Statistical Convergence Applied to Korovkin-type Approximation Theory

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Abstract: We present two general sequences of positive linear operators. The first is introduced by using a class of dependent random variables, and the second is a mixture between two linear operators of discrete type. Our goal is to study their statistical convergence to the approximated function. This type of convergence can replace classical results provided by Bohman-Korovkin theorem. A particular case is delivered.

Key–Words: Positive linear operator, Bohman-Korovkin theorem, statistical convergence, Bernstein operator, Baskakov operator.

1 Introduction

Since the fifties, positive linear operators (PLOs) play an important role in approximating a real valued function. Korovkin-type theorems furnish useful tools in order to establish whether a sequence of PLOs is an approximation process this meaning that it converges strongly to the identity operator. The genuine Bohman-Korovkin's theorem asserts: if the positive linear operators L_n, $n \in \mathbb{N}$, map $C([a,b])$ into itself such that $(L_n e_k)_{n\geq 1}$ converges to e_k uniformly on $[a,b]$, $k \in \{0,1,2\}$, for the test functions

$$e_0(x) = 1,\ e_1(x) = x,\ e_2(x) = x^2,$$

then $(L_n f)_{n\geq 1}$ converges to f uniformly on $[a,b]$ for each $f \in C([a,b])$. Here $C([a,b])$ is the space of all real-valued and continuous functions defined on the interval $[a,b]$.

Lately, new research directions targeting this area have been developed. One of them is given by replacing the uniform convergence by statistical convergence. The remembering of this concept will be made in the next section.

In this note we focus on the presentation of some linear positive processes and on their study in terms of statistical convergence.

2 Preliminaries

Following H. Fast [4], a sequence of real numbers $(x_n)_{n\geq 1}$ is said to be statistical convergent to a real number L, if, for every $\varepsilon > 0$,

$$\delta(\{n \in \mathbb{N} :\ |x_n - L| \geq \varepsilon\}) = 0,$$

where

$$\delta(S) = \lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} \chi_S(j)$$

is the density of the set $S \subseteq \mathbb{N}$ and χ_S stands for the characteristic function on S. For $\delta(S)$ also uses the term *asymptotic density*. We use the notation $st - \lim\limits_{n\to\infty} x_n = L$.

The following characterization of statistical convergence (cf. [6, *Lemma 1.1*]) takes place. A sequence of real numbers $(x_n)_{n\geq 1}$ converges statistically to $L \in \mathbb{R}$ if and only if there is a set of indices $M = \{n_j :\ n_j < n_{j+1},\ j \in \mathbb{N}\} \subset \mathbb{N}$ with the property

$$\delta(M) = 1 \quad \text{and} \quad \lim_{j\to\infty} x_{n_j} = L.$$

The main idea of statistical convergence of a sequence is that the majority, in a certain sense, of its elements converges and we are not interested in what happens to the remaining elements. Actually, the sequences that come from the real life sources are not convergent in the strictly mathematical sense. The advantage of replacing the uniform convergence by statistical convergence consists in the fact that the second convergence is efficient in summing divergent sequences which may have unbounded subsequences.

In Approximation Theory by linear positive operators, the statistical convergence has been examined for the first time in 2002 by A.D. Gadjiev and Cihan Orhan. Bohman-Korovkin criterion via statistical convergence will be read as follows.

Theorem 1 ([5, *Theorem 1*]). *If the sequence of positive linear operators $L_n : C([a,b]) \to B([a,b])$ satisfies the conditions*

$$st - \lim_{n\to\infty} \|L_n e_j - e_j\| = 0,\ j \in \{0,1,2\},$$

then, for any function $f \in C([a,b])$, we have

$$st - \lim_{n\to\infty} \|L_n f - f\| = 0.$$

In the above $B([a,b])$ stands for the real valued functions bounded on the domain $[a,b]$. We get $C([a,b]) \le B([a,b])$ and $B([a,b])$ is endowed with the uniform norm (or the sup-norm) $\|\cdot\|$, where

$$\|f\| = \sup_{f\in B([a,b])} |f(x)|.$$

3 Two classes of operators

We follows closely the construction of the operators given at [2]. Let J be given interval of the real line. Since an affine substitution maps (a,b), $-\infty \le a < b \le \infty$, onto $(0,1)$, $\mathbb{R}_+^* = (0,\infty)$ or $\mathbb{R}$, it is enough to consider these intervals as being $int(J)$.

Let I_n, $n \in \mathbb{N}$, be the sets of indices such that $I_n \subset I_{n+1}$ holds. We consider two variants. I_n is finite, thus a model can be chosen $\{0,1,\dots,s_n\}$ or $\{-s_n,\dots,0,\dots,s_n\}$. I_n is infinite, thus our model can be considered $\mathbb{N}_0 = \{0\}\cup\mathbb{N}$ or $\mathbb{Z}$. For each integer $n \ge 1$ we consider a net on J namely $(kn^{-\beta})_{k\in I_n}$, where $\beta > 0$ is a fixed number.

We start from a sequence $(L_n)_{n\ge1}$ of linear positive operators of discrete type given by the formula

$$(L_n f)(x) = \sum_{k\in I_n} a_{n,k}(x) f\left(\frac{k}{n^\beta}\right),\ x \in J, \qquad (1)$$

where $a_{n,k} \in C(J)$ and $a_{n,k} \ge 0$ for every (n,k) belonging to $\mathbb{N}\times I_n$. Here F belongs to a vectorial subspace of $\mathbb{R}^J$ such that the operators L_n, $n \ge 1$, are well defined. Regarding the above operators we require the following conditions to be fulfilled for each $n \in \mathbb{N}$

$$L_n e_0 = e_0,\ L_n e_1 = e_1, \qquad (2)$$

$$L_n e_2 = e_2 + \varphi_n, \qquad (3)$$

where $\varphi_n \in C(J)$. Operators satisfying relations (2) are called of Markov type. Further on, let X be a non constant real random variable on a probability space $(\Omega,\mathcal{F},P)$. Denoting by ψ its probability density function, we assume that $\psi \in L_2(\mathbb{R})$ and ψ has a compact support included in J. This implies $\psi \in L_1(\mathbb{R})$. Also, ψ being a density function, one has $\psi \ge 0$ and

$$\|\psi\|_1 = \int_{\mathbb{R}} \psi(t)dt = 1.$$

We set

$$E(X) = e,\ Var(X) = \sigma^2,$$

the expectation and the variance of X, respectively. Starting from X we generate the random variables $X_{n,k}$ defined by

$$X_{n,k} = \frac{1}{n^\beta}(X + k - e),\ (n,k) \in \mathbb{N}\times I_n. \qquad (4)$$

Since X is non-constant, by examining (4) we deduce that for any $(k_1,k_2) \in I_n \times I_n$, the variables X_{n,k_1}, X_{n,k_2} are not independent. All these variables represent scaled versions of the same variable X, they being obtained from it by contractions $(n^{-\beta},\ n\in\mathbb{N})$ and by translations $((k-e)n^{-\beta},\ k \in I_n)$.

We get

$$E(X_{n,k}) = \frac{k}{n^\beta} \quad \text{and} \quad Var(X) = \frac{\sigma^2}{n^{2\beta}}. \qquad (5)$$

The expectations of $X_{n,k}$, $k \in I_n$, represent exactly the mesh of L_n operator.

Letting

$$S := \{f : \mathbb{R}\to\mathbb{R},\ E(|f\circ X_{n,k}|) < \infty, \forall(n,k)\in\mathbb{N}\times I_n\},$$

we introduce the operators $\Lambda_n : S \to C(J)$, $n \in \mathbb{N}$, as follows

$$\Lambda_n f = \sum_{k\in I_n} a_{n,k} E(f\circ X_{n,k}) = \sum_{k\in I_n} a_{n,k} \int_\Omega f\circ X_{n,k} dP,$$

this meaning

$$(\Lambda_n f)(x) = n^\beta \sum_{k\in I_n} a_{n,k}(x) \int_{\mathbb{R}} f(t)\psi(n^\beta t - k + e)dt, \qquad (6)$$

$x \in J$.

It is easy to see that Λ_n operators are linear and positive.

To present the next class of operators we return to relation (1). The operators L_n, $n \ge 1$, are fully determined if we specify the interval J, the set of indices I_n, the system of nodes considered and the functions $a_{n,k}$, $(n,k) \in \mathbb{N}\times I_n$. This time we consider a completely arbitrary network nodes $(x_{n,k})$. Also, in relation (3) we assume that

$$\varphi_n(x) = \frac{\theta_n(x)}{u(n)},\ \theta_n \in C(J)$$

and

$$u(n) = \mathcal{O}(n^\alpha),\ n \to \infty,$$

for some constant $\alpha > 0$. In short this information will be write as follows

$$L_n : \langle J, I_n, x_{n,k}, a_{n,k}, \theta_n, u\rangle,\ (n,k) \in \mathbb{N} \times I_n.$$

Let be two sequences of this type, namely

$$L_n^{(1)} : \langle [0,1], I_n, x_{n,k}, a_{n,k}^{(1)}, \theta_{1,n}, u_1\rangle,\ (n,k) \in \mathbb{N}\times I_n,$$

$$L_n^{(2)} : \langle [0,\infty), J_n, x_{n,k}, a_{n,k}^{(2)}, \theta_{2,n}, u_2\rangle,\ (n,k) \in \mathbb{N}\times J_n,$$

such that $0 \in I_n \cap J_n$ and $x_{n,0} = 0$. Regarding these sequences we impose the following admissible condition to be satisfied: for any $n \in \mathbb{N}$, a function $\widetilde{\theta}_n \in C([0,\infty))$ exists such that

$$x_{n,p}\theta_{2,p}(x) = u_2(p)\frac{\widetilde{\theta}_n(x)}{u_1(n)},\ p \in I_n,\ x \geq 0. \quad (7)$$

If this condition is fulfilled then the pair $(L_n^{(1)}, L_n^{(2)})$ forms a compatible couple of approximation processes, see [1].

Finally we consider a function $\lambda \in C([0,\infty))$ such that $0 \leq \lambda(x) \leq 1$ for every $x \geq 0$. The announced sequence of operators are defined as follows

$$(L_{n,\lambda}f)(x) = \sum_{p\in I_n}\sum_{k\in J_p} a_{n,p}^{(1)}(\lambda(x))a_{p,k}^{(2)}(x) \times f(x_{n,p}x_{p,k} + (1-x_{n,p})x), \quad (8)$$

$x \geq 0$, where f belongs to a space such that the operators are well defined.

A particular case can be obtained by choosing $L_n^{(1)} \equiv B_n$ and $L_n^{(2)} \equiv V_n$, Bernstein and Baskakov operator of n^{th} order, respectively. We recall

$$(B_nf)(x) = \sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}f\left(\frac{k}{n}\right),$$

$x \in [0,1]$,

$$(V_nf)(x) = \sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}f\left(\frac{k}{n}\right),$$

$x \geq 0$.

We identify $x_{n,k} = k/n$, $u_1(n) = n \in \mathbb{N}$, $u_2(p) = p \in \mathbb{N}_0$, $\theta_{2,p}(x) = x + x^2$, $x \geq 0$. Taking $\widetilde{\theta}_n(x) = x^2 + x$, $x \geq 0$. condition (7) is fulfilled. In this case the operators $L_{n,\lambda}$ have been introduced and studied by F. Altomare and E.M. Mangino [3]. We also mention that the operators defined by (8) are positive and linear.

4 Approximation properties

The main results will be read as follows.

Theorem 2 *Let Λ_n, $n \in \mathbb{N}$, be the operators given at (6) such that*

$$st - \lim_{n\to\infty} \|\varphi_n\|_K = 0. \quad (9)$$

For any function f continuous on a compact $K \subset J$, the following relation

$$st - \lim_{n\to\infty} \|\Lambda_n f - f\|_K = 0 \quad (10)$$

takes place, where the norm $\|\cdot\|_K$ is computed only for functions restricted to K.

Proof. At first step we estimate $\Lambda_n e_j$, $j \in \{0,1,2\}$.

$$\Lambda_n e_0 = \sum_{k\in I_n} a_{n,k} = L_n e_0 = e_0, \quad (11)$$

see (2).

$$\Lambda_n e_1 = \sum_{k\in I_n} a_{n,k}E(X_{n,k}) = \sum_{k\in I_n} a_{n,k}e_1\left(\frac{k}{n^\beta}\right) = L_ne_1 = e_1, \quad (12)$$

see (5) and (2).

$$\Lambda_n e_2 = \sum_{k\in I_n} a_{n,k}E(X_{n,k}^2) = \sum_{k\in I_n} a_{n,k}(Var(X_{n,k}) + E^2(X_{n,k})) = \frac{\sigma^2}{n^{2\beta}}L_ne_0 + L_ne_2 = e_2 + \varphi_n + \frac{\sigma^2}{n^{2\beta}}, \quad (13)$$

see (5) and (3).

Relying on what we have achieved in the previous step, we verify the accomplishment of conditions required by Theorem 1. For $j = 0$ and $j = 1$, it is clear $\|\Lambda_n e_j - e_j\|_K = 0$. Also, by using (13), we get

$$\|\Lambda_n e_2 - e_2\|_K = \left\|\varphi_n + \frac{\sigma^2}{n^{2\beta}}\right\|_K \leq \|\varphi_n\|_K + \frac{\sigma^2}{n^{2\beta}},$$

consequently $st - \lim_{n\to\infty} \|\Lambda_n e_2 - e_2\|_K = 0$. We used relation (9).

Since the requirements of Theorem 1 are satisfied, the identity (10) takes place and the proof is ended.

Theorem 3 *Let $L_{n,\lambda}$, $n \in \mathbb{N}$, be the operators given at (8), such that the functions $\widetilde{\theta}_n$ satisfy the property*

$$\exists\, C > 0,\ |\widetilde{\theta}_n(x)| \le C,\ x \ge 0,\ n \in \mathbb{N},$$

where C is independent of n. For any function f continuous on a compact $K \subset [0, \infty)$, the following relation

$$st - \lim_{n\to\infty} \|L_{n,\lambda} f - f\|_K = 0 \tag{14}$$

takes place.

Proof. By using [1, *Theorem 1*] the following identities hold

$$L_{n,\lambda} e_j = e_j,\ j \in \{0,1\},\ L_{n,\lambda} e_2 = e_2 + \frac{\lambda\widetilde{\theta}_n}{u_1(n)}.$$

These imply $\|L_{n,\lambda} e_j - e_j\|_K = 0$ for $j = 0$ and $j = 1$. Moreover,

$$\|L_{n,\lambda} e_2 - e_2\|_K = \frac{1}{|u_1(n)|}\|\lambda\|_K \|\widetilde{\theta}\|_K.$$

Since $0 \le \lambda \le 1$, $\widetilde{\theta}_n$, $n \in \mathbb{N}$, are equi-bounded and $u_1(n) = \mathcal{O}(n^\alpha)$, we have

$$st - \lim_{n\to\infty} \|L_{n,\lambda} e_2 - e_2\|_K = 0.$$

By applying Theorem 1, relation (14) follows.

Comments. At first glance, it is a discrepancy between the intricate construction of the operators and the new results presented. The described constructions have a higher expanse then the section which collects our new results. The explanation is as follows. The described constructions in Section 3 target very general classes of operators. These classes were investigated in [1], [2] from the point of view of approximation properties relating to Korovkin processes. Our new results aimed the statistical approach, research direction that was not even created when the mentioned classes have been defined.

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