int_{B_{2R}}\xi^{p}u^{m(\alpha-1)}\left|\nabla u^m\right|^{p}dxd\tau$$

$$\leq p\mu_1 m^{1-p}\int_0^t\int_{B_{2R}}\xi^{p-1}u^{m\alpha}\left|\nabla u^m\right|^{p-1}|\nabla\xi|dxd\tau$$

$$+\frac{p}{\alpha m+1}\int_0^t\int_{B_{2R}}\xi^{p-1}\xi_t u^{m\alpha+1}dxd\tau$$

$$+c\int_0^t\int_{B_{2R}}u^{\alpha m+1}dxd\tau. \tag{26}$$

Since

$$\left|\nabla(\xi u^{m(\frac{\alpha-1}{p}+1)})\right|^{p}$$

$$=\left|\frac{M_5+u^{m(\frac{\alpha-1}{p}+1)}\nabla\xi}{2}\right|^{p}2^{p}$$

$$\leq 2^{p-1}[{M_5}^{p}+|u^{m(\frac{\alpha-1}{p}+1)}\nabla\xi|^{p}],$$

where$M_5=[\nabla(\xi u^{m(\frac{\alpha-1}{p}+1)})-u^{m(\frac{\alpha-1}{p}+1)}\nabla\xi]$.
then

$$\int_0^t\int_{B_{2R}}\xi^{p}u^{m(\alpha-1)}\left|\nabla u^m\right|^{p}dxd\tau$$

$$=\int_0^t\int_{B_{2R}}\xi^{p}\left|(u^m)^{\frac{\alpha-1}{p}}\nabla u^m\right|^{p}dxd\tau$$

$$=\frac{p}{\alpha-1+p}\int_0^t\int_{B_{2R}}\left|M_5\right|^{p}dxd\tau$$

$$\geq\frac{p}{\alpha-1+p}2^{1-p}[\int_0^t\int_{B_{2R}}|\nabla(\xi u^{m(\frac{\alpha-1}{p}+1)})|^{p}dxd\tau$$

$$-\int_0^t\int_{B_{2R}}|u^{m(\frac{\alpha-1}{p}+1)}\nabla\xi|^{p}]dxd\tau. \tag{27}$$

By Young inequality,

$$\xi^{p-1}u^{m\alpha}\left|\nabla u^m\right|^{p-1}$$

$$\leq\varepsilon\xi^{p}u^{m(\alpha-1)}\left|\nabla u^m\right|^{p}+c(\varepsilon)u^{m(\alpha-1a+p)},$$

$$\int_0^t\int_{B_{2R}}\xi^{p-1}u^{m\alpha}\left|\nabla u^m\right|^{p-1}|\nabla\xi|dxd\tau$$

$$\leq\varepsilon\int_0^t\int_{B_{2R}}\xi^{p}u^{m(\alpha-1)}\left|\nabla u^m\right|^{p}dxdt$$

$$+c(\varepsilon)\int_0^t\int_{B_{2R}}u^{m(\alpha-1+p)}|\nabla\xi|^{p}dxdt. \tag{28}$$

By (26)-(28), since

$$\varepsilon\int_0^t\int_{B_{2R}}\xi^{p}u^{m(\alpha-1)}\left|\nabla u^m\right|^{p}dxdt$$

can be compensated by

$$\alpha\int_0^t\int_{B_{2R}}\xi^p u^{m(\alpha-1)}\left|\nabla u^m\right|^p dxd\tau,$$

and noticing that $0\leq\xi\leq 1$, $m(p-1)<1$,

$$\xi^p\geq\xi^{\frac{(\alpha m+1)p}{m(\alpha-1+p)}},$$

we have

$$\begin{aligned}&\sup_{0<t<T}\int_{B_{2R}}(\xi u^{m(\frac{\alpha-1}{p}+1)})^{\frac{(\alpha m+1)p}{m(\alpha-1+p)}}dx\\&+\int_0^T\int_{B_{2R}}\left|\nabla(\xi u^{\frac{m}{p}(\alpha-1+p)})\right|^p dxdt\\&\leq c\int_0^T\int_{B_{2R}}|\nabla\xi|^p u^{m(\alpha-1+p)}dxdt\\&+c\int_0^T\int_{B_{2R}}|\xi_t|\,u^{m\alpha+1}dxdt\\&+c\int_0^T\int_{B_{2R}}|\nabla\xi|u^{m\alpha+1}dxdt\\&+c\int_0^T\int_{B_{2R}}u^{m\alpha+1}dxdt.\end{aligned}\tag{29}$$

Let $\nu=\xi\,u^{m(\frac{\alpha-1}{p}+1)}$. By Sobolev inequality (referring to [7], page 62), we have

$$\begin{aligned}&\int_0^T\int_{B_{2R}}\nu^r dxdt\\&\leq c\left(\sup_{0<t<T}\int_{B_{2R}}\nu^{\frac{(\alpha m+1)p}{m}(\alpha-1+p)}dx\right)^{M_6}\\&\int_0^T(\int_{B_{2R}}|\nabla\nu|^p\,dx)^{\frac{\delta r}{p}}dt,\end{aligned}\tag{30}$$

where

$$M_6=\frac{m(\alpha-1+p)}{(\alpha m+1)p}(1-\delta)r,$$

$$\delta=(\frac{m(\alpha-1+p)}{\alpha m+1}-\frac{1}{r})(\frac{p}{N}-\frac{1}{p}+\frac{m(\alpha-1+p)}{\alpha m+1})^{-1}.$$

Thus, if we choose

$$r=p(1+\frac{1}{N}\frac{(\alpha m+1)p}{m(\alpha-1+p)}),\ \frac{\delta\,r}{p}=1,$$

noticing that for any $a\geq 0, b\geq 0$,

$$a^{\frac{p}{N}}b\leq(a+b)^{1+\frac{p}{N}}$$

is always true, then from (29), (30)

$$\begin{aligned}&\int_0^T\int_{B_{2R}}\xi^r u^{m(\alpha-1+p)+\frac{(\alpha m+1)p}{N}}dxdt\\&\leq c(\sup_{0<t<T}\int_{B_{2R}}\xi^{\frac{(\alpha m+1)p}{m(\alpha-1+p)}}u^{\alpha m+1}dx)^{\frac{p}{N}}\\&\int_0^T\int_{B_{2R}}\left|\nabla(\xi u^{\frac{m(\alpha-1+p)}{p}})\right|^p dxdt\\&\leq c[\sup_{0<t<T}\int_{B_{2R}}\xi^{\frac{(\alpha m+1)p}{m(\alpha-1+p)}}u^{\alpha m+1}dx\\&+\int_0^T\int_{B_{2R}}\left|\nabla(\xi u^{\frac{m(\alpha-1+p)}{p}})\right|^p dxdt]^{1+\frac{p}{N}}\\&\leq c[\int_0^T\int_{B_{2R}}|\nabla\xi|^p\,u^{m(\alpha-1+p)}dxdt\\&+\int_0^T\int_{B_{2R}}|\xi_t|\,u^{\alpha m+1}dxdt]^{1+\frac{p}{N}}\\&+c\int_0^T\int_{B_{2R}}(1+|\nabla\xi|)u^{\alpha m+1}dxd\tau.\end{aligned}\tag{31}$$

Now for $s\in[\frac{1}{2},1)$, let $R_l=2R(s+\frac{1-s}{2^l})$,

$$T_l=\frac{T}{2}-\frac{T}{2}(s+\frac{1-s}{2^l}),Q_{R_l}=B_{R_l}\times(T_l,T),$$

$l=0,\ 1,\ 2,\cdots$. Suppose that ξ_l is the cut functions on Q_{R_l} which satisfy

$$\xi_l=1,in\ Q_{R_{L+1}};\quad \xi_l=0,in\ \bar{S}_T\backslash Q_{R_l}.$$

$$|\nabla\xi_l|\leq\frac{2^{l+1}}{(1-s)R},\quad 0\leq|\xi_{lt}|\leq\frac{2^{l+1}}{(1-s)T}.$$

Denote $\gamma=1+\frac{p}{N}$ and choose α such that

$$m\alpha+1=\frac{N}{p}(1-m(p-1))+\gamma^l,$$

which implies

$$m(\alpha-1+p)+\frac{(\alpha m+1)p}{N}=\frac{N}{p}(1-m(p-1))+\gamma^{l+1}.$$

Applying (31) to Q_{R_l}, one obtains,

$$\begin{aligned}&\iint_{Q_{R_l}}u^{\frac{N}{p}(1-m(p-1))+\gamma^{l+1}}dxdt\\&\leq[c^l((1-s)R)^{-p}\left(\iint_{Q_{R_l}}M_7dxdt\right)^{\frac{m(\alpha-1+p)}{m\alpha+1}}\end{aligned}$$

$$+c^l(1-s)^{-1})T^{-1}\iint_{Q_{R_l}} M_7dxdt]^\gamma$$
$$+c\iint_{Q_{R_l}}(1+|\nabla\xi_l|)u^{\alpha m+1}dxdt$$
$$\leq c2^{l+1}(1-s)^{-1}R^{-1}\iint_{Q_{R_l}} M_7dxdt. \quad (32)$$

where $M_7=u^{\frac{N}{p}(1-m(p-1))+\gamma^l}$.

Hence, from (32), by the standard Moser's iteration, using Lemma 5, we have

$$\sup_{Q_{s2R}} u\leq\left[c(1-s)^{-\frac{N+p}{p}}T^{-\frac{N+p}{p}}\iint_{Q_{2R}} M_8dxdt\right]^{\frac{1}{\gamma}}$$
$$\leq(\sup_{Q_{2R}} u)^{1-\frac{k}{p\gamma}}(c\left((1-s)T)^{-\frac{N+p}{p}}\iint_{Q_{2R}} udxdt\right)^{\frac{1}{\gamma}}$$
$$\leq\frac{1}{4}\sup_{Q_{2R}} u+c((1-s)T)^{-\frac{N+p}{k}}\left(\iint_{Q_{2R}} udxdt\right)^{\frac{p}{k}}.$$

where $M_8=u^{\frac{N}{p}(m(p-1)+1)+\gamma}$.

Hence by lemma 2, we obtain the conclusion.

Lemma 7 *Suppose $1<p<2$, $m(p-1)<1$, $f(x,t)\leq 0$, $R\geq 1$,then the solution u of Cauchy problem of equation (1) with the initial value (24) satisfies*

$$\sup_{0<\tau<t}\int_{B_R} u(x,\tau)dx$$
$$\leq c\int_{B_{2R}} u(x,0)dx+cR^{\frac{N}{1-m(p-1)}}M_9. \quad (33)$$

where $M_9=[t^{\frac{1}{p-(m+1)(p-1)}}+t^{\frac{1}{1-m(p-1)}}+t^{\frac{p}{p-m(p-1)}}]$.

Proof Let ξ be the cut function on B_{2R}, and ξ satisfy that $\xi=1$ on B_{s2R}, $|\nabla\xi|\leq(1-s)^{-1}R^{-1}$, $s\in[\frac{1}{2},1)$. For any $t>0$, we have

$$\int_{B_{s2R}} u(x,t)dx\leq\int_{B_{2R}} u_0dx$$
$$+\frac{c}{(1-s)R}\int_0^t\int_{B_{2R}}|\nabla u^m|^{p-1}\xi^{p-1}dxd\tau$$
$$+\frac{c}{(1-s)R}\int_0^t\int_{B_{2R}} u^{\frac{(m-1)(p-1)}{p}}\xi^{p-1}dxd\tau$$
$$+\int_0^t\int_{B_{2R}} f(x,t)\xi^pdxd\tau, \quad (34)$$

where the constant c depends on μ_1, m.

We choose testing function

$$\varphi=\xi^pt^{\frac{\beta p}{p-1}}u^{m(1-\frac{\alpha p}{p-1})}$$

in (25), where α,β are constants to be chosen later, then

$$\int_0^t\int_{B_{2R}}\xi^p\tau^{\frac{\beta p}{p-1}}u^{m(1-\frac{\alpha p}{p-1})}u_\tau dxd\tau$$
$$=M_{10}\int_0^t\int_{B_{2R}}\xi^p\tau^{\frac{\beta p}{p-1}}u_\tau^{m(1-\frac{\alpha p}{p-1})+1}dxd\tau$$
$$=M_{10}\int_{B_{2R}}\xi^pt^{\frac{\beta p}{p-1}}u^{m(1-\frac{\alpha p}{p-1})+1}dx$$
$$-M_{10}\frac{\beta p}{p-1}\int_0^t\int_{B_{2R}}\xi^p\tau^{\frac{\beta p}{p-1}-1}u^{m(1-\frac{\alpha p}{p-1})+1}dxd\tau,$$

where $M_{10}=[m(1-\frac{\alpha p}{p-1})+1]^{-1}$. According to the assumptions (6) and (7),

$$\int_0^t\int_{B_{2R}}\xi^p\tau^{\frac{\beta p}{p-1}}u^{m(1-\frac{\alpha p}{p-1})}\mathrm{div}(\vec{a}(u,\nabla u)dxd\tau$$
$$=-\int_0^t\int_{B_{2R}}\tau^{\frac{\beta p}{p-1}}\nabla(\xi^pu^{m(1-\frac{\alpha p}{p-1})})\cdot\vec{a}(u,\nabla u)dxd\tau$$
$$=-p\int_0^t\int_{B_{2R}}\tau^{\frac{\beta p}{p-1}}\xi^{p-1}u^{m(1-\frac{\alpha p}{p-1})}\nabla\xi\cdot\vec{a}(u,\nabla u)dxd\tau$$
$$+M_{11}\int_0^t\int_{B_{2R}}\tau^{\frac{\beta p}{p-1}}\xi^pu^{-\frac{\alpha pm}{p-1}}\nabla u^m\cdot\vec{a}(u,\nabla u)dxd\tau$$
$$=-I_1+I_2$$

where $M_{11}=(\frac{\alpha p}{p-1}-1)$.

$$I_1\leq p\int_0^t\int_{B_{2R}} M_{12}|\nabla\xi|M_{13}dxd\tau$$
$$=\frac{\mu_1p}{m^{p-1}}\int_0^t\int_{B_{2R}} M_{12}|\nabla\xi||\nabla u^m|^{p-1}dxd\tau$$
$$+p\int_0^t\int_{B_{2R}}\xi^{p-1}\tau^{\frac{\beta p}{p-1}}u^{m(1-\frac{\alpha p}{p-1})+1}|\nabla\xi|dxd\tau$$
$$\leq\int_0^t\int_{B_{2R}} M_{14}(\varepsilon|\nabla u^m|^p+c(\varepsilon)u^{mp}|\nabla\xi|^p)dxd\tau$$
$$+M_{15}\int_0^t\int_{B_{2R}}\tau^{\frac{\beta p}{p-1}}u^{m(1-\frac{\alpha p}{p-1})+\frac{(m-1)(p-1)}{p}}dxd\tau.$$

where

$$M_{12}=\xi^{p-1}\tau^{\frac{\beta p}{p-1}}u^{m(1-\frac{\alpha p}{p-1})},$$

$$M_{13}=(\mu_1 u^{(m-1)(p-1)}|\nabla u|^{p-1}+u),$$
$$M_{14}=\xi^p\tau^{\frac{\beta p}{p-1}}u^{-\frac{\alpha mp}{p-1}}, M_{15}=\frac{p}{(1-s)R}.$$

while I_2 is equal to

$$M_{16}\int_0^t\int_{B_{2R}}\tau^{\frac{\beta p}{p-1}}\xi^p u^{-\frac{\alpha pm}{p-1}}\nabla u^m\cdot\vec{a}(u,\nabla u)dxd\tau$$
$$\geq M_{16}\nu_0\int_0^t\int_{B_{2R}}\tau^{\frac{\beta p}{p-1}}\xi^p M_{17}|\nabla u|^p dxd\tau$$
$$-M_{16}\mu_0\int_0^t\int_{B_{2R}}\tau^{\frac{\beta p}{p-1}}\xi^p u^{-\frac{\alpha pm}{p-1}+m-1}M_{18}dxd\tau$$
$$\geq\frac{\nu_0}{m^{p-2}}M_{16}\int_0^t\int_{B_{2R}}\tau^{\frac{\beta p}{p-1}}\xi^p u^{-\frac{\alpha pm}{p-1}}|\nabla u^m|^p dxd\tau$$
$$-2m\mu_0(\frac{\alpha p}{p-1}-1)\int_0^t\int_{B_{2R}}\tau^{\frac{\beta p}{p-1}}u^{-\frac{\alpha pm}{p-1}+mp}dxd\tau$$

where $M_{16}=m(\frac{\alpha p}{p-1}-1)$,

$$M_{17}=u^{-\frac{\alpha pm}{p-1}+m-1+(m-1)(p-1)},$$
$$M_{18}=(u^{m(p-1)+1}+1).$$

By the above inequalities, we have

$$[\frac{\nu_0}{m^{p-1}}(\frac{\alpha p}{p-1}-1)-\varepsilon]\int_0^t\int_{B_{2R}}M_{19}|\nabla u^m|^p dxd\tau$$
$$+c\int_0^t\int_{B_{2R}}\xi^p\tau^{\frac{\beta p}{p-1}-1}u^{m(1-\frac{\alpha p}{p-1})+1}dxd\tau$$
$$=c\int_{B_{2R}}\xi^p t^{\frac{\beta p}{p-1}}u^{m(1-\frac{\alpha p}{p-1})+1}dx$$
$$+\frac{c(\varepsilon)}{[(1-s)R]^p}\int_0^t\int_{B_{2R}}\xi^p\tau^{\frac{\beta p}{p-1}}u^{-\frac{\alpha mp}{p-1}}u^{mp}dxd\tau$$
$$+\frac{p}{(1-s)R}\int_0^t\int_{B_{2R}}\tau^{\frac{\beta p}{p-1}}u^{m(1-\frac{\alpha p}{p-1})+1}dxd\tau$$
$$+2m\mu_0(\frac{\alpha p}{p-1}-1)\int_0^t\int_{B_{2R}}\tau^{\frac{\beta p}{p-1}}u^{-\frac{\alpha pm}{p-1}+mp}dxd\tau$$
$$+\int_0^t\int_{B_{2R}}\xi^p\tau^{\frac{\beta p}{p-1}}u^{m(1-\frac{\alpha p}{p-1})}f(x,t)dxd\tau. \qquad (35)$$

where $M_{19}=\tau^{\frac{\beta p}{p-1}}\xi^p u^{-\frac{\alpha pm}{p-1}}$.

The later choice of α can guarantee that we can choose small $\varepsilon>0$ such that

$$\frac{\nu_0}{m^{p-1}}(\frac{\alpha p}{p-1}-1)-\varepsilon>0, \qquad (36)$$

then, noticing that $0\geq f(x,t)\in L^\infty(S_T)$, $R\geq 1$, we have

$$\int_0^t\int_{B_{2R}}\xi^p u^{-\frac{\alpha pm}{p-1}}\tau^{\frac{\beta p}{p-1}}|\nabla u^m|^p dxd\tau$$
$$\leq c[\int_{B_{2R}}t^{\frac{\beta p}{p-1}}u^{m(1-\frac{\alpha p}{p-1})+1}dx$$
$$+\int_0^t\int_{B_{2R}}\tau^{\frac{\beta p}{p-1}}u^{-\frac{\alpha mp}{p-1}}u^{mp}dxd\tau$$
$$+\int_0^t\int_{B_{2R}}\tau^{\frac{\beta p}{p-1}}u^{m(1-\frac{\alpha p}{p-1})+1}dxd\tau]. \qquad (37)$$

At the same time, by Hölder inequality

$$\int_0^t\int_{B_{2R}}|\nabla u^m|^{p-1}\xi^{p-1}dxd\tau$$
$$=\int_0^t\int_{B_{2R}}|\nabla u^m|^{p-1}\xi^{p-1}\tau^\beta u^{\alpha m}u^{-\alpha m}\tau^{-\beta}dxd\tau$$
$$\leq\left(\int_0^t\int_{B_{2r}}\xi^p\tau^{\frac{\beta p}{p-1}}|\nabla u^m|^p u^{-\frac{\alpha pm}{p-1}}dxd\tau\right)^{\frac{p-1}{p}}$$
$$\left(\int_0^t\int_{B_{2R}}\tau^{-p\beta}u^{\alpha pm}dxd\tau\right)^{\frac{1}{p}}. \qquad (38)$$

By (37), (38),

$$\int_0^t\int_{B_{2R}}|\nabla u^m|^{p-1}\xi^{p-1}dxd\tau$$
$$\leq c\{\int_{B_{2R}}t^{\frac{\beta p}{p-1}}u^{m(1-\frac{\alpha p}{p-1})+1}dx$$
$$+\int_0^t\int_{B_{2R}}\tau^{\frac{\beta p}{p-1}}u^{-\frac{\alpha mp}{p-1}}u^{mp}dxd\tau$$
$$+\int_0^t\int_{B_{2R}}\tau^{\frac{\beta p}{p-1}}u^{m(1-\frac{\alpha p}{p-1})+1}dxd\tau\}^{\frac{p-1}{p}}$$
$$\left(\int_0^t\int_{B_{2R}}\tau^{-p\beta}u^{\alpha pm}dxd\tau\right)^{\frac{1}{p}}. \qquad (39)$$

It is able to make more explicitly estimates to the above inequality by considering the two cases what follows.

(1) If $\frac{1}{mp}>p-1$, we choose $\alpha=\frac{1}{mp}, \beta=\frac{1}{2p}$ in (39). Then

$$\int_0^t\int_{B_{2R}}|\nabla u^m|^{p-1}\xi^{p-1}dxd\tau$$
$$\leq c\{\int_0^t\int_{B_{2R}}\tau^{\frac{1}{2(p-1)}}u^{mp-\frac{1}{p-1}}dxd\tau$$

$$+\int_{B_{2R}} t^{\frac{1}{2(p-1)}} u^{m(1-\frac{1}{m(p-1)}+1} dx$$

$$+\int_0^t \int_{B_{2R}} \tau^{\frac{1}{2(p-1)}-1} u^{m(1-\frac{1}{m(p-1)})+1} dxd\tau\}^{\frac{p-1}{p}}$$

$$\left(\int_0^t \int_{B_{2R}} \tau^{-\frac{1}{2}} u dxd\tau\right)^{\frac{1}{p}}. \quad (40)$$

Since

$$\int_0^t \int_{B_{2R}} \tau^{\frac{1}{2(p-1)}-1} u^{m(1-\frac{1}{m(p-1)})+1} dxd\tau$$

$$= \int_0^t \int_{B_{2R}} \tau^{\frac{1}{2(p-1)}-1+\frac{1}{2}(m(1-\frac{1}{m(p-1)})+1)} M_{20} dxd\tau$$

$$\leq \left(\int_0^t \int_{B_{2R}} u\tau^{-\frac{1}{2}} dxd\tau\right)^{m(1-\frac{1}{m(p-1)})+1}$$

$$\left(\int_0^t \int_{B_{2R}} M_{21} dxd\tau\right)^{\frac{1}{p-1}-m}$$

$$\leq \left(\int_0^t \int_{B_{2R}} u\tau^{-\frac{1}{2}} dxd\tau\right)^{m(1-\frac{1}{m(p-1)})+1}$$

$$R^{\frac{N(1-m(p-1))}{p-1}} t^{\frac{2-(m+1)(p-1)}{2(p-1)}},$$

where

$$M_{20} = \tau^{-\frac{1}{2}(m(1-\frac{1}{m(p-1)})+1)} u^{m(1-\frac{1}{m(p-1)})+1},$$

$$M_{21} = \tau^{[\frac{1}{2(p-1)}-1+\frac{1}{2}(m(1-\frac{1}{m(p-1)})+1)]\frac{1}{\frac{1}{p-1}-m}}.$$

then

$$\left[\int_0^t \int_{B_{2R}} \tau^{\frac{1}{2(p-1)}-1} u^{m(1-\frac{1}{m(p-1)})+1} dxd\tau\right]^{\frac{p-1}{p}}$$

$$\leq \left(\int_0^t \int_{B_{2R}} u\tau^{-\frac{1}{2}} dxd\tau\right)^{\frac{m(p-1)+p-2}{p}}$$

$$R^{\frac{N(1-m(p-1))}{p}} t^{\frac{2-(m+1)(p-1)}{2p}},$$

$$\left[\int_{B_{2R}} t^{\frac{1}{2(p-1)}-1} u^{m(1-\frac{1}{m(p-1)})+\frac{(m-1)(p-1)}{p}} dx\right]^{\frac{p-1}{p}}$$

$$\leq c\left[\left(\int_{B_{2R}} u dx\right)^{m(1-\frac{1}{m(p-1)})+1} t^{\frac{1}{2(p-1)}} M_{21}\right]^{\frac{p-1}{p}}$$

$$\leq c\left(\int_{B_{2R}} u dx\right)^{\frac{m(p-1)+p-2}{p}} R^{\frac{1-m(p-1)}{p}N} t^{\frac{1}{2p}},$$

where $M_{21} = R^{N(\frac{1}{p-1}-m)}$.

$$\left(\int_0^t \int_{B_{2R}} \tau^{\frac{1}{2(p-1)}} u^{mp-\frac{1}{p-1}} dxd\tau\right)^{\frac{p-1}{p}}$$

$$\leq \left[\int_0^t \int_{B_{2R}} M_{22} dxdt\right]^{(mp-\frac{1}{p-1})\frac{p-1}{p}}$$

$$\times \left[\int_0^t \int_{B_{2R}} M_{23} dxd\tau\right]^{(\frac{1}{p-1}+1-pm)\frac{p-1}{p}}$$

$$\leq c\left(\int_0^t \int_{B_R} \tau^{-\frac{1}{2}} u dxd\tau\right)^{m(p-1)-\frac{1}{p}}$$

$$R^{N(1-mp+m)} t^{\frac{2-m(p-1)}{2}}.$$

where

$$M_{22} = (\tau^{-\frac{1}{2}(pm-\frac{1}{p-1})} u^{mp-\frac{1}{p-1}})^{\frac{1}{pm-\frac{1}{p-1}}}$$

$$M_{23} = (\tau^{\frac{1}{2(p-1)}+\frac{1}{2}(pm-\frac{1}{p-1})})^{\frac{1}{\frac{1}{p-1}+1-pm}}$$

we have

$$\int_0^t \int_{B_{2R}} |\nabla u^m|^{p-1} \xi^{p-1} dxd\tau$$

$$\leq c\{\int_0^t \int_{B_{2R}} \tau^{\frac{1}{2(p-1)}} u^{mp-\frac{1}{p-1}} dxd\tau$$

$$+\int_{B_{2R}} t^{\frac{1}{2(p-1)}} u^{m(1-\frac{1}{m(p-1)}+1} dx$$

$$+\int_0^t \int_{B_{2R}} \tau^{\frac{1}{2(p-1)}-1} u^{m(1-\frac{1}{m(p-1)})+1} dxd\tau\}^{\frac{p-1}{p}}$$

$$\left(\int_0^t \int_{B_{2R}} \tau^{-\frac{1}{2}} u dxd\tau\right)^{\frac{1}{p}}$$

$$\leq \{\left(\int_0^t \int_{B_{2R}} u\tau^{-\frac{1}{2}} dxd\tau\right)^{\frac{m(p-1)+p-2}{p}}$$

$$R^{\frac{N(1-m(p-1))}{p}} t^{\frac{2-(m+1)(p-1)}{2p}}$$

$$+\left(\int_{B_{2R}} u dx\right)^{\frac{m(p-1)+p-2}{p}} R^{\frac{1-m(p-1)}{p}N} t^{\frac{1}{2p}}$$

$$+\left(\int_0^t \int_{B_R} \tau^{-\frac{1}{2}} u dxd\tau\right)^{m(p-1)-\frac{1}{p}}$$

$$R^{N(1-mp+m)} t^{\frac{2-m(p-1)}{2}}\}$$

$$\left(\int_0^t \int_{B_{2R}} \tau^{-\frac{1}{2}} u dxd\tau\right)^{\frac{1}{p}}$$

$$\le cR^{N(1-m(p-1))}t^{\frac{2-m(p-1)}{2}}\left(\int_0^t\int_{B_{2R}}\tau^{-\frac{1}{2}}udxd\tau\right)^{m(p-1)}$$
$$+cR^{\frac{N(m-mp+1)}{p}}t^{\frac{2-(m+1)(p-1)}{2p}}\left(\int_0^t\int_{B_{2R}}\tau^{-\frac{1}{2}}udxd\tau\right)^{\frac{(m+1)(p-1)}{p}}$$
$$+c\left(\int_{B_{2R}}udx\right)^{\frac{m(p-1)+p-2}{p}}R^{\frac{1-m(p-1)}{p}N}t^{\frac{1}{2p}}\left(\int_0^t\int_{B_{2R}}\tau^{-\frac{1}{2}}udxd\tau\right)^{\frac{1}{p}}. \tag{41}$$

By (34), (41),

$$\sup_{0<\tau<t}\int_{B_{s2R}}u(x,\tau)dx$$
$$\le\int_{B_{2R}}u(x,0)dx+c\int_0^t\int_{B_{2R}}|\nabla u^m|^{p-1}\xi^{p-1}dxd\tau$$
$$+c\int_0^t\int_{B_{2R}}udxd\tau$$
$$\le cR^{N(1-m(p-1))}t^{\frac{2-m(p-1)}{2}}\left(\int_0^t\int_{B_{2R}}\tau^{-\frac{1}{2}}udxd\tau\right)^{m(p-1)}$$
$$+cR^{\frac{N(m-mp+1)}{p}}t^{\frac{2-(m+1)(p-1)}{2p}}\left(\int_0^t\int_{B_{2R}}\tau^{-\frac{1}{2}}udxd\tau\right)^{\frac{(m+1)(p-1)}{p}}$$
$$+c\left(\int_{B_{2R}}udx\right)^{\frac{m(p-1)+p-2}{p}}R^{\frac{1-m(p-1)}{p}N}t^{\frac{1}{2p}}\left(\int_0^t\int_{B_{2R}}\tau^{-\frac{1}{2}}udxd\tau\right)^{\frac{1}{p}}$$
$$\le cR^{N(1-m(p-1))}t\left(\sup_{0<\tau<t}\int_{B_{2R}}udxd\tau\right)^{m(p-1)}$$
$$+cR^{\frac{N(m-mp+1)}{p}}t^{\frac{1}{p}}\left(\sup_{0<\tau<t}\int_{B_{2R}}udxd\tau\right)^{\frac{(m+1)(p-1)}{p}}$$
$$+\int_{B_{2R}}u(x,0)dx+ct\sup_{0<\tau<t}\int_{B_{2R}}u^{\frac{m(p-1)}{p}}dxd\tau$$
$$\le c(\varepsilon)[R^Nt^{\frac{1}{1-m(p-1)}}+M_{24}+t^{\frac{p}{p-m(p-1)}}R^N]$$
$$+\varepsilon\sup_{0<\tau<t}\int_{B_{2R}}udx$$
$$\le c(\varepsilon)R^N[t^{\frac{1}{1-m(p-1)}}+t^{\frac{1}{p-(m+1)(p-1)}}+t^{\frac{p}{p-m(p-1)}}]$$
$$+\varepsilon\sup_{0<\tau<t}\int_{B_{2R}}udx. \tag{42}$$

where $M_{24}=R^{\frac{N[1-m(p-1)]}{p-(m+1)(p-1)}}t^{\frac{1}{p-(m+1)(p-1)}}$.

(2) If $\frac{1}{mp}\le p-1$, we choose $\alpha=p-1,\beta=\frac{p-1}{2}$ in (39). Then, using the fact $R\ge 1$,

$$\int_0^t\int_{B_{2R}}|\nabla u^m|^{p-1}\xi^{p-1}dxd\tau$$
$$\le c\{\int_{B_{2R}}t^{\frac{p}{2}}u^{1-m(p-1)}dx+M_{25}\}^{\frac{p-1}{p}}$$
$$\times\left(\int_0^t\int_{B_{2R}}u^{mp(p-1)}\tau^{-\frac{p(p-1)}{2}}dxd\tau\right)^{\frac{1}{p}}.$$

where

$$M_{25}=\int_0^t\int_{B_{2R}}\tau^{\frac{p}{2}}dxd\tau$$
$$+\int_0^t\int_{B_{2R}}\tau^{\frac{p-2}{2}}u^{m+1-mp}dxd\tau$$

Since

$$\left(\int_0^t\int_{B_{2R}}\tau^{\frac{p-2}{2}}u^{m+1-mp}dxdt\right)^{\frac{p-1}{p}}$$
$$\le\left[\int_0^t\int_{B_{2R}}M_{26}dxd\tau\right]^{\frac{(m+1-mp)(p-1)}{p}}$$
$$\times\left[\int_0^t\int_{B_{2R}}M_{27}dxd\tau\right]^{\frac{m(p-1)^2}{p}}$$
$$\le c\left(\int_0^t\int_{B_{2R}}u\tau^{-\frac{1}{2}}dxd\tau\right)^{\frac{(m+1-mp)(p-1)}{p}}M_{28},$$

where

$$M_{26}=\left(u^{m+1-mp}\tau^{-\frac{m+1-mp}{2}}\right)^{\frac{1}{m+1-mp}},$$
$$M_{27}=\tau^{(\frac{p-2}{2}+\frac{m+1-mp}{2})\frac{1}{m(p-1)}},$$
$$M_{28}=t^{\frac{(p-1)^2(m+1)}{2p}}R^{\frac{mN(p-1)^2}{p}}.$$
$$\left(\int_0^t\int_{B_{2R}}u^{mp(p-1)}\tau^{-\frac{p(p-1)}{2}}dxd\tau\right)^{\frac{1}{p}}$$
$$\le c\left(\int_0^t\int_{B_{2R}}u\tau^{-\frac{1}{2}}dxd\tau\right)^{m(p-1)}$$

$$t^{\frac{mp-mp^2-p^2+p+2}{2p}}R^{\frac{N[1-mp(p-1)]}{p}},$$

and

$$\left[\int_{B_{2R}} t^{\frac{p}{2}}u^{1-m(p-1)}dx\right]^{\frac{p-1}{p}}$$

$$\leq ct^{\frac{p(p-1)}{2p}}R^{\frac{Nm(p-1)^2}{p}}(\int_{B_{2R}} udx)^{[1-m(p-1)]\frac{p-1}{p}},$$

we have

$$\int_0^t\int_{B_{2R}} |\nabla u^m|^{p-1}\xi^{p-1}dxd\tau$$

$$\leq c\left(\int_0^t\int_{B_{2R}} u\tau^{-\frac{1}{2}}dxd\tau\right)^{m(p-1)}$$

$$t^{\frac{m+2-mp}{2}}R^{\frac{N[1-m(p-1)]}{p}}$$

$$+C\left(\int_0^t\int_{B_{2R}} u\tau^{-\frac{1}{2}}dxd\tau\right)^{\frac{(p-1)(m+1)}{p}}$$

$$R^{\frac{N[1-m(p-1)]}{p}}t^{\frac{m+3-mp-p}{2p}}$$

$$+ct^{\frac{2-mp(p-1)}{2p}}R^{\frac{N}{p}}\left(\int_{B_{2R}} udx\right)^{[1-m(p-1)]\frac{p-1}{p}}$$

$$\left(\int_0^t\int_{B_{2R}} u\tau^{-\frac{1}{2}}dxd\tau\right)^{m(p-1)}. \tag{43}$$

By (34), (43),

$$\int_{B_{s2R}} u(x,t)dx$$

$$\leq \int_{B_{2R}} u_0dx + \int_0^t\int_{B_{2R}} |\nabla u^m|^{p-1}\xi^{p-1}dxd\tau$$

$$+c\int_0^t\int_{B_{2R}} u^{\frac{m(p-1)}{p}}dxd\tau$$

$$\leq \int_{B_{2R}} u(x,0)dx$$

$$+ctR^{\frac{N[1-m(p-1)]}{p}}\left(\sup_{0<\tau<t}\int_{B_{2R}} udx\right)^{m(p-1)}$$

$$+ct^{\frac{1}{p}}R^{\frac{N[1-m(p-1)]}{p}}\left(\sup_{0<\tau<t}\int_{B_{2R}} udx\right)^{\frac{(m+1)(p-1)}{p}}$$

$$+ct^{\frac{1}{p}}R^{\frac{N}{p}}\left(\sup_{0<\tau<t}\int_{B_{2R}} udx\right)^{[1-m(p-1)]\frac{p-1}{p}+m(p-1)}$$

$$+ct\sup_{0<\tau<t}\int_{B_{2R}} u^{\frac{m(p-1)}{p}}dxd\tau$$

$$\leq \int_{B_{2R}} u(x,0)dx + \varepsilon\sup_{0<\tau<t}\int_{B_{2R}} udx$$

$$+c(\varepsilon)[t^{\frac{1}{1-m(p-1)}}R^{\frac{N}{p}} + t^{\frac{1}{p-(m+1)(p-1)}}R^{\frac{N[1-m(p-1)]}{p-(m+1)(p-1)}}$$

$$+t^{\frac{1}{1-m(p-1)}}R^{\frac{N}{1-m(p-1)}} + t^{\frac{p}{p-m(p-1)}}R^{N}]$$

$$\leq \int_{B_{2R}} u(x,0)dx + \varepsilon\sup_{0<\tau<t}\int_{B_{2R}} udx$$

$$+c(\varepsilon)R^{\frac{N}{1-m(p-1)}}[t^{\frac{1}{1-m(p-1)}}$$

$$+t^{\frac{1}{p-(m+1)(p-1)}} + t^{\frac{p}{p-m(p-1)}}]. \tag{44}$$

From (42), (44), noticing that $R \geq 1$, we have

$$\sup_{0<\tau<t}\int_{B_R} u(x,\tau)dx \leq c\int_{B_{2R}} u(x,0)dx$$

$$+cR^{\frac{N}{1-m(p-1)}}[t^{\frac{1}{p-(m+1)(p-1)}} + t^{\frac{1}{1-m(p-1)}} + t^{\frac{p}{p-m(p-1)}}].$$

We know that (33) is true.

By Lemma 6, Lemma 7, it is easy to deduce the following Lemma 8, Lemma 9.

Lemma 8 *Let $1 < p < 2$, $m(p-1) < 1$ and $R > R_0 \geq 1$, $f(x,t) \leq 0$. Then the solution u of Cauchy problem (1)-(2) satisfies*

$$\sup_{x\in B_R} |u(x,t)| \leq c(N,p,R_0)[t^{-\frac{N}{k}}\left(\int_{B_{4R}} u_0dx\right)^{\frac{p}{k}}$$

$$+ R^{\frac{Np}{k[1-m(p-1)]}}t^{-\frac{N}{k}}[t^{\frac{1}{p-(m+1)(p-1)}}$$

$$+ t^{\frac{1}{1-m(p-1)}} + t^{\frac{p}{p-m(p-1)}}].$$

Lemma 9 *Let $1 < p < 2$, $m(p-1) < 1$. There exists a constant c such that*

$$\int_0^t\int_{B_{2R}} |\nabla u^m|^{p-1}dxd\tau$$

$$\leq cR^{N(1-m(p-1))}t\left(\sup_{0<\tau<t}\int_{B_{2R}} udxd\tau\right)^{m(p-1)}$$

$$+cR^{\frac{N(m-mp+1)}{p}}t^{\frac{1}{p}}\left(\sup_{0<\tau<t}\int_{B_{2R}} udxd\tau\right)^{\frac{(m+1)(p-1)}{p}}$$

$$+ct^{\frac{1}{p}}R^{\frac{N}{p}}\left(\sup_{0<\tau<t}\int_{B_{2R}} udx\right)^{[1-m(p-1)]\frac{p-1}{p}+m(p-1)}. \tag{45}$$

Lemma 10 *Let $\frac{N(m+1)}{Nm+1} < p < 2$, $1 < m < \frac{1}{p-1}$.*

Then the solution u of Cauchy problem of equation (1) with the initial value (24) satisfies

$$\nabla u^m \in L^p_{loc}(S_T).$$

It implies that

$$|\nabla u^m|^{p-2}\nabla u^m \in L^{\frac{p}{p-1}}_{loc}(S_T). \quad (46)$$

Proof Let η be the smooth cut function in $B_{4R}\times(\frac{t_1}{8}, t)$ which satisfies $\eta = 1$ on $B_{2R} \times (\frac{t_1}{4}, t)$ and

$$|\nabla\eta| \leq \frac{4}{R}, \quad 0 \leq \eta_t \leq \frac{4}{t_1}.$$

Multiplying (1) by $u^m\eta^p$ and integrating by parts,

$$\int_{\frac{t_1}{4}}^t \int_{B_{4R}} \mathrm{div}\vec{a}(u,\nabla u)u^m\eta^p dxd\tau$$

$$= -\int_{\frac{t_1}{4}}^t \int_{B_{2R}} \vec{a}(u,\nabla u)M_{29}dxd\tau$$

$$m\int_{\frac{t_1}{4}}^t \int_{B_{4R}} \vec{a}(u,\nabla u)u^{m-1}\eta^p\nabla udxd\tau$$

$$\geq m\nu_0\int_{\frac{t_1}{4}}^t \int_{B_{4R}} \eta^p u^{m-1}u^{(m-1)(p-1)}|\nabla u|^p dxd\tau$$

$$-m\mu_0\int_{\frac{t_1}{4}}^t \int_{B_{2R}} u^{m-1}\eta^p(u^{m(p-1)+1}+1)dxd\tau$$

$$\geq m^{1-p}\nu_0\int_{\frac{t_1}{4}}^t \int_{B_{4R}} \eta^p|\nabla u^m|^p dxd\tau$$

$$-m\mu_0\int_{\frac{t_1}{4}}^t \int_{B_{2R}} u^{m-1}\eta^p(u^{m(p-1)+1}+1)dxd\tau,$$

$$\geq m^{1-p}\nu_0\int_{\frac{t_1}{4}}^t \int_{B_{4R}} \eta^p|\nabla u^m|^p dxd\tau$$

$$-2m\mu_0\int_{\frac{t_1}{4}}^t \int_{B_{2R}} u^{mp}dxd\tau,$$

where $M_{29} = (mu^{m-1}\eta^p\nabla u + p\eta^{p-1}u^m\nabla\eta)$.

Using Young inequality,

$$p\int_{\frac{t_1}{4}}^t \int_{B_{4R}} u^m\eta^{p-1}\vec{a}(u,\nabla u)\cdot\nabla\eta dxd\tau$$

$$\leq \int_{\frac{t_1}{4}}^t \int_{B_{2R}} u^m\eta^{p-1}M_{30}dxd\tau$$

$$= \frac{p\mu_1}{m^{p-1}}\int_{\frac{t_1}{4}}^t \int_{B_{4R}} u^m\eta^{p-1}|\nabla\eta||\nabla u^m|^{p-1}dxd\tau$$

$$+p\int_{\frac{t_1}{4}}^t \int_{B_{4R}} u^{m+\frac{m(p-1)}{p}}\eta^{p-1}|\nabla\eta|dxd\tau$$

$$= \varepsilon\int_{\frac{t_1}{4}}^t \int_{B_{4R}} \eta^p|\nabla u^m|^p dxd\tau$$

$$+c(\varepsilon)\int_{\frac{t_1}{4}}^t \int_{B_{4R}} u^{mp}|\nabla\eta|^p dxd\tau$$

$$+p\int_{\frac{t_1}{4}}^t \int_{B_{4R}} u^{m+\frac{m(p-1)}{p}}\eta^{p-1}|\nabla\eta|dxd\tau$$

$$\leq \varepsilon\int_{\frac{t_1}{4}}^t \int_{B_{4R}} \eta^p|\nabla u^m|^p dxd\tau$$

$$+\frac{c(\varepsilon)}{R^p}\int_{\frac{t_1}{4}}^t \int_{B_{4R}} u^{mp}dxd\tau$$

$$+\frac{c}{R}\int_{\frac{t_1}{4}}^t \int_{B_{4R}} u^{m+1}dxd\tau,$$

where

$$M_{30} = (\mu_1 u^{(m-1)(p-1)}|\nabla u|^{p-1} + u^{\frac{m(p-1)}{p}}).$$

By the above inequalities, we have

$$\int_{\frac{t_1}{4}}^t \int_{B_{2R}} |\nabla u^m|^p\, dxd\tau$$

$$\leq c(\frac{1}{R^p}+1)\int_{\frac{t_1}{8}}^t \int_{B_{4R}} u^{mp}dxd\tau$$

$$+c(\frac{1}{t_1}+\frac{1}{R})\int_{\frac{t_1}{8}}^t \int_{B_{4R}} u^{m+1}dxd\tau. \quad (47)$$

By Lemma 8, for $R > 1$,

$$c(\frac{1}{R^p}+1)\int_{\frac{t_1}{8}}^t \int_{B_{4R}} u^{mp}dxd\tau$$

$$\leq c\sup_{x\in B_{4R}} |u(x,t)|^{mp}tR^N,$$

$$\leq ctR^N[t^{-\frac{N}{k}}\left(\int_{B_{4R}} u_0dx\right)^{\frac{p}{k}}$$

$$+R^{\frac{Np}{k(1-mp+m)}}t^{\frac{1}{1-mp+m}}]^{mp}$$

$$\leq ct^{1-\frac{Nmp}{k}}\left(\int_{B_{4R}} u_0dx\right)^{\frac{mp^2}{k}}$$

$$+R^{\frac{mNp^2}{k(1-mp+m)}+N}t^{\frac{mp}{1-mp+m}}. \quad (48)$$

$$\int_{\frac{t}{8}}^t \int_{B_{4R}} u^{m+1}dxd\tau$$

$$\le c[t^{-\frac{N}{k}}\left(\int_{B_{4R}} u_0 dx\right)^{\frac{p}{k}} + R^{\frac{Np}{k(1-mp+m)}} t^{\frac{1}{1-mp+m}}]^{m+1} t R^N$$

$$\le c[t^{1-\frac{N(m+1)}{k}}\left(\int_{B_{4R}} u_0 dx\right)^{\frac{p(m+1)}{k}} + R^{\frac{N(m+1)p}{k(1-mp+m)}} t^{\frac{1+m}{1-mp+m}+1}] R^N. \quad (49)$$

Substituting (48), (49) into (47), we get the lemma.

3 Proof of Theorem 4

In this section, we will prove Theorem 4.

Lemma 11 *If $1 < p \le \frac{(m+1)N}{mN+1}$, $m(p-1) < 1$, $f(x,t) \le 0$ and the constant α satisfies*

$$\frac{1}{m} < \alpha < \max\{1 + \frac{2}{m} - p, 1 - m(p-1)\}, \quad (50)$$

then the solution of Cauchy problem of equation (1) with the initial value (22) has the following properties.

(1) *For any given $R > 0$,*

$$\int_0^T \int_{B_R} \frac{u^{m(\alpha-1)}}{(1+u^{m\alpha})^2} |\nabla u^m| \, dxdt \le c, \quad (51)$$

(2)

$$\int_0^T \int_{B_R} u^{m(p-1)+\frac{p}{N}-\alpha} dxdt < c. \quad (52)$$

Proof: (1) By Definition 1, for any

$$\psi(x) \in C_0^\infty(\mathbb{R}^N), \varepsilon \in (0,T),$$

we have

$$\int_{\mathbb{R}^N} \int_0^{u(x,T)} \frac{s^{m\alpha}}{1+s^{m\alpha}} ds \psi(x)^p dx + \int_\varepsilon^T \int_{\mathbb{R}^N} \frac{\alpha u^{m(\alpha-1)}}{(1+u^{m\alpha})^2} \vec{a}(u,\nabla u) \cdot \nabla u^m \psi^p dxdt$$

$$= -p \int_\varepsilon^T \int_{\mathbb{R}^N} \frac{u^{m\alpha}}{1+u^{m\alpha}} \vec{a}(u,\nabla u) \cdot \nabla\psi \psi^{p-1} dxdt + \int_{\mathbb{R}^N} \int_0^{u(x,\varepsilon)} \frac{s^{m\alpha}}{1+s^{m\alpha}} ds \psi(x)^p dx + \int_\varepsilon^T \int_{\mathbb{R}^N} \frac{u^{m\alpha}}{1+u^{m\alpha}} f(x,t) \psi(x)^p dxdt$$

$$\le -p \int_\varepsilon^T \int_{\mathbb{R}^N} \frac{u^{m\alpha}}{1+u^{m\alpha}} \vec{a}(u,\nabla u) \cdot \nabla\psi \psi^{p-1} dxdt + \int_{\mathbb{R}^N} \int_0^{u(x,\varepsilon)} \frac{s^{m\alpha}}{1+s^{m\alpha}} ds \psi(x)^p dx. \quad (53)$$

Noticing that

$$\int_\varepsilon^T \int_{\mathbb{R}^N} \frac{\alpha u^{m(\alpha-1)}}{(1+u^{m\alpha})^2} \vec{a}(u,\nabla u) \cdot \nabla u^m \psi^p dxdt \ge \alpha m^{1-p} \nu_0 \int_\varepsilon^T \int_{\mathbb{R}^N} M_{31} |\nabla u^m|^p \psi^p(x) dxdt - m\alpha\mu_0 \int_\varepsilon^T \int_{\mathbb{R}^N} u^{m-1}(u^{m(p-1)+1}+1) M_{31} dxdt,$$

where $M_{31} = \frac{u^{m(\alpha-1)}}{(1+u^{m\alpha})^2}$.

$$-p \int_\varepsilon^T \int_{\mathbb{R}^N} \frac{u^{m\alpha}}{1+u^{m\alpha}} \vec{a}(u,\nabla u) \cdot \nabla\psi \psi^{p-1} dxdt \ge -pm^{1-p}\mu_1 \int_0^T \int_{\mathbb{R}^N} \frac{u^{m\alpha}}{1+u^{m\alpha}} M_{32} dxdt - p \int_0^T \int_{\mathbb{R}^N} \frac{u^{m\alpha}}{1+u^{m\alpha}} u^{\frac{(m-1)(p-1)}{p}} |\nabla\psi| \psi^{p-1} dxdt$$

where $M_{32} = |\nabla u^m|^{p-1} |\nabla\psi| \psi^{p-1}$.

$$\int_0^T \int_{\mathbb{R}^N} \frac{u^{m\alpha}}{1+u^{m\alpha}} |\nabla u^m|^{p-1} |\nabla\psi| \psi^{p-1} dxdt \le \epsilon \int_0^T \int_{\mathbb{R}^N} \frac{u^{m(\alpha-1)}}{(1+u^{m\alpha})^2} |\nabla u^m|^p \psi^p dxdt + c(\epsilon) \int_0^T \int_{\mathbb{R}^N} \frac{u^{m(p-1+\alpha)}}{(1+u^{m\alpha})^{2-p}} |\nabla\psi|^p dxdt,$$

$$p \int_0^T \int_{\mathbb{R}^N} \frac{u^{m\alpha}}{1+u^{m\alpha}} u^{\frac{(m-1)(p-1)}{p}} |\nabla\psi| \psi^{p-1} dxdt \le p \int_0^T \int_{\mathbb{R}^N} \frac{u^{m(p+\alpha-1)}}{(1+u^{m\alpha})^{2-p}} |\nabla\psi|^p dxdt + p(p-1) \int_0^T \int_{\mathbb{R}^N} \frac{u^{m\alpha-1}}{(1+u^{m\alpha})^p} \psi^p dxdt.$$

As $\varepsilon \to 0$,

$$\int_{\mathbb{R}^N} \int_0^{u(x,\varepsilon)} \frac{s^{m\alpha}}{1+s^{m\alpha}} ds \psi^p(x) dx \le \int_{\mathbb{R}^N} u(x,\varepsilon) \psi^p(x) dx \to \int_{R^N} \psi^p(x) d\mu,$$

then, we have

$$\sup_{0<t<T}\int_{\mathbb{R}^N} u(x,t)\psi(x)^p dx$$
$$+\int_0^T\int_{\mathbb{R}^N}\frac{u^{m(\alpha-1)}}{(1+u^{m\alpha})^2}\left|\nabla u^m\right|^p\psi^p dxdt$$
$$=c[1+\int_0^T\int_{\mathbb{R}^N}\frac{u^{m(p-1+\alpha)}}{(1+u^{m\alpha})^{2-p}}\left|\nabla\psi\right|^p dxdt$$
$$+\int_0^T\int_{\mathbb{R}^N}\frac{u^{m\alpha-1}}{(1+u^{m\alpha})^p}\psi^p dxdt$$
$$+\int_0^T\int_{\mathbb{R}^N}\frac{u^{m\alpha-1}(u^{m(p-1)}+1)}{(1+u^{m\alpha})^2}\psi^p dxdt\quad]$$
$$\leq c[1+\int_0^T\int_{\mathbb{R}^N}\frac{u^{m(p-1+\alpha)}}{(1+u^{m\alpha})^{2-p}}\left|\nabla\psi\right|^p dxdt$$
$$+\int_0^T\int_{\mathbb{R}^N}u^{m\alpha-1}\psi^p dxdt$$
$$+\int_0^T\int_{\mathbb{R}^N}u^{m\alpha-1}(u^{m(p-1)}+1)\psi^p dxdt]. \quad (54)$$

By (50), using Young inequality, we have

$$\sup_{0<t<T}\int_{\mathbb{R}^N} u(x,t)\psi(x)^p dx$$
$$\leq c\left[1+\int_0^T\int_{\mathbb{R}^N}\frac{u^{m(p-1+\alpha)}}{(1+u^{m\alpha})^{2-p}}\left|\nabla\psi\right|^p dxdt\right],$$

using Young inequality again,

$$\sup_{0<t<T}\int_{\mathbb{R}^N} u(x,t)\psi(x)^p dx$$
$$\leq c+\epsilon\int_0^T\int_{\mathbb{R}^N}u\psi^p dxdt$$
$$+c(\epsilon)\int_0^T\int_{\mathbb{R}^N}\left(\frac{|\nabla\psi|}{\psi^{1-m(p-1)-\alpha}}\right)^{\frac{p}{1-m(p-1)-\alpha}}dxdt$$
$$\leq c+\epsilon\sup_{0<t<T}\int_{\mathbb{R}^N}u(x,t)\psi(x)^p dx$$
$$+c(\epsilon)\int_0^T\int_{\mathbb{R}^N}\left(\frac{|\nabla\psi|}{\psi^{1-m(p-1)-\alpha}}\right)^{\frac{p}{1-m(p-1)-\alpha}}dxdt.$$

Now, choosing $\alpha<1-m(p-1)$ such that

$$\frac{p}{1-m(p-1)-\alpha}\geq K\geq N+1,$$

where K is large enough such that

$$1-m(p-1)-\frac{p}{K}>0.$$

Let $X\in C_0^\infty(B_{2R})$,$X\mid_{B_R}=1$,$h\geq 2-m(p-1)-\alpha$,$\psi=X^h$. By the above formula, we have

$$\sup_{0<t<T}\int_{\mathbb{R}^N}u(x,t)\psi^p(x)dx\leq c. \quad (55)$$

Combining (54) with (55), we get (51), i.e.

$$\int_0^T\int_{B_R}\frac{u^{m(\alpha-1)}}{(1+u^{m\alpha})^2}\left|\nabla u^m\right|dxdt\leq c.$$

Now, let

$$w=u^{\frac{m(p-1-\alpha)}{p}}.$$

By Sobolev inequality,

$$\left(\int_{\mathbb{R}^N}\psi^p w^r dx\right)^{\frac{1}{\gamma}}$$
$$\leq c(\int_{\mathbb{R}^N}\left|\nabla\psi w\right|^p dx)^{\frac{\theta}{p}}$$
$$\left(\int_{B_{2R}}w^{\frac{p}{p-1-\alpha}}dx\right)^{\frac{(1-\theta)(p-1-\alpha)}{p}}, \quad (56)$$

where,

$$\theta=(\frac{p-1-\alpha}{p}-\frac{1}{\gamma})(\frac{1}{N}-\frac{1}{p}+\frac{p-1-\alpha}{p})^{-1}.$$

For $\gamma=\frac{p(p-1+\frac{p}{N}-\alpha)}{p-1-\alpha}$, by (56), we have

$$\int_0^T\int_{\mathbb{R}^N}\psi^p w^r dxdt$$
$$\leq\int_0^T\int_{\mathbb{R}^N}\left|\nabla(\psi w)\right|^p dxdt$$
$$\sup_{0<t<T}\left(\int_{B_{2R}}w^{\frac{p}{p-1-\alpha}}dx\right)^{\frac{(\gamma-\theta)(p-1-\alpha)}{p}}.$$

Hence, by (55), (51), we have

$$\int_0^T\int_{\mathbb{R}^N}\psi^p u^{m(p-1)+\frac{p}{N}-\alpha}dxdt$$
$$\leq c(1+\int_0^T\int_{R^N}\left|\nabla\psi\right|^p u^{m(p-1-\alpha)}dxdt).$$

We can prove (52) in a similar way.

Lemma 12 *If* $1<p\leq\frac{(m+1)N}{mN+1}$, *then the solution of Cauchy problem (1)-(22) satisfies*

$$\iint_{S_T}[u\xi_t-\vec{a}(u,\nabla u)\cdot\nabla\xi+f(x,t)\xi]dxdt=0, \quad (57)$$

where $\xi \in C_0^\infty(R^N \times (-T,T))$.

Proof Let

$$\psi_k(x,t) = \eta_k(x,t) = \eta_k(|x|^2)\xi(x,t),$$

where $\xi \in C_0^\infty(R^N \times (-T,T))$,$\eta \in c^\infty(R)$; $\eta(s) = 1$ when $s \geq 2$; $\eta(s) = 0$ when $s \leq 1$.

Let $\eta_k(x) = \eta(k|x|^2)$. By the definition of weak solution,

$$\int_0^T \int_{\mathbb{R}^N} [u(\xi\eta_k)_t - \vec{a}(u,\nabla u)\cdot\nabla(\xi\eta_k)$$
$$+ f(x,t)\xi\eta_k]dxdt = 0.$$

To prove the lemma, it is enough to prove that

$$\lim_{k\to\infty} \iint_{S_T} \vec{a}(u,\nabla u)\cdot\nabla\eta_k\xi dxdt = 0. \quad (58)$$

Denoting $D_k = \{x : k^{-1} < |x|^2 < 2k^{-1}\}$, clearly mes$D_k \leq ck^{\frac{-N}{2}}$. Hence, by Hölder inequality and Lemma 11, we have

$$k^{\frac{1}{2}} \int_0^T \int_{D_k} |\nabla u^m|^{p-1} dxdt$$

$$\leq k^{\frac{1}{2}} \left(\int_0^T \int_{D_k} \frac{u^{m(\alpha-1)}}{(1+u^{m\alpha})^2} |\nabla u^m|^p dxdt\right)^{\frac{p-1}{p}}$$

$$\times \left(\int_0^T \int_{D_k} (1+u^{m\alpha})^{2(p-1)} u^{m(p-1)(1-\alpha)} dxdt\right)^{\frac{1}{p}}$$

$$\leq ck^{\frac{1}{2}} \left(\int_0^T \int_{D_k} u_1^{m(p-1)(1+\alpha)} dxdt\right)^{\frac{1}{p}}$$

$$\leq c_1 \left(\int_0^T \int_{D_k} u_1^{m(p-1)+\frac{p}{N}-\alpha} dxdt\right)^{\frac{m(p-1)(1+\alpha)}{(m(p-1)+\frac{p}{N}-\alpha)p}}$$

$$k^{\frac{1}{2}-\frac{p-N\alpha-\alpha Nm(p-1)}{2p(m(p-1)+\frac{p}{N}-\alpha)}}, \quad (59)$$

where $u_1 = \max\{u, 1\}$. Since $1 < p \leq \frac{(m+1)N}{mN+1}$, if $p < \frac{(m+1)N}{mN+1}$, we have

$$\frac{1}{2} - \frac{p - N\alpha - \alpha Nm(p-1)}{2p(m(p-1) + \frac{p}{N} - \alpha)} < 0.$$

Thus the right hand side of the inequality (59) tends to the zero as $k \to \infty$. At the same time,

$$|\iint_{S_T} \vec{a}(u,\nabla u)\nabla\eta_k\xi dxdt|$$

$$\leq \iint_{S_T} [\mu_1 u^{(m-1)(p-1)}|\nabla u|^{p-1}$$
$$+\mu(u)u^{\frac{(m-1)(p-1)}{p}}]|\nabla\eta_k|\xi dxdt, \quad (60)$$

its right hand side also tends to the zero as $k \to \infty$. By (7), one knows that (58) is true.

If $p = \frac{(m+1)N}{mN+1}$, (58) is obtained by (41), (43). Thus we get lemma 12.

The proof Theorem 4 Suppose to the contrary that Cauchy problem of equation (1) with the initial value (22) has a solution. Then by lemma 12, we have

$$\iint_{S_T} [u\xi_t - \vec{a}(u,\nabla u)\cdot\nabla\xi + f\xi]dxdt = 0, \quad (61)$$

where $\xi \in C_0^\infty(\mathbb{R}^N \times (-T,T))$.

Let $\eta_h(t) = 1 - \int_{-\infty}^{t-\tau-2h} j_h(s)ds$, where

$$j_h \in C_0^1(-2h, 2h), j_h \geq 0, \int_R j_h(s)\, ds = 1,$$
$$\tau \in (0,T),\ 2h < T - \tau.$$

Clearly, $\eta_h \in C^\infty(R)$. If $t < \tau + h,\ 0 \leq \eta_h \leq 1$; if $t < T$, $\lim_{h\to 0}\eta_h(t) = 0$.

For any $\forall\chi \in C_0^\infty(R^N)$, we choose $\xi = \chi(x)\eta_h(t)$ in (57), then

$$-\int_0^T \int_{\mathbb{R}^N} j_h(t-\tau-2h)u\chi dxdt$$
$$-\int_0^T \int_{\mathbb{R}^N} [\vec{a}(u,\nabla u)\cdot\nabla\chi\eta_h - f\xi]dxdt = 0.$$

Let $h \to 0^+$. We have

$$\int_{\mathbb{R}^N} u(x,\tau)\chi(x)dx$$
$$= -\int_0^\tau \int_{\mathbb{R}^N} [\vec{a}(u,\nabla u)\cdot\nabla\chi\eta_h - f\xi]dxds,$$

which implies that, for $\forall\chi \in C_0^\infty(\mathbb{R}^N)$,

$$\lim_{\tau\to 0} \int_{\mathbb{R}^N} u(x,\tau)\chi(x)dx = 0.$$

it contradicts (22). So, there is not the solution for the Cauchy problem of equation (1) with the initial value (22).

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