Matrix Inequalities in the Theory of Mixed Quermassintegrals and the L_p-Brunn-Minkowski Theory

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Abstract: The Brunn-Minkowski theory is central to convex geometric analysis, and mixed quermassintegrals and mixed p-quermassintegrals play a very important role in this theory. During the past quarter of a century both duals and L_p extensions of this theory have been developed. It is the aim of this present work to continue the development of analogues, for positive definite symmetric matrices, of some of the fundamental notations, invariants, and inequalities of mixed quermassintegrals, mixed p-quermassintegrals and L_p Brunn-Minkowski theory.

Key–Words: Mixed determinant, Ordinary Quermassintegral, Mixed Quermassintegral, Mixed p-Quermassintegral Matrix L_p-sum, Minkowski inequality, L_p-Brunn Minkowski inequality.

1 Introduction

This paper establishes important matrix inequalities that are analogous to some fundamental inequalities in convex geometry. The two fundamental inequalities are the Minkowski and Brunn-Minkowski inequalities. The notions of mixed determinants, matrix L_p-sum, and symmetric matrix polynomials for positive definite symmetric matrices are quoted and used to establish these inequalities.

2 Mixed Determinants

Definition 2.1 (Mixed Determinant [1]) *If $A_1,\ldots,A_r$ are $n \times n$ positive definite symmetric matrices and $\lambda_1,\ldots,\lambda_r$ are nonnegative real numbers, then of fundamental importance is the fact that the determinant of $\lambda_1 A_1 + \cdot + \lambda_r A_r$ is a homogeneous polynomial of degree n in $\lambda_1,\ldots,\lambda_r$ given by*

$$D(\lambda_1 A_1 + \cdots + \lambda_r A_r) = \sum \lambda_{i_1},\ldots,\lambda_{i_n} D(A_{i_1},\ldots,A_{i_n})$$

where the sum is taken over all n-tuples of positive integers $(i_1,\ldots,i_n)$ whose entries do not exceed r. The coefficient $D(A_{i_1},\ldots,A_{i_n})$ is the mixed determinant of the matrices $A_{i_1},\ldots,A_{i_n}$ and is uniquely determined by the requirement that it be symmetric in its arguments.

The mixed determinant $D(A_1, A_2,\ldots,A_n)$ of $n \times n$ matrices $A_1, A_2,\ldots,A_n$ can be regarded as the arthimetic mean of the determinants of all possible matrices that have exactly one row from the corresponding rows of $A_1, A_2,\ldots,A_n$ (Pranayanuntana [1]).

Properties of Mixed Determinants
The following properties for mixed determinants are well known (see for example Pranayanuntana [1]). For $n \times n$ matrices $A_1,\ldots,A_n, B$, B' and scalars $\lambda_1,\ldots,\lambda_n$:

(2.1) $D(A_1,\ldots,A_n) = D(A_{\pi(1)},\ldots,A_{\pi(n)})$ where π is a permutation on $\{1, 2,\ldots,n\}$.

(2.2) $D(A_1,\ldots,A_{n-1}, B + B') = D(A_1,\ldots,A_{n-1}, B) + D(A_1,\ldots,A_{n-1}, B')$

(2.3) $D(\lambda_1 A_1,\ldots,\lambda_n A_n) = \lambda_1 \cdots \lambda_n D(A_1,\ldots, A_n)$

Notation: $\forall A, B \in M_n^{s,+}$, where $M_n^{s,+}$ is the space of $n \times n$, symmetric, positive, definite matrices, and $0 \le i \le n$, we let

$$D(A, n-i; B, i) = D(\underbrace{A,\ldots,A}_{n-i \text{ copies}}, \underbrace{B,\ldots,B}_{i \text{ copies}})$$

for notational simplification purposes.

We now state an important and useful theorem for our work.

Theorem 2.2 (Aleksandrov [4, 5]) *Let $A_1, \ldots, A_n$ be real symmetric $n \times n$ matrices where $A_2, \ldots, A_n$ are positive definite. Then*

$$D^2(A_1, A_2, A_3, \ldots, A_n) \geq D(A_1, A_1, A_3, \ldots, A_n)D(A_2, A_2, A_3, \ldots, A_n).$$

Equality holds if and only if $A_1 = \lambda A_2$ where $\lambda > 0$ is a real number.

The form of this theorem most suitable for our purposes, states that:

$$D^s(A, s+t; \Phi)D^t(B, s+t; \Phi) \leq D^{s+t}(A, s; B, t; \Phi)$$

where A, B are positive definite symmetric matrices and Φ is any $(n-s-t)$-tuple of positive definite symmetric matrices. Equality holds if and only if $A = \lambda B$ where $\lambda > 0$ a real number.

A very useful inequality can be obtained by repeated applications of the Aleksandrov inequality:

Lemma 2.3 *For $A_1, \ldots, A_n \in M_n^{s,+}$, $D(A_1) \cdots D(A_n) \leq D^n(A_1, \ldots, A_n)$. Equality holds if and only if A_i, $i = 1, 2, \ldots, n$ are scalar multiples of each other; that is, $A_i = c_{ij}A_j$, where $c_{ij} > 0$, $i \neq j$.*

A special case of this general inequality is the Minkowski inequality.

Theorem 2.4 (Minkowski [1]) *If A and $B \in M_n^{s,+}$ then $D_1(A, B) \geq D^{\frac{(n-1)}{n}}(A)D^{\frac{1}{n}}(B)$, with equality if and only if $A = cB$, $c > 0$, and $D_1(A, B) = D(A, n-1; B, 1)$.*

We now prove the matrix analog of the Brunn-Minkowski theorem from convex geometry.

Theorem 2.5 *If $A, B \in M_n^{s,+}$ then $D^{\frac{1}{n}}(A+B) \geq D^{\frac{1}{n}}(A) + D^{\frac{1}{n}}(B)$, with equality if and only if $A = cB$, where c is a nonzero scalar.*

Proof.

$$\begin{aligned} & D(A+B) \\ &= D_1(A+B, A+B) \\ &= D(A+B, n-1; A+B) \\ &= D(A+B, n-1; A) + D(A+B, n-1; B) \\ &\geq D(A+B)^{\frac{(n-1)}{n}} D^{\frac{1}{n}}(A) \\ &\quad + D(A+B)^{\frac{(n-1)}{n}} D^{\frac{1}{n}}(B). \end{aligned}$$

Thus we obtain $D^{\frac{1}{n}}(A+B) \geq D^{\frac{1}{n}}(A) + D^{\frac{1}{n}}(B)$. □

We now prove a Uniqueness Theorem similar to one proved by Pranayanuntana [1].

Theorem 2.6 (Uniqueness Theorem) *Suppose A, B, $C \in M_n^{s,+}$ then:*

1. *$D_1(A, C) = D_1(B, C)$ for all $C \in M_n^{s,+}$ implies $A = B$*
2. *$D_1(A, B) = D_1(A, C)$ for all $A \in M_n^{s,+}$ implies $B = C$.*

Proof. The proof of parts 1 and 2 are very similar, and so we only give the proof of part 1. The Minkowski inequality states that $D_1^n(A, B) \geq D^{n-1}(A)D(B)$ for $A, B \in M_n^{s,+}$, with equality if and only if $A = cB$, $c > 0$. Since $D_1(A, C) = D_1(B, C)$, using $C = A$, we have $D(A) = D_1(A, A) = D_1(B, A) \geq D^{\frac{n-1}{n}}(B)D^{\frac{1}{n}}(A)$, with equality if and only if $A = cB$, $c > 0$. Since A is positive definite, $D(A) > 0$ and $D^{\frac{1}{n}}(A) > 0$, and then the last inequality becomes $D^{\frac{n-1}{n}}(A) \geq D^{\frac{n-1}{n}}(B)$ and therefore $D(A) \geq D(B)$, with equality if and only if $A = cB$, $c > 0$. Similarly, we can show that $D(B) \geq D(A)$, with equality if and only if $A = cB$, $c > 0$. We then conclude that $D(A) = D(B)$ and $A = cB$, $c > 0$. This is possible if and only if $c = 1$, and consequently $A = B$.

3 Symmetric Polynomials Inequality of Elementary

Definition 3.1 *The kth elementary symmetric polynomials $s_k(x)$ on variables $x = (x_1, \ldots, x_n)$ are defined by $s_1(x) = \sum_{1 \leq i \leq n} x_i$,*

$$s_2(x) = \sum_{1 \leq i < j \leq n} x_i x_j, \; s_3(x) = \sum_{1 \leq i < j < k \leq n} x_i x_j x_k, \ldots$$

$$s_k(x) = \sum_{1 \leq i < \cdots < i_k \leq n} \prod_{l=1}^{k} x_{i_l}, \ldots, \quad s_n(x) = \prod_{1 \leq i \leq n} x_i$$

The elementary symmetric polynomial functions evaluated at $(\lambda_1, \ldots, \lambda_n)$, where λ_i are the eigenvalues of A, are related to the characteristic polynomial of a matrix. Precisely, if $p_A(t) = D(tI - A)$ is the characteristic polynomial of the $n \times n$ matrix A, then

$$p_A(t) = t^n - s_1(\lambda)t^{n-1} + s_2(\lambda)t^{n-2} - \cdots \pm s_n(\lambda)$$

where $s_k(\lambda) = s_k(\lambda_1, \ldots, \lambda_n)$.

Definition 3.2 ([4]) *Let A be an $n \times n$ matrix. For the index set $\alpha \subseteq \{1, \ldots, n\}$, we denote the principal submatrix that lies in the rows and columns of A indexed by α as $A[\alpha, \alpha]$, or briefly, $A[\alpha]$.*
The determinant of such a principal submatrix is called a principal minor.

We denote the sum of the $\binom{n}{k}$ different $k \times k$ principal minors of $A \in M_n^{s,+}$ by $E_k(A)$. Therefore $E_k(A) := \sum_{\substack{|\alpha|=k \\ \alpha\subseteq J}}^{J} A[\alpha]$, where $J = \{1, \ldots, n\}$
In particular $E_1(A) = \sum_{i=1}^{n} a_{ii} = tr(A)$ and $E_n(A) = det(A) s_k(x_1, \ldots, x_n)$ evaluated at the eigenvalues $(\lambda_1, \ldots, \lambda_n)$ of A equals $E_k(A)$.

Definition 3.3 (Operator Monotone [7]) *A real-valued continuous function $f(t)$ defined on a real interval Ω is said to be operator monotone if $A \leq B \Rightarrow f(A) \leq f(B)$ for all symmetric matrices A, B of all sizes whose eigenvalues are contained in Ω.*

Definition 3.4 (Operator Convex/Concave [7]) *A real-valued continuous function $f(t)$ defined on a real interval Ω is called operator convex if for any $0 < \varepsilon < 1$, $f(\varepsilon A+(1-\varepsilon)B) \leq \varepsilon f(A)+(1-\varepsilon)f(B)$ holds for all symmetric matrices A, B of all sizes with eigenvalues in Ω. f is called operator concave if $-f$ is operator convex.*

Definition 3.5 (Positive Map [7]) *A map $\Phi : M_m \to M_n$ is called positive if it maps positive semidefinite matrices to positive semidefinite matrices: $A \geq 0 \Rightarrow \Phi(A) \geq 0$. M_m and M_n are the spaces of $m \times m$ and $n \times n$ matrices respectively.*

Definition 3.6 (Unital Map [6]) *A map $\Phi : M_n \to M_n$ is called unital if $\Phi(I_m) = I_n$.*

Theorem 3.7 (Operator Monotone and Operator Concave Functions [7]) *A nonnegative continuous function on $[0, \infty)$ is operator monotone if and only if it is operator concave.*

Theorem 3.8 ([7]) *Let Φ be a unital positive linear map from M_m to M_n and f an operator monotone function on $[0, \infty)$. Then for every $A \geq 0 \in M_n^s$, $f(\Phi(A)) \geq \Phi(f(A))$.*

Theorem 3.9 *The map $A \mapsto D^{1/(n-i)}(A, n-i; I, i)$ from $M_n^{s,+}$ to $(0, \infty)$ is operator concave.*

Proof. Since $D^{1/(n-i)}(\lambda A, n-i; I, i) = \lambda D^{1/(n-i)}(A, n-i; I, i)$. It suffices to prove that $D^{1/(n-1)}(A+B, n-i; I, i) \geq D^{1/(n-i)}(A, n-i; I, i) + D^{1/(n-i)}(B, n-i; I, i)$. This can be obtained by applying the Aleksandrov inequality (2.4) to the expansion of $D(A, n-i; I, i)$ as follows:

$$\begin{aligned}
&D(A+B, n-i; I, i) \\
&= \sum_{k=0}^{n-i} \binom{n-i}{k} D(A, n-i-k; B, k; I, i) \\
&\geq \sum_{k=0}^{n-i} \binom{n-i}{k} \\
&\quad D^{\frac{n-i-k}{n-i}}(A, n-i; I, i) D^{\frac{k}{n-i}}(B, n-i; I, i) \\
&= \Big(D^{1/(n-i)}(A, n-i; I, i) \\
&\qquad + D^{1/(n-i)}(B, n-i; I, i) \Big)^{n-i}
\end{aligned}$$

□

The scalar matrix operator $f : A \mapsto D^{1/(n-i)}(A, n-i; I, i)$ is operator concave, and by Theorem 3.7 is operator monotone. It is easy to see that $\Phi : A \mapsto A \circ I$ is a unital positive linear map. Here $\circ$ denotes the Hadamard product of A and I.
Therefore by Theorem 3.8 we have

$$\begin{aligned}
&D^{1/(n-i)}(A \circ I, n-i; I, i) I \\
&\quad \geq D^{1/(n-i)}(A, n-i; I, i) I \circ I \\
&\quad = D^{1/(n-i)}(A, n-i; I, i) I.
\end{aligned}$$

This implies

$$\begin{aligned}
&D^{1/(n-i)}(A \circ I, n-i; I, i) \\
&\quad \geq D^{1/(n-i)}(A, n-i; I, i)
\end{aligned} \tag{3.8a}$$

Since $n - i \geq 1$, $t \mapsto t^{n-i}$ is an increasing function on $(0, \infty)$, then (3.8a) is equivalent to

$$D(A \circ I, n-i; I, i) \geq D(A, n-i; I, i) \tag{3.8b}$$

Theorem 3.10 ([2]) *Let $A \in M_n^{s,+}$. Then*

$$s_{n-i}(a_{11}, \ldots, a_{nn}) \geq s_{n-i}(\lambda_1, \ldots, \lambda_n) \tag{3.9a}$$

$0 \leq i \leq n-i$, where a_{ii} and λ_i, $i = 1, 2, \ldots, n$, are diagonal entries and eigenvalues of A, respectively.

Proof. The inequality follows from the fact that $A \mapsto D(A, n-i; I, i)$ is invariant under similarity transformation, particularly the diagnolization transformation $A = P \wedge P^{-1}$. Therefore (3.8b) is equivalent to

$$D([a_{ij}\delta_{ij}], n-i; I, i) \geq D(\wedge, n-i; I, i) \tag{3.9b}$$

where $\delta_{ij} = 1$ if $i = j$ and zero otherwise. This gives the desired result (3.9a). □

4 Applications of Symmetric Polynomials: The Projection Operator

We define the matrix equivalent of the projection operator in a manner analogous to Lutwak. [8].

Definition 4.1 (Projection Operator) *The projection operator is defined through the following limit:*

$$(\mathscr{C}_{n-i}\, A) \cdot B := \lim_{t \to 0} \frac{E_i(A + tB) - E_i(A)}{t}$$

where $A,\ B \in M_n^{s,+}$ and $E_i(A)$ is the ith symmetric polynomial $s_i(\lambda_1, \dots, \lambda_n)$, λ_i are eigenvalues of A, and c_{n-i} is the projection operator, $1 \le i \le n$. Our goal is to obtain a formula for $\mathscr{C}\, n - i$. We can simplify the calculation of the limit by writing the E_i, $1 \le i \le n$, in terms of the trace function applied to the appropriate powers of matrices.
First, recall that

$$\begin{aligned}
\det(\lambda I - A) &= \prod_{i=1}^{n} (\lambda - \lambda_i) \\
&= \lambda^n - \left(\sum_{1 \le i \le n} \lambda_i \right) \lambda^{n-1} + \\
&\quad \left(\sum_{1 \le i < j \le n} \lambda_i \lambda_j \right) \lambda^{n-2} \\
&\quad - \left(\sum_{1 \le i < j < k \le n} \lambda_i \lambda_j \lambda_k \right) \lambda^{n-3} \\
&\quad + \left(\sum_{1 \le i < j < k < l \le n} \lambda_i \lambda_j \lambda_k \lambda_l \right) \\
&\quad \times \lambda^{n-4} - \cdots \pm \prod_{1 \le i \le n} \lambda_i \\
&= \lambda^n + c_1 \lambda^{n-1} + c_2 \lambda^{n-2} + \cdots + c_n
\end{aligned}$$

where λ_i, $1 \le i \le n$, and c_i, $1 \le i \le n$, are constants. Equating the coefficients of λ^{n-1} of the last two lines of the equations immediately above, yields

$$c_1 = -\sum_{i=1}^{n} \lambda_i = -tr\, A.$$

Finkbeiner [10] has shown that the other c_i, $2 \le i \le n$, can be determined similarly to obtain the following recursive set of equations

$$\begin{aligned}
c_2 &= -2^{-1}[c_1 tr(A) + tr(A^2)], \\
c_3 &= -3^{-1}[c_2 tr(A) + c_1 tr(A^2) + tr(A^3)] \\
&\vdots \\
c_n &= -n^{-1}[c_{n-1} tr(A) + c_{n-2} tr(A^2) + \cdots \\
&\quad + c_1 tr(A^{n-1}) + tr(A^n)]
\end{aligned} \tag{4.1}$$

Consequently, $E_1(A) = -c_1$, $E_2(A) = c_2, \dots$, $E_n(A) = (-1)^n c_n$, where the $E_i(A)$, $i = 1, \dots, n$ are the ith elementary symmetric polynomials $s_i(\lambda_1, \dots, \lambda_n)$, λ_i are eigenvalues of A, and the c_i, $1 \le i \le n$, are given in (4.1)*. Using the formulas for the $E_i(A)$, $1 \le i \le n$, the limit definition for $\mathscr{C}_{n-i}$ and the Uniqueness Theorem, we find that*

$$\mathscr{C}_{n-i}\, A = (-1)^{i+1}[A^{i-1} + c_1 A^{i-2} + c_2 A^{i-3} + \cdots + c_{i-1} I]$$

Writing $c_0 = 1$, this formula can be written as

$$\mathscr{C}_{n-i}\, A = -A[(-1)^i \sum_{k=1}^{i-1} c_{k-1} A^{i-k-1}] + E_{i-1}(A)$$

A recursive formula for $\mathscr{C}_{n-1}$ can then be written as follows:

$$\begin{aligned}
&\mathscr{C}_{n-1}\, A = I, \\
&\mathscr{C}_{n-i}\, A = E_{i-1}(A) I - A\, \mathscr{C}_{n-i+1}(A),
\end{aligned}$$

where $1 < i \le n$.

5 Quermassintegrals of Mixed Projection Bodies

Definition 5.1 (Ordinary Quermassintegrals) *For $A \in M_n^{s,+}$, $0 \le i \le n$, the ithe ordinary Quermassintegral of A, denoted by $W_i(A)$, is the mixed determinant $D(A, n-i; I, i)$, with $n - i$ copies of A and i copies of the identity matrix I.*

Definition 5.2 (Mixed Quermassintegrals [8]) *The mixed Quermassintegrals $W_0(A, B)$, $W_1(A, B)$, $\dots$, $W_{n-1}(A, B)$, of A and $B \in M_n^{s,+}$ are defined by*

$$(n - i) W_i(A, B) = \lim_{\varepsilon \to 0^+} \frac{W_i(A + \varepsilon B) - W_i(A)}{\varepsilon}$$

It is easy to see that since $W_i(\lambda A) = \lambda^{n-i} W_i(A)$ for all $0 \le i \le n-1$, $W_i(A, A) = W_i(A)$.
For $A,\ B \in M_n^{s,+}$ we have ([1])

$$D(A + B) = \sum_{i=0}^{n} \binom{n}{i} D(A, n-i; B, i) \tag{5.2}$$

We can also expand $W_i(A+\varepsilon B)$ as follows:

$$\begin{aligned} W_i(A+\varepsilon B) &= D(A+\varepsilon B, n-i; I, i) \\ &= \sum_{k=0}^{n-i} \binom{n-i}{k} \varepsilon^k \\ &\quad \times D(A, n-i-k; B, k; I, i) \\ &= D(A, n-i; I, i) + \\ &\quad \sum_{k=1}^{n-i} \binom{n-i}{k} \varepsilon^k \\ &\quad \times D(A, n-i-k; B, k; I, i) \\ &= W_i(A) + \sum_{k=1}^{n-i} \binom{n-i}{k} \varepsilon^k \\ &\quad \times D(A, n-i-k; B, k; I, i) \end{aligned}$$

Consequently;

$$\begin{aligned} W_i(A,B) &= \frac{1}{(n-i)} \\ &\quad \lim_{\varepsilon\to 0^+} \frac{W_i(A+\varepsilon B) - W_i(A)}{\varepsilon} \\ &= \frac{1}{(n-i)} \\ &\quad \lim_{\varepsilon\to 0^+} \frac{\sum_{k=1}^{n-i} \binom{n-i}{k} \varepsilon^k D(A, n-i-k; B, k; I, i)}{\varepsilon} \\ &= D(A, n-i-1; B, 1; I, i) \end{aligned}$$

It clearly follows that for all $A \in M_n^{s,+}$, $W_{n-1}(A,B) = W_{n-1}(B)$, since $W_{n-1}(A,B) = D(B, \underbrace{I,\ldots,I}_{n-1}) = W_{n-1}(B)$. We recognize the mixed Quermassintegral $W_0(A,B)$ as $D_1(A,B)$.
Since $(n-i)W_i(A,B)$ of definition 5.2 is a directional derivative, we can rewrite it as

$$(n-i)W_i(A,B) = \frac{d}{d\varepsilon} W_i(A+\varepsilon B)\bigg|_{\varepsilon=0} \tag{5.3}$$

Let $\frac{\partial W_i(A)}{\partial A}$ denote the gradient of the functional $W_i(A)$ with respect to the matrix A ([1]). Then the directional derivative in Definition 5.2 and (5.3) can be written as

$$\begin{aligned} (n-i)W_i(A,B) &= \frac{d}{d\varepsilon} W_i(A+\varepsilon B)\bigg|_{\varepsilon=0} \\ &= \frac{\partial}{\partial A} W_i(A) \cdot B \end{aligned} \tag{5.4}$$

The ith symmetric polynomial of eigenvalues $(\lambda_1, \ldots, \lambda_n)$ of A is $E_i(A) = \binom{n}{n-i} W_{n-i}(A)$ and from the definition of the Projection Operator we conclude that

$$\begin{aligned} (\mathscr{C}_{n-i}\, A) \cdot B &= \binom{n}{n-i} i W_{n-i}(A,B) \\ &= \binom{n}{n-i} \frac{\partial}{\partial A} W_{n-i}(A) \cdot B \end{aligned} \tag{5.5}$$

If we extend our definition of the Projection Operator and mixed Quermassintegrals to all $n \times n$ matrices in M_n then (5.5) gives:

$$\mathscr{C}_{n-1}\, A = \binom{n}{n-1} \frac{\partial W_{n-i}(A)}{\partial A} \tag{5.6}$$

and from (4.2) we have the following recursive formulae for $\frac{\partial W_{n-i}(A)}{\partial A}$:

$$\begin{aligned} \frac{\partial W_{n-1}(A)}{\partial A} &= \frac{1}{n} I, \\ \frac{\partial W_{n-i}(A)}{\partial A} &= \frac{1}{\binom{n}{n-i}} \Bigg(\binom{n}{n-i+1} W_{n-i+1}(A) I \\ &\qquad - A \binom{n}{n-i+1} \frac{\partial W_{n-i}}{\partial A}(A) \Bigg) \\ &= \frac{\binom{n}{n-i+1}}{\binom{n}{n-i}} \times \\ &\quad \left(W_{n-i+1}(A) I - A \frac{\partial W_{n-i+1}}{\partial A}(A) \right) \\ &= \frac{i}{(n-i+1)} \\ &\quad \left(W_{n-i+1}(A) I - A \frac{\partial W_{n-i+1}}{\partial A}(A) \right) \end{aligned}$$

where $1 < i \leq n$.
From Definition 5.2, with $B = I$, we have the following relationship between $W_i(A)$, $0 \leq i \leq n$:

$$\begin{aligned} W_{i+1}(A) &= \frac{1}{(n-i)} \lim_{\varepsilon\to 0^+} \frac{W_i(A+\varepsilon I) - W_i(A)}{\varepsilon} \\ &= \frac{1}{(n-i)} \left[\frac{\partial W_i(A)}{\partial (A)_{\hat{i}\hat{j}}} \right] \cdot [\delta_{\hat{i}\hat{j}}] \\ &= \frac{1}{(n-i)} \sum_{\hat{i},\hat{j}} \frac{\partial}{\partial (A)_{\hat{i}\hat{j}}} (A) \delta_{\hat{i}\hat{j}} \\ &= \frac{1}{(n-i)} \sum_{\hat{i}} \frac{\partial W_i(A)}{\partial (A)_{\hat{i}\hat{i}}} \end{aligned} \tag{5.7}$$

But according to Pranayanuntana [1, 2, 3], $W_i(A)$

can be seen from the following expansion:

$$D(A+\varepsilon I) = \sum_{i=0}^{n} \binom{n}{i} \varepsilon^i D(A, n-i; I, i)$$
$$= \sum_{i=0}^{n} \binom{n}{i} \varepsilon^i W_i(A) \tag{5.8}$$

Differenting (5.8) i times with respect to ε and setting $\varepsilon = 0$, we obtain

$$\frac{d^i}{d\varepsilon^i} D(A+\varepsilon I)\bigg|_{\varepsilon=0} = \frac{n!}{(n-1)!} D(A, n-i; I, i) \tag{5.9}$$

and consequently

$$D(A, n-i; I, i) = \frac{(n-i)!}{n!} \frac{d^i}{d\varepsilon^i} D(A+\varepsilon I)\bigg|_{\varepsilon=1} \tag{5.10}$$

For positive definite symmetric matrix A, there always exists an orthogonal matrix P such that $A = P \wedge P^{-1} = P \wedge P^T$, where P is the matrix whose columns form an orthonormal eigenbasis of A and $\wedge$ is the diagonal matrix whose diagonal entries are the corresponding eigenvalues of A. Equation (5.10) then yields

$$\begin{aligned} W_i(A) &= D(A, n-i; I, i) \\ &= \frac{(n-i)!}{n!} \frac{d^i}{d\varepsilon^i} D(P \wedge P^{-1} + \varepsilon P I P^{-1})\bigg|_{\varepsilon=0} \\ &= \frac{(n-i)!}{n!} \frac{d^i}{d\varepsilon^i} D(\wedge + \varepsilon I)\bigg|_{\varepsilon=0} \\ &= D(\wedge, n-i; I, i) \\ &= W_i(\lambda) \\ &= \frac{1}{\binom{n}{n-i}} \sum_{1} \lambda_{j_1 \cdots \lambda_{j_{n-i}}} \end{aligned} \tag{5.11}$$

where the sums are taken over all $(n-i)$-tuple of positive integers $(j_1, \ldots, j_{n-i})$ whose entries do not exceed n, with λ_{j_k}, $1 \leq k \leq n$, from the set of all n positive eigenvalues of A. Equation (5.11) tells us that $W_i(\cdot)$ is invariant under similarity transformation $A = P \wedge P^{-1}$. Therefore from this invariant relation and (5.7) we have the following important relation between $W_i(A)$, $0 \leq i \leq n$:

$$W_{i+1}(A) = W_{i+1}(\wedge) = \frac{1}{(n-i)} \sum_j \frac{\partial W_i}{\partial \lambda_j}(\wedge) \tag{5.11}$$

As an illustration, for $A \in M_3$, we have

$$\begin{aligned} W_0(A) &= D(A) = \lambda_1\lambda_2\lambda_3 \\ W_1(A) &= \frac{1}{3} \sum_j \frac{\partial}{\partial \lambda_j}(\lambda_1\lambda_2\lambda_3) \\ &= \frac{1}{3}(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3) \\ W_2(A) &= \frac{1}{2} \sum_j \frac{\partial}{\partial \lambda_j} \quad W_1(A) = \frac{1}{3}(\lambda_1 + \lambda_2 + \lambda_3) \\ W_3(A) &= \sum_j \frac{\partial}{\partial \lambda_j} W_2(A) = \frac{1}{3} \sum_j \frac{\partial}{\partial_j}(\lambda_1 + \lambda_2 + \lambda_3) \\ &= 1 \end{aligned}$$

6 L_p-Sum and Scalar Multiplication of Matrices

Definition 6.1 (L_p-Sum of Matrices) *For matrices A, $B \in M_n^{s,+}$ and $p \geq 1$ we define the L_p-sum of A and B as:*

$$A +_p B = (A^p + B^p)^{1/p}.$$

The commutativity of $+_p$ is obvious. For the associativity:

$$\begin{aligned} (A +_p B) +_p C &= [(A +_p B)^p + C^p]^{\frac{1}{p}} \\ &= [A^p + B^p + C^p]^{\frac{1}{p}} \\ &= [A^p + (B^p + C^p)]^{\frac{1}{p}} \\ &= [A^p + (B +_p C)^p]^{\frac{1}{p}} \\ &= A +_p (B +_p C). \end{aligned}$$

Definition 6.2 (L_p Scalar Multiplication) *For $p \geq 1$, $B \in M_n^{s,+}$ and scalar λ, we define the L_p scalar multiplication of λ and B as*

$$\lambda \cdot B = \lambda^{1/p} B.$$

Consequently for scalars α, β and matrices A, $B \in M_n^{s,+}$:

$$\alpha \cdot A +_p \beta \cdot B = (\alpha A^p + \beta B^p)^{1/p}.$$

7 Mixed p-Quermassintegrals

We define the matrix equivalent of mixed p-Quermassintegral in a manner analogous to Lutwak [9].

Definition 7.1 (Mixed p-Quermassintegrals) *For A, $B \in M_n^{s,+}$, $p \geq 1$, $0 \leq i < n-1$, the mixed p-Quermassintegrals of A, B, denoted $W_{p,i}(A,B)$, we define by*

$$\frac{n-i}{p}W_{p,i}(A,B) = \lim_{\varepsilon\to 0^+} \frac{W_i(A+_p \varepsilon\cdot B) - W_i(A)}{\varepsilon}.$$

Clearly, if $p=1$, $A+_p\varepsilon B$ and consequently the mixed p-Quermassintegral $W_{p,i}(A,B)|_{p=1} = W_i(A,B)$ in this particular case. It is easy to see that $W_{p,i}(A,A) = W_i(A)$ for all $p \geq 1$:

$$\begin{aligned}
&\frac{n-i}{p}W_{p,i}(A,A)\\
&\quad= \lim_{\varepsilon\to 0^+} \frac{W_i(A+_p \varepsilon\cdot A) - W_i(A)}{\varepsilon}\\
&\quad= \lim_{\varepsilon\to 0^+} \frac{W_i\left([A^p(1+\varepsilon)]^{\frac{1}{p}}\right) - W_i(A)}{\varepsilon}\\
&\quad= \lim_{\varepsilon\to 0^+} \frac{\left[1+\frac{n-i}{p}\varepsilon + \mathcal{O}(\varepsilon^2)\right]W_i(A) - W_i(A)}{\varepsilon}\\
&\quad= \frac{n-i}{p}W_i(A)
\end{aligned}$$

and we conclude that $W_{p,i}(A,A) = W_i(A)$ for all $p \geq 1$.

Theorem 7.2 *(a) For all A, $B \in M_n^{s,+}$ and $\alpha,\beta > 0$, $W_{p,i}(\alpha A, \beta B) = \alpha^{n-i-p}\beta^p W_{p,i}(A,B)$, and when $p = n-i$ and $\beta = 1$, $W_{p,i}(\alpha A, B) = W_{p,i}(A,B)$.*

(b) For all Q, A, $B \in M_n^{s,+}$, $W_{p,i}(Q, A+_p B) = W_{p,i}(Q,A) + W_{p,i}(Q,B)$.

(c) For all A, $B \in M_n^{s,+}$, $\gamma > 0$, $W_{p,i}(A, \gamma\cdot B) = \gamma W_{p,i}(A,B)$.

Proof. For $\alpha,\beta > 0$ and A, $B \in M_n^{s,+}$ we have

$$\begin{aligned}
&W_{p,i}(\alpha A, \beta B)\\
&\quad= \left(\frac{p}{n-i}\right)\lim_{\varepsilon\to 0^+}\frac{W_i(\alpha A +_p \varepsilon\cdot\beta B) - W_i(\alpha A)}{\varepsilon}\\
&\quad= \left(\frac{p}{n-i}\right)\lim_{\varepsilon\to 0^+}\frac{W_i(\alpha(A +_p \varepsilon\cdot\frac{\beta}{\alpha} B)) - W_i(\alpha A)}{\varepsilon}\\
&\quad= \alpha^{n-i}\left(\frac{p}{n-i}\right)\lim_{\varepsilon\to 0^+}\frac{W_i(A +_p (\frac{\beta^p\varepsilon}{\alpha^p})\cdot B)) - W_i(A)}{\varepsilon}.
\end{aligned}$$

Now let $\widetilde{\varepsilon} = \frac{\beta^p\varepsilon}{\alpha^p}$,

$$\begin{aligned}
&W_{p,i}(\alpha A, \beta B)\\
&\quad= \frac{\beta^p\alpha^{n-i}}{\alpha^p}\left(\frac{p}{n-i}\right)\lim_{\tilde{\varepsilon}\to 0^+}\frac{W_i(A +_p \widetilde{\varepsilon}\cdot B) - W_i(A)}{\widetilde{\varepsilon}}\\
&\quad= \alpha^{n-i-p}\beta^p\left(\frac{p}{n-i}\right)\lim_{\tilde{\varepsilon}\to 0^+}\frac{W_i(A +_p \widetilde{\varepsilon}\cdot B) - W_i(A)}{\widetilde{\varepsilon}}\\
&\quad= \alpha^{n-i-p}\beta^p W_{p,i}(A,B).
\end{aligned}$$

This shows that the functional $W_{p,i} : M_n^{s,+}\times M_n^{s,+} \to (0,\infty)$ is Minkowski homogeneous of degree $n-i-p$ in its first argument and Minkowski homogeneous of degree p in its second argument. Trivially, when $p = n-i$, $\beta = 1$, $\alpha > 0$, then $W_{p,i}(\alpha A, B) = W_{p,i}(A,B)$. For part (b) take Q, A, $B \in M_n^{s,+}$,

$$\begin{aligned}
&W_{p,i}(Q, A+_p B)\\
&\quad= \left(\frac{p}{n-i}\right)\\
&\quad\lim_{\varepsilon\to 0^+}\frac{W_i(Q +_p \varepsilon\cdot(A+B)) - W_i(Q)}{\varepsilon}\\
&\quad= \left(\frac{p}{n-i}\right)\\
&\quad\lim_{\varepsilon\to 0^+}\frac{W_i((Q +_p \varepsilon\cdot A) +_p \varepsilon\cdot B) - W_i(Q +_p \varepsilon\cdot A)}{\varepsilon}\\
&\quad+ \left(\frac{p}{n-i}\right)\\
&\quad\lim_{\varepsilon\to 0^+}\frac{W_p(Q +_p \varepsilon\cdot A) - W_i(Q)}{\varepsilon}
\end{aligned}$$

Write $\tilde{Q} = Q +_p \varepsilon\cdot A$ in the first limit,

$$\begin{aligned}
&W_{p,i}(Q, A +_p \varepsilon\cdot B)\\
&\quad= \left(\frac{p}{n-i}\right)\lim_{\varepsilon\to 0^+}\frac{W_i(\tilde{Q} +_p \varepsilon\cdot B) - W_i(\tilde{Q})}{\varepsilon}\\
&\qquad+ \left(\frac{p}{n-i}\right)\lim_{\varepsilon\to 0^+}\frac{W_p(Q +_p \varepsilon\cdot A) - W_i(Q)}{\varepsilon}\\
&\quad= W_{p,i}(Q,A) + W_{p,i}(Q,B).
\end{aligned}$$

Part (c) is part (a) with $\alpha = 1$ and $\gamma = \beta^p$. □

This result shows that the mixed p-Quermassintegral is linear, with respect to L_p-sum and scalar multiplication, in its second argument.

Definition 7.3 (Jointly Concave Map [7]) *Let P_n be the set of positive semidefinite matrices in M_n^s. A map $\Psi : P_n\times P_n \to P_m$ is called* jointly concave *if*

$$\Psi(\lambda A + (1-\lambda)B, \lambda C + (1-\lambda)D) \geq$$
$$\lambda\Psi(A,C) + (1-\lambda)\Psi(B,D).$$

for all A, B, C, $D \geq 0$ and $0 < \lambda < 1$.

The following lemma proved by Zhan [7] is very useful in the proof of Theorem 8.1.

Lemma 7.4 ([7]) *For $0 < r < 1$ the map*

$$(A, B) \mapsto A^r \circ B^{1-r}$$

is jointly concave in A, $B \geq 0$.

8 Some Useful Inequalities

Theorem 8.1 ([7]) *For A, B, C, $D \geq 0$ and p, $q > 1$ with $1/p + 1/q = 1$,*

$$A \circ B + C \circ D \leq (A +_p C) \circ (B +_q D),$$

where $A \circ B := [a_{ij} b_{ij}] \in M_n$.

Proof. This is just the mid-point joint concavity case $\lambda = 1/2$ of Lemma 7.4 with $r = 1/p$. ☐

Theorem 8.2 *For X, $Y > 0$ that is X, $Y \in M_n^{s,+}$ and $\varepsilon \in [0, 1]$ we have*

$$(1 - \varepsilon)X + \varepsilon Y \leq ((1 - \varepsilon)X^p + \varepsilon Y^p)^{1/p}$$

$$=: (1 - \varepsilon) \cdot X +_p \varepsilon \cdot Y$$

Proof. This is just Theorem 8.1 with $A = (1 - \varepsilon)^{1/p} X$, $B = (1 - \varepsilon)^{1/q} 1_n$, $C = \varepsilon^{1/p} Y$ and $D = \varepsilon^P 1/q 1_n$, where $1_n = [1]_{n \times n}$, that is 1_n is the $n \times n$ matrix that has all of its entries equal to 1. ☐

The following theorem proved by Horn [6] is useful in the proof of Corollary 8.4.

Theorem 8.3 ([6]) *If A, $B \in M_n$ are positive definite symmetric, then if $A \geq B$, then $detA \geq detB$ and $trA \geq trB$; and more generally, if $A \geq B$, then $\lambda_k(A) \geq \lambda_k(B)$ for all $k = 1, 2, \ldots, n$ if the respective eigenvalues of A and B are arranged in the same (increasing or decreasing) order.*

Corollary 8.4 *For any A, $B > 0 \in M_n^{s,+}$, $\varepsilon \in [0, 1]$*

$$W_i((1 - \varepsilon) \cdot A +_p \varepsilon \cdot B) \geq W_i((1 - \varepsilon)A + \varepsilon B), \quad 0 \leq i \leq n - 1. \tag{8.4}$$

Proof. Applying Theorem 8.3 to the inequality in Theorem 8.2 and using the fact that for any matrix $A \in M_n^{s,+}$,

$$\begin{aligned}
\binom{n}{0} W_0(A) &= E_n(A) = \lambda_1 \lambda_2 \cdots \lambda_n, \\
\binom{n}{0} W_1(A) &= E_{n-1}(A) = \lambda_{i_1} \cdots \lambda_{i_{n-1}}, \\
&\vdots \\
\binom{n}{n-3} W_{n-3}(A) &= E_3(A) = \sum \lambda_{i_1} \lambda_{i_2} \lambda_{i_3}, \\
\binom{n}{n-2} W_{n-2}(A) &= E_2(A) = \sum \lambda_{i_1} \lambda_{i_2}, \\
\binom{n}{n-1} W_{n-1}(A) &= E_1(A) = \sum \lambda_{i_1}
\end{aligned}$$

where $E_i(A)$, $1 \leq i \leq n$ is the ith symmetric polynomial of eigenvalues $(\lambda_1, \ldots, \lambda_n)$ of A. ☐

Theorem 8.5 *For any A, $B > 0 \in M_n^{s,+}$, $\varepsilon \in [0, 1]$*

$$\begin{aligned}
&W_i((1 - \varepsilon)A + \varepsilon B) \\
&\quad \geq ((1 - \varepsilon) W_i(A)^{\frac{1}{n-i}} + \varepsilon (B)^{\frac{1}{n-i}})^{n-i}, \\
&\quad 0 \leq i \leq n - 1.
\end{aligned} \tag{8.5a}$$

Equality holds if and only if $A = cB$ with a real number $c > 0$.

Proof. It suffices to prove that

$$W_i(A + B) \geq (W_i(A)^{\frac{1}{n-i}} + W_i(B)^{\frac{1}{n-i}})^{n-i}, \quad 0 \leq i \leq n - 1. \tag{8.5b}$$

with equality if and only if $A = cB$, $c > 0$. The inequality is obtained by applying the Aleksandrov inequality to the expansion of $W_i(A + B)$ as follows:

$$\begin{aligned}
&W_i(A + B) \\
&= D(A + B, n - i; I, i) \\
&= \sum_{k=0}^{n-i} \binom{n-i}{k} D(A, n - i - k; B, k; I, i) \\
&\geq \sum_{k=0}^{n-i} \binom{n-i}{k} D^{\frac{n-i-k}{n-i}}(A, n - i; I, i) \\
&\qquad D^{\frac{k}{n-i}}(B, n - i; I, i) \\
&= (D^{1/(n-i)}(A, n - i; I, i) + D^{1/(n-i)} \\
&\qquad (B, n - i; I, i))^{n-i} \\
&= (W_i^{1/(n-i)(A)} + W_i^{1/(n-i)}(B))^{n-i}
\end{aligned} \tag{8.5c}$$

which is (8.5b). For the equality part, it can be easily seen that if $A = cB$, $c > 0$ then

$$W_i^{1/(n-i)}(A+B) = W_i^{1/(n-i)}(A) + W_i^{1/(n-i)}(B).$$

Therefore, we only need to prove that

$$W_i^{1/(n-i)}(A+B) = W_i^{1/(n-i)}(A) + W_i^{1/(n-i)}(B)$$

implies $A = cB, c > 0$.
Suppose $A \neq cB$ then the Aleksandrov inequality (Theorem 2.4) is strict, and applying it in the process of getting (8.5c) yields

$$W_i^{1/(n-i)}(A+B) > W_i^{1/(n-i)}(A) + W_i^{1/(n-i)}(B)$$

which means $W_i^{1/(n-i)}(A+B) = W_i^{1/(n-i)}(A) + W_i^{1/(n-i)}(B)$ implies $A = cB$, $c > 0$. □

Theorem 8.6 *If $p > 1$, $\alpha \in [0,1]$, A_0, $B_0 > 0 \in M_n^{s,+}$ with $W_i(A_0) = W_i(B_0) = 1$ then*

$$W_i(\alpha \cdot A_0 +_p (1-\alpha)\cdot B_0) \geq 1, \quad 0 \leq i \leq n-1. \tag{8.6a}$$

Equality holds if and only if $A_0 = B_0$.

Proof. Applying (8.4) and (8.5a) with $\alpha = 1 - \varepsilon$ to A_0, B_0 we obtain

$$\begin{aligned} &W_i(\alpha \cdot A_0 +_p (1-\alpha)\cdot B_0) \\ &\quad \geq W_i(\alpha A_0 + (1-\alpha)B_0) \geq 1, \\ &\quad\quad 0 \leq i \leq n-i. \end{aligned}$$

To see the equality part of (5.3.5) we first set $A_0 = B_0$,

$$\begin{aligned} W_i(\alpha &\cdot A_0 +_p (1-\alpha)\cdot B_0) \\ &= W_i(\alpha \cdot A_0 +_p (1-\alpha)\cdot A_0) \\ &= W_i[(\alpha A_0^p + (1-\alpha)A_0^p)^{1/p}] \\ &= W_i[(A_0^p)^{1/p}] \\ &= W_i(A_0) \\ &= 1. \end{aligned}$$

This proves that $A_0 = B_0$ implies $W_i(\alpha \cdot A_0 +_p (1-\alpha)\cdot B_0) = 1$.
Now we set $W_i(\alpha \cdot A_0 +_p (1-\alpha)\cdot B_0) = 1$, from (8.6b), we see that this implies $W_i(\alpha A_0 + (1-\alpha)B_0) = 1$, which in turn, by Theorem 8.5 implies $A_0 = cB_0$, but since we have $W_i(A_0) = W_i(B_0)$ therefore

$$c = 1 \text{ and } A_0 = B_0.$$

This proves that $W_i(\alpha \cdot A_0 +_p (1-\alpha)\cdot B_0) = 1$ implies $A_0 = B_0$. This completes the proof. □

Theorem 8.7 (Brunn-Minkowski Inequality for L_p-Sum of Matrices) *If A, $B \in M_n^{s,+}$, $0 \leq i \leq n-1$, $p \geq 1$, then*

$$W_i^{\frac{p}{n-i}}(A +_p B) \geq W_i^{\frac{p}{n-i}}(A) + W_i^{\frac{p}{n-i}}(B).$$

with equality if and only if $A = c$, B, $c > 0$.

Proof. We apply Theorem 8.6 with

$$\begin{aligned} A_0 &= \frac{1}{W_i(A)^{\frac{p}{n-i}}} \cdot A, \\ B_0 &= \frac{1}{W_i(B)^{\frac{p}{n-i}}} \cdot B, \\ \alpha &= \frac{W_i(A)^{\frac{p}{n-i}}}{W_i(A)^{\frac{p}{n-i}} + W_i(B)^{\frac{p}{n-i}}} \\ &W_i(\alpha \cdot A_0 +_p (1-\alpha)\cdot B_0) \geq 1 \end{aligned}$$

to obtain

$$\begin{aligned} W_i\Bigg(&\frac{W_i(A)^{\frac{p}{n-i}}}{(W_i(A)^{\frac{p}{n-i}} + W_i(B)^{\frac{p}{n-i}})\cdot A} \\ &+ \frac{W_i(B)^{\frac{p}{n-i}}}{p(W_i(A)^{\frac{p}{n-i}} + W_i(B)^{\frac{p}{n-i}})} \cdot \frac{1}{(W_i(B)^{\frac{p}{n-i}})} \cdot B\Bigg)^{\frac{p}{n-i}} \\ &\geq 1 \end{aligned}$$

or

$$W_i\left(\frac{1}{(W_i(A)^{\frac{p}{n-i}})} \cdot A +_p \frac{1}{(W_i(A)^{\frac{p}{n-i}} + W_i(B)^{\frac{p}{n-i}})\cdot B}\right)^{\frac{p}{n-i}} \geq 1$$

$$\begin{aligned} W_i^{\frac{p}{n-i}}\Bigg\{\Bigg[&\left(\frac{1}{(W_i(A)^{\frac{p}{n-i}} + W_i(B)^{\frac{p}{n-i}})^{1/p}} A\right)^p \\ &+ \left(\frac{1}{(W_i(A)^{\frac{p}{n-i}} + W_i(B)^{\frac{p}{n-i}})^{1/p}} B\right)^p\Bigg]^{1/p}\Bigg\} \geq 1 \end{aligned}$$

$$W_i^{p/(n-i)}\left[\left(\frac{A^p + B^p}{(w_i(A)^{\frac{p}{n-i}} + W_i(B)^{\frac{p}{n-i}})}\right)^{1/p}\right] \geq 1$$

$$W_i^{p/(n-i)}\left[\frac{(A^p+B^p)^{1/p}}{(W_i(A)^{\frac{p}{n-i}})} + W_i(B)^{\frac{p}{n-i}}\right]^{1/p} \geq 1$$

$$\begin{aligned} &\left(\frac{1}{(W_i(A)^{\frac{p}{n-i}} + W_i(B)^{\frac{p}{n-i}})^{(n-i)/p}}\right) \\ &\quad W_i^{p/(n-i)}[(A^p+B^p)^{1/p}] \geq 1 \end{aligned}$$

that is,

$$W_i(A +_p B)^{\frac{p}{n-i}} \geq W_i(A)^{\frac{p}{n-i}} + W_i(B)^{\frac{p}{n-i}}.$$

The sufficiency of the equality part can be seen by directly substituting $A = c \cdot B$, $c > 0$ and the necessity of the equality part can be proved by contradiction as follows:
Suppose $A \neq c \cdot B$ then $A_0 \neq B_0$ which in turn by Theorem 8.6, implies $W_i(\alpha \cdot A_0 +_p (1-\alpha) \cdot B_0) > 1$ or

$$W_i(A +_p B)^{\frac{p}{n-i}} > W_i(A)^{\frac{p}{n-i}} + W_i(B)^{\frac{p}{n-i}}$$

which is a contradiction. This completes the proof. □

Theorem 8.8 Minkowski Inequality for L_p-Sum of Matrices *If A, $B \in M_n^{s,+}$, $0 \le i \le n-1$, $p \ge 1$, then $W_{p,i}^{n-i}(A, B) \ge W_i^{n-i-p}(A) W_i^p(B)$ with equality if and only if $A = c \cdot B$, $c > 0$.*

Proof. Theorem 8.7 implies

$$\begin{aligned}
&W_i^{p/(n-i)}((1-\varepsilon) \cdot A +_p \varepsilon \cdot B) \\
&\ge W_i^{p/(n-i)}((1-\varepsilon) \cdot A) + W_i^{p/(n-i)}(\varepsilon \cdot B) \\
&= W_i^{p/(n-i)}((1-\varepsilon)^{1/p} A) + W_i^{p/(n-i)}(\varepsilon^{1/p} B) \\
&= ((1-\varepsilon)^{(n-i)/p} W_i(A))^{p/(n-i)} \\
&\quad + (\varepsilon^{(n-i)/p} W_i(B))^{p/(n-i)} \\
&= (1-\varepsilon) W_i^{p/(n-i)}(A) + \varepsilon W_i^{p/(n-i)}(B),
\end{aligned}$$

and since

$$\begin{aligned}
&\lim_{\varepsilon \to 0} \frac{W_i((1-\varepsilon) \cdot A +_p \varepsilon \cdot B) - W_i(A)}{\varepsilon} \\
&= \lim_{\lambda \to 1} \frac{W_i(\lambda \cdot A +_p (1-\lambda) \cdot B) - W_i(A)}{1-\lambda} \\
&= \lim_{\lambda \to 1} \frac{\lambda^{\frac{n-i}{p}} W_i\left(A +_p \frac{1-\lambda}{\lambda} \cdot B\right) - W_i(A)}{1-\lambda} \\
&= \lim_{\varepsilon \to 0} \frac{(1+\varepsilon)^{\frac{i-n}{p}} W_i(A +_p \epsilon \cdot B) - W_i(A)}{\epsilon} (1+\varepsilon), \\
&\text{where } \epsilon = \frac{1-\lambda}{\lambda} \\
&= \lim_{\epsilon \to 0} \frac{g(\epsilon) f(\epsilon) - g(0) f(0)}{\epsilon} (1+\epsilon),
\end{aligned}$$

(where $f(\epsilon) = W_i(A +_p \epsilon \cdot B)$ and $g(\epsilon) = (1+\epsilon)^{\frac{i-n}{p}}$)

$$\begin{aligned}
&= (gf)'(0) \\
&= g(0) f'(0) + g'(0) f(0) \\
&= 1 \frac{n-i}{p} W_{p,i}(A, B) + \frac{i-n}{p} W_i(A).
\end{aligned}$$

Then

$$\begin{aligned}
&W_{p,i}(A, B) \\
&= W_i(A) + \frac{p}{n-i} \\
&\lim_{\varepsilon \to 0} \frac{W_i((1-\varepsilon) \cdot A +_p \varepsilon \cdot B) - W_i(A)}{\varepsilon} \\
&\ge W_i(A) + \frac{p}{n-i} \\
&\lim_{\varepsilon \to 0} \frac{[(1-\varepsilon) W_i^{\frac{p}{n-i}}(A) + \varepsilon W_i^{\frac{p}{n-i}}(B)]^{\frac{(n-i)}{p}} - W_i(A)}{\varepsilon} \\
&= W_i(A) + \frac{p}{n-i} \left[\frac{n-i}{p}\right] \\
&[W_i(A)^{\frac{p}{(n-i)}\left(\frac{(n-i)}{p} - 1\right)} W_i(B)^{\frac{p}{(n-i)}} - W_i(A)] \\
&= W_i(A)^{1 - \frac{p}{(n-i)}} W_i(B)^{\frac{p}{(n-i)}}.
\end{aligned}$$

This gives the inequality part of Theorem 8.8. The sufficiency of the equality part can be seen by directly substituting $A = c \cdot B$, $c > 0$ using the fact that $W_{p,i}(B, B) = W_i(B)$.

The necessity of the equality part can be shown as follows:

$$W_{p,i}^{n-i}(A, B) = W_i^{n-i-p}(A) W_i^p(B)$$

for A, $B \in M_n^{s,+}$, $0 \le i \le n-i$, $p > 1$, and

$$W_{p,i}(Q, A +_p B) = W_{p,i}(Q, A) + W_{p,i}(Q, B)$$

then

$$W_{p,i}(Q, A +_p B) = W_i^{\frac{n-i-p}{n-i}}(Q)[W_i^{\frac{p}{n-i}}(A) + W_i^{\frac{p}{n-i}}(B)].$$

We now set $A +_p B$ equal to Q and use the fact that $W_{p,i}(Q, Q) = W_i(Q)$ to obtain

$$W_i(A +_p B) = W_i^{\frac{n-i-p}{n-i}}(A + pB)[W_i^{\frac{p}{n-i}}(A) + W_i^{\frac{p}{n-i}}(B)],$$

or

$$W_i^{\frac{p}{n-i}}(A +_p B) = W_i^{\frac{p}{n-i}}(A) + W_i^{\frac{p}{n-i}}(B)$$

which is the equality part of Theorem 8.7 and that is if and only if $A = c \cdot B$, $c > 0$. This proves that $W_{p,i}^{n-i}(A, B) = W_i^{n-i-p}(A) W_i^p(B)$ implies $A = cB$, $c > 0$. This completes the proof. □

Furthermore, we can also show that the inequalities of Theorems 8.7 and 8.8 are equivalent. Since we have already shown that Theorem 8.7 implies Theorem 8.8, it suffices to show that Theorem 8.8 implies Theorem 8.7.

Since

$$W_{p,i}(A,B) \geq W_i^{\frac{n-i-p}{n-i}}(A) W_i^{\frac{p}{n-i}}(B),$$

for A, $B \in M_n^s$, $0 \leq i \leq n-1$, $p > 1$, and

$$W_{p,i}(Q, A +_p B) = W_{p,i}(Q,A) + W_{p,i}(Q,B)$$

then

$$W_{p,i}(Q, A +_p B) \geq W_i^{\frac{n-i-p}{n-i}}(Q)[W_i^{\frac{p}{n-i}}(A) + W_p^{\frac{p}{n-i}}(B)].$$

We now set $A +_p B$ equal to Q and use the fact that $W_{p,i}(Q,Q) = W_i(Q)$ to obtain

$$W_i(A +_p B) \geq W_i^{\frac{n-i-p}{n-i}}(A +_p B) \left[W_i^{\frac{p}{n-i}}(A) + W_i^{\frac{p}{n-i}}(B)\right],$$

or

$$W_i^{\frac{p}{n-i}}(A +_p B) \geq W_i^{\frac{p}{n-i}}(A) + W_i^{\frac{p}{n-i}}(B)$$

which is the inequality of Theorem 8.7 □

The limiting cases of Theorems 8.7 and 8.8 for the case where $p = 1$ hold due to the Aleksandrov inequality (Theorem 2.4).

The well known Fundamental Inequality of Mixed Quermassintegrals (see [7]) stated below is the limiting case of Theorem 8.8.

Theorem 8.9 Fundamental Inequality of Mixed Quermassintegrals *For A, $B \in M_n^{s,+}$ and $0 \leq i < n-1$, $W_i^{n-i}(A,B) \geq W_i^{n-i-1}(A)W_i(B)$ with equality if and only if $A = c \cdot B$, $c > 0$.*

Theorem 8.10 *Suppose $0 \leq i < n$ and A, $B \in M_n^{s,+}$ are such that $W_i(A) \leq W_i(B)$. Then*

(a) If $W_i(A) \geq W_{p,i}(A,B)$, for some $p > 1$, then $A = B$.

(b) If $W_i(A) \geq W_{p,i}(B,A)$, for some p such that $n-i > p > 1$ then $A = B$.

(c) If $W_i(B) \geq W_{p,i}(A,B)$, for some $p > n-i$, then $A = B$.

Proof.

(a) Since $W_i(A) \geq W_{p,i}(A,B)$ it follows from Theorem 8.8 that

$$W_i^{n-i}(A) \geq W_{p,i}^{n-i}(A,B) \geq W_i^{n-i-p}(A)W_i^p(B)$$

with equality in the right inequality if and only if $A = c\dot{B}$, $c \geq 0$. This string of inequalities implies that

$$W_i^p(A) \geq W_i^p(B)$$

or simply

$$W_i(A) \geq W_i(B).$$

But the hypothesis $W_i(A) \leq W_i(B)$ shows that there is in fact equality in both inequalities and that

$$W_i(A) = W_i(B).$$

We conclude that $A = B$.

(b) Since $W_i(A) \geq W_{p,i}(B,A)$ for some p, $n-i > p > 1$, it follows from Theorem 8.8 that

$$W_i^{n-i}(A) \geq W_{p,i}^{n-i}(B,A) \geq W_i^{n-i-p}(B)W_i^p(A)$$

with equality if and only if $A = c\dot{B}$, $c \geq 0$. This last inequality implies that

$$W_i^{n-i-p}(A) \geq W_i^{n-i-p}(B)$$

or simply

$$W_i(A) \geq W_i(B).$$

The condition $W_i(A) \leq W_i(B)$ implies that

$$W_i(A) = W_i(B)$$

and hence $A = B$.

(c) This follows identically from the proof of parts (a) and (b). □

Theorem 8.11 *Suppose A, $B_n \in M_n^{s,+}$. If $0 \leq i < n$, and $n-i \neq p > 1$ and if $W_{p,i}(A,Q) = W_{p,i}(B,Q)$ for all $Q \in M_n^{s,+}$, then $A = B$.*

Proof. Set $Q = A$, and get $W_i(A) = W_{p,i}(A,A) = W_{p,i}(B,A)$. Set $Q = B$, and get $W_i(B) = W_{p,i}(B,B) = W_{p,i}(A,B)$. From parts (b) and (c) of the last theorem we obtain $A = B$.

Theorem 8.12 *Suppose A, $B = M_n^{s,+}$ and $0 \leq i < n-1$. If $p = n-i$ and $W_i(A) \geq W_{p,i}(B,A)$, then $A = c \cdot B$, $c > 0$.*

Proof. From the hypothesis and Theorem 8.8 we have $W_i^{n-i}(A) \geq W_{p,i}^{n-i}(B,A) \geq W_i^p(B)W_i^p(A)$ with equality in the right inequality implying that $A = cB$, $c > 0$. Since $p = n - i$, we will have $W_i^{n-i}(A) \geq W_{n-i,i}^{n-i}(B,A) \geq W_i^{n-i}(A)$ so $W_{n-i,i}^{n-i}(B,A) = W_i^{n-i}(A)$. Hence, $A = c \cdot B$, $c > 0$. $\square$

Theorem 8.13 *Suppose* A, $B \in M_n^{s,+}$. *If* $0 \leq i < n-1$, $p = n - i$ *and* $W_{p,i}(A,Q) \geq W_{p,i}(B,Q)$ *for all* $Q \in M_n^{s,+}$. *Then* $W_{p,i}(A,Q) = W_{p,i}(B,Q)$ *for all* $Q \in M_n^{s,+}$.

Proof. From Theorem 8.8 and the hypothesis in this theorem we will have $W_{p,i}(B,Q) \geq W_i(Q) \geq W_{p,i}(A,Q)$, $0 \leq i < n-1$, $p = n-i$, for all $Q \in M_n^{s,+}$. Since $W_{p,i}(A,Q) \geq W_{p,i}(B,Q)$ for all $Q \in M_n^{s,+}$, combining these inequalities yield $W_{p,i}(A,Q) = W_{p,i}(B,Q)$ for all $Q \in M_n^{s,+}$. $\square$

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