Modeling of A Transport Network and its Spectral Analysis

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Abstract: In the present paper, we study a regular triangle bi-directed transport network by the approach of the operator semigroup theory and linear operator spectral theory. First we established a model of the partial differential equations for the transport problem. And then we prove the well-posedness of the network system. By the detailed spectral analysis of the system operator, we prove that the spectrum of the system operator is composed of isolated eigenvalue of finite multiplicity, all root vectors are incomplete in the state space. Finally, we discuss some operation strategies for the transport networks based on the spectral distribution.

Key–Words: triangle transport network, semigroup theory, spectral analysis, operation strategies.

1 Introduction

With the development of the modern technology and the world economy, many mathematical models have been developed to study the complex transport phenomena, for instance, [1] introduced in detail the existing traffic flow model, which can perfectly reproduce stop-and-go traffic, phase transitions, local clusters, traffic waves; [2] introduced the car-following model (also called Gips Model) and [3] studied its mathematical prosperities and signal control problem. However, these models cannot be directly used to investigate network flow. In the 1950s James Lighthill and Gerald Whitham in [4] and Richards in [5] proposed to apply fluid dynamics concepts to traffic. In a single road, this nonlinear model is based on the conservation of cars described by the scalar hyperbolic conservation law. We refer to [6, 7] for more details and comments on the single road models, [8, 9] for an updated account of the theory of general hyperbolic conservation laws and to [10, 11] for a standard introduction to the main ideas of numerical solution. We observed that in all these classical works on traffic flows, only a single road was taken into account. More recently, in [12, 13, 14, 15], some models have been proposed for traffic flow on highway networks.

Networks have been widely applied in classical natural sciences, for example, food-webs, electrical power grids, cellular and metabolic networks, chemical processes, neural networks, telephone call graphs, ecological webs, financial networks and the World-Wide Web in recent years. Much progress has been made in understanding the structure of these networks, and we refer to [16] for a survey on these development. Several discrete or combinatorial interactions in networks have been treated in graph theory, mostly with applications to Markov process [17]. In past decades the investigations of dynamic behavior of the dynamic graphs in which the edges do not only link vertices but also serve as a transmission media on which time-and space-depending process take place also have made greatly progress, we infer to [18, 19, 20] and the book [21].

In the present paper, we also are interested in the transport problem that will be described by the dynamical graphs. We study a triangular transport network, in which two vertices are connected by parallel edges, and on each edge the dynamic behavior of the system is described by the partial differential equation. Obviously, such a network is different from that in the works mentioned above. We mainly discuss the structure property of the transport network by employing the operator semigroup theory [22, 23] and linear operator spectral analysis approach [24]. From applicable point of view, we also consider the problem of operating strategy for the traffic problem.

The rest is organized as follows: In section 2, we establish the partial differential equation model for the network system under consideration, and give the complete description of the network at the junction. In section 3, we prove well-posedness of the system by linear operator semigroup theory, as a practice problem, we also consider the existence of the positive solution of the system. In section 4, we carry out a complete spectral analysis for the system operator. We describe the spectral distribution and the multiplicity of eigenvalue, and prove the incompleteness of the eigenvectors. In section 5, we give a simple analysis for operating strategy based on the dominant eigenvalue of $A + B$. In section 6, we conclude the result of the present paper.

2 Mathematical modeling for a transport problem

In this section we shall establish a mathematical model for triangle transport network in a region. Here we mainly consider the non-fixed site operation strategy. Such an operating strategy is similar to the mini bus (or taxi) whose character is waving-stop. With this operating strategy, the passenger can get on and get off at anywhere in the transport line. It forms a transport network. The most important character of this operation strategy is that the number of passengers on the transport line always varies at any time and at anywhere. Based on this fact, we choose the number of passengers on each transport line as the main research object. For simplicity, we also use linear equations to describe the process.

2.1 Mathematical modeling

Suppose that $a_j, j = 1, 2, 3$ are the transport sites in a region (see Figure 1), and the distance between sites a_i and a_j is 1, $i, j = 1, 2, 3$, which only is a normalized form.

At first we consider the transport flow on the line $\overrightarrow{a_1a_2}$. Suppose that the running speed of the vehicle on this line is c_{12} that is a constant for simplicity. Let s denote the distance from a point z on the line $\overrightarrow{a_1a_2}$ to a_1 (see Figure 1). In what follows, we will identify z and s.

We denote by $X_{12}(s, t)$ the number of passengers on the vehicle at time t and at position s. Suppose that the probability of passengers getting off at position s is $\mu_{12}(s)$, the probability of passengers getting on is $\nu_{12}(s)$. Then in the small time Δt, the number change of passengers on the vehicle satisfies the following balance relation

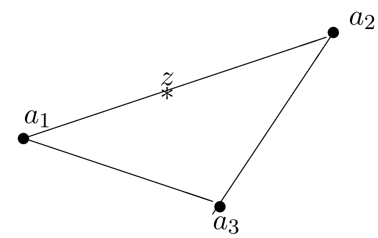

Figure 1: Triangular-shape transport line (the metric graph)

$$\begin{aligned} & X_{12}(s + c_{12}\Delta t, t + \Delta t) - X_{12}(s, t) \\ = \ & (\nu_{12}(s) - \mu_{12}(s))X_{12}(s, t)\Delta t + O(\Delta^2 t) \end{aligned}$$

Hence, the mean change rate is given by

$$\begin{aligned} & \frac{X_{12}(s + c_{12}\Delta t, t + \Delta t) - X_{12}(s, t)}{\Delta t} \\ = \ & (\nu_{12}(s) - \mu_{12}(s))X_{12}(s, t) + O(\Delta t). \end{aligned}$$

Taking $\Delta t \to 0$, we get a partial differential equation

$$\begin{aligned} & \frac{\partial X_{12}(s, t)}{\partial t} + c_{12}\frac{\partial X_{12}(s, t)}{\partial s} \\ = \ & (\nu_{12}(s) - \mu_{12}(s))X_{12}(s, t). \end{aligned}$$

Next we consider number change of passengers on the return vehicle, herein we regard a_2 as start point of the transport line. Denote by r a point on line $\overrightarrow{a_2a_1}$, which is also the distance from the point to a_2. Let $X_{21}(r, t)$ denote the number of passengers on the vehicle at time t and at position r. Assume that the running speed of the return vehicle is c_{21}; the probabilities of passengers getting off and getting on at position r are $\mu_{21}(r)$ and $\nu_{21}(r)$, respectively. Thus the number change of passengers in the small time Δt satisfies the relation

$$\begin{aligned} & X_{21}(r + c_{21}\Delta t, t + \Delta t) - X_{21}(r, t) \\ = \ & (\nu_{21}(r) - \mu_{21}(r))X_{21}(r, t)\Delta t + O(\Delta^2 t). \end{aligned}$$

From above we get a partial differential equation

$$\begin{aligned} & \frac{\partial X_{21}(r, t)}{\partial t} + c_{21}\frac{\partial X_{21}(r, t)}{\partial r} \\ = \ & (\nu_{21}(r) - \mu_{21}(r))X_{21}(r, t). \end{aligned}$$

In a similar manner, we can discuss the other transport lines. Similarly we can get the following partial differential equations

$$\begin{aligned} \frac{\partial X_{13}(s,t)}{\partial t} + c_{13}\frac{\partial X_{13}(s,t)}{\partial s} &= (\nu_{13}(s) - \mu_{13}(s))X_{13}(s,t), \\ \frac{\partial X_{31}(r,t)}{\partial t} + c_{31}\frac{\partial X_{31}(r,t)}{\partial r} &= (\nu_{31}(r) - \mu_{31}(r))X_{31}(r,t), \\ \frac{\partial X_{23}(s,t)}{\partial t} + c_{23}\frac{\partial X_{23}(s,t)}{\partial s} &= (\nu_{23}(s) - \mu_{23}(s))X_{23}(s,t), \\ \frac{\partial X_{32}(r,t)}{\partial t} + c_{32}\frac{\partial X_{32}(r,t)}{\partial r} &= (\nu_{32}(r) - \mu_{32}(r))X_{32}(r,t). \end{aligned}$$

with $s \in (0,1), r \in (0,1)$, where $X_{ij}(s,t)$ is the number of passengers on the vehicle from a_i to a_j, c_{ij} is the running speed of the vehicle, and $\mu_{ij}(r)$ and $\nu_{ij}(r)$ are the probabilities of passengers getting off and getting on, respectively.

The partial differential equations above describe the number change of passengers on the vehicle along each transport line. In what follows, we consider the number change of passengers at each site $a_j, j = 1,2,3$.

At each site, the passengers are composed of the following four parts:

1) the passengers coming from the other sites;

2) the new passengers who come from outside of the system (it is called the input);

3) the passengers departing for the other sites;

4) the passengers going out of the system (it is called the output).

At the site a_1, the numbers of arrival passengers are $X_{21}(1,t)$ and $X_{31}(1,t)$ respectively; the input number of passengers is $u_1(t)$. Therefore, the number of all passengers coming in is

$$X_{21}(1,t) + X_{31}(1,t) + u_1(t).$$

The numbers of passengers departing for the other sites are $X_{12}(0,t)$ and $X_{13}(0,t)$ respectively, and the number of passengers going out of the system is $\xi_1(t)$. Assume that the site does not keep passengers and let α_1 be the probability of arrival passengers leaving the system and β_1 be the distribution rate of passengers for different directions. Then we have

$$\xi_1(t) = \alpha_1[X_{21}(1,t) + X_{31}(1,t)],$$

$$X_{12}(0,t) = \beta_1[(1-\alpha_1)(X_{21}(1,t)+X_{31}(1,t))+u_1(t)],$$

and

$$X_{13}(0,t) = (1-\beta_1)[(1-\alpha_1)(X_{21}(1,t)+X_{31}(1,t))+u_1(t)].$$

Clearly, the relations meet the flow balance condition

$$X_{12}(0,t) + X_{13}(0,t) + \xi_1(t) = X_{21}(1,t) + X_{31}(1,t) + u_1(t).$$

At the site a_2, the numbers of passengers departing for the other sites are $X_{21}(0,t)$ and $X_{23}(0,t)$, the numbers of arrival passengers are $X_{12}(1,t)$ and $X_{32}(1,t)$ respectively, the number of passengers from outside is $u_2(t)$ and the number of passengers going out of the system is $\xi_2(t)$. Let α_2 be the probability of arrival passengers leaving the system and β_2 be the distribution rate of passengers for different directions. Thus we have

$$\xi_2(t) = \alpha_2[X_{12}(1,t) + X_{32}(1,t)],$$

$$X_{21}(0,t) = \beta_2[(1-\alpha_2)(X_{12}(1,t)+X_{32}(1,t))+u_2(t)],$$

and

$$X_{23}(0,t) = (1-\beta_2)[(1-\alpha_2)(X_{12}(1,t)+X_{32}(1,t))+u_2(t)].$$

Similarly, at the site a_3, let α_3 be the probability of passengers going out of the system and β_3 be the distribution rate of passengers for different directions. Then we have

$$\xi_3(t) = \alpha_3[X_{13}(1,t) + X_{23}(1,t)],$$

$$X_{31}(0,t) = \beta_3[(1-\alpha_3)(X_{13}(1,t)+X_{23}(1,t))+u_3(t)],$$

and

$$X_{32}(0,t) = (1-\beta_3)[(1-\alpha_3)(X_{13}(1,t)+X_{23}(1,t))+u_3(t)].$$

where $u_3(t)$ is the input of passengers and $\xi_3(t)$ is the output of passengers.

In addition we assume that distributions of passengers on each transport line at the initial moment are respectively

$$\begin{aligned} X_{12}(s,0) &= x_{12}(s), \quad X_{21}(r,0) = x_{21}(r), \\ X_{13}(s,0) &= x_{13}(s), \quad X_{31}(r,0) = x_{31}(r), \\ X_{23}(s,0) &= x_{23}(s), \quad X_{32}(r,0) = x_{32}(r). \end{aligned}$$

Thus, a full description of mathematical model for operation strategy of the non-fixed site is given by

$$\begin{cases} \frac{\partial X_{12}(s,t)}{\partial t} + c_{12}\frac{\partial X_{12}(s,t)}{\partial s} = (\nu_{12}(s) - \mu_{12}(s))X_{12}(s,t), \\ \frac{\partial X_{21}(r,t)}{\partial t} + c_{21}\frac{\partial X_{21}(r,t)}{\partial r} = (\nu_{21}(r) - \mu_{21}(r))X_{21}(r,t), \\ \frac{\partial X_{13}(s,t)}{\partial t} + c_{13}\frac{\partial X_{13}(s,t)}{\partial s} = (\nu_{13}(s) - \mu_{13}(s))X_{13}(s,t), \\ \frac{\partial X_{31}(r,t)}{\partial t} + c_{31}\frac{\partial X_{31}(r,t)}{\partial r} = (\nu_{31}(r) - \mu_{31}(r))X_{31}(r,t), \\ \frac{\partial X_{23}(s,t)}{\partial t} + c_{23}\frac{\partial X_{23}(s,t)}{\partial s} = (\nu_{23}(s) - \mu_{23}(s))X_{23}(s,t), \\ \frac{\partial X_{32}(r,t)}{\partial t} + c_{32}\frac{\partial X_{32}(r,t)}{\partial r} = (\nu_{32}(r) - \mu_{32}(r))X_{32}(r,t), \\ X_{12}(0,t) = \beta_1[(1-\alpha_1)(X_{21}(1,t)+X_{31}(1,t))+u_1(t)], \\ X_{13}(0,t) = (1-\beta_1)[(1-\alpha_1)(X_{21}(1,t)+X_{31}(1,t))+u_1(t)], \\ X_{21}(0,t) = \beta_2[(1-\alpha_2)(X_{12}(1,t)+X_{32}(1,t))+u_2(t)], \\ X_{23}(0,t) = (1-\beta_2)[(1-\alpha_2)(X_{12}(1,t)+X_{32}(1,t))+u_2(t)], \\ X_{31}(0,t) = \beta_3[(1-\alpha_3)(X_{13}(1,t)+X_{23}(1,t))+u_3(t)], \\ X_{32}(0,t) = (1-\beta_3)[(1-\alpha_3)(X_{13}(1,t)+X_{23}(1,t))+u_3(t)], \\ X_{12}(s,0) = x_{12}(s), \quad X_{21}(r,0) = x_{21}(r), \\ X_{13}(s,0) = x_{13}(s), \quad X_{31}(r,0) = x_{31}(r), \\ X_{23}(s,0) = x_{23}(s), \quad X_{32}(r,0) = x_{32}(r). \end{cases}$$

with $s \in (0,1), r \in (0,1)$, observing that the equations have nonhomogeneous boundary conditions, we can divide the system into two parts: one is a system with homogeneous boundary conditions and nonzero initial data, i.e.,

$$\begin{cases} \frac{\partial y_{12}(s,t)}{\partial t} + c_{12}\frac{\partial y_{12}(s,t)}{\partial s} = (\nu_{12}(s) - \mu_{12}(s))y_{12}(s,t), \\ \frac{\partial y_{21}(r,t)}{\partial t} + c_{21}\frac{\partial y_{21}(r,t)}{\partial r} = (\nu_{21}(r) - \mu_{21}(r))y_{21}(r,t), \\ \frac{\partial y_{13}(s,t)}{\partial t} + c_{13}\frac{\partial y_{13}(s,t)}{\partial s} = (\nu_{13}(s) - \mu_{13}(s))y_{13}(s,t), \\ \frac{\partial y_{31}(r,t)}{\partial t} + c_{31}\frac{\partial y_{31}(r,t)}{\partial r} = (\nu_{31}(r) - \mu_{31}(r))y_{31}(r,t), \\ \frac{\partial y_{23}(s,t)}{\partial t} + c_{23}\frac{\partial y_{23}(s,t)}{\partial s} = (\nu_{23}(s) - \mu_{23}(s))y_{23}(s,t), \\ \frac{\partial y_{32}(r,t)}{\partial t} + c_{32}\frac{\partial y_{32}(r,t)}{\partial r} = (\nu_{32}(r) - \mu_{32}(r))y_{32}(r,t), \\ y_{12}(0,t) = \beta_1[(1-\alpha_1)(y_{21}(1,t)+y_{31}(1,t))], \\ y_{13}(0,t) = (1-\beta_1)[(1-\alpha_1)(y_{21}(1,t)+y_{31}(1,t))], \\ y_{21}(0,t) = \beta_2[(1-\alpha_2)(y_{12}(1,t)+y_{32}(1,t))], \\ y_{23}(0,t) = (1-\beta_2)[(1-\alpha_2)(y_{12}(1,t)+y_{32}(1,t))], \\ y_{31}(0,t) = \beta_3[(1-\alpha_3)(y_{13}(1,t)+y_{23}(1,t))], \\ y_{32}(0,t) = (1-\beta_3)[(1-\alpha_3)(y_{13}(1,t)+y_{23}(1,t))], \\ y_{12}(s,0) = x_{12}(s), \quad y_{21}(r,0) = x_{21}(r), \\ y_{13}(s,0) = x_{13}(s), \quad y_{31}(r,0) = x_{31}(r), \\ y_{23}(s,0) = x_{23}(s), \quad y_{32}(r,0) = x_{32}(r). \end{cases} \tag{1}$$

with $s \in (0,1), r \in (0,1)$, and the other is a system with nonhomogeneous boundary conditions and zero initial data, i.e.,

$$\begin{cases} \frac{\partial z_{12}(s,t)}{\partial t} + c_{12}\frac{\partial z_{12}(s,t)}{\partial s} = (\nu_{12}(s) - \mu_{12}(s))z_{12}(s,t), \\ \frac{\partial z_{21}(r,t)}{\partial t} + c_{21}\frac{\partial z_{21}(r,t)}{\partial r} = (\nu_{21}(r) - \mu_{21}(r))z_{21}(r,t), \\ \frac{\partial z_{13}(s,t)}{\partial t} + c_{13}\frac{\partial z_{13}(s,t)}{\partial s} = (\nu_{13}(s) - \mu_{13}(s))z_{13}(s,t), \\ \frac{\partial z_{31}(r,t)}{\partial t} + c_{31}\frac{\partial z_{31}(r,t)}{\partial r} = (\nu_{31}(r) - \mu_{31}(r))z_{31}(r,t), \\ \frac{\partial z_{23}(s,t)}{\partial t} + c_{23}\frac{\partial z_{23}(s,t)}{\partial s} = (\nu_{23}(s) - \mu_{23}(s))z_{23}(s,t), \\ \frac{\partial z_{32}(r,t)}{\partial t} + c_{32}\frac{\partial z_{32}(r,t)}{\partial r} = (\nu_{32}(r) - \mu_{32}(r))z_{32}(r,t), \\ z_{12}(0,t) = \beta_1[(1-\alpha_1)(z_{21}(1,t) + z_{31}(1,t)) + u_1(t)], \\ z_{13}(0,t) = (1-\beta_1)[(1-\alpha_1)(z_{21}(1,t) + z_{31}(1,t)) + u_1(t)], \\ z_{21}(0,t) = \beta_2[(1-\alpha_2)(z_{12}(1,t) + z_{32}(1,t)) + u_2(t)], \\ z_{23}(0,t) = (1-\beta_2)[(1-\alpha_2)(z_{12}(1,t) + z_{32}(1,t)) + u_2(t)], \\ z_{31}(0,t) = \beta_3[(1-\alpha_3)(z_{13}(1,t) + z_{23}(1,t)) + u_3(t)], \\ z_{32}(0,t) = (1-\beta_3)[(1-\alpha_3)(z_{13}(1,t) + z_{23}(1,t)) + u_3(t)], \\ z_{12}(s,0) = 0, \quad z_{21}(r,0) = 0, \\ z_{13}(s,0) = 0, \quad z_{31}(r,0) = 0, \\ z_{23}(s,0) = 0, \quad z_{32}(r,0) = 0. \end{cases} \tag{2}$$

with $s \in (0,1), r \in (0,1)$, in the present paper, we mainly discuss the system (1).

2.2 Discussion

In the transport network, we have assumed that the passengers can get on and get off at anywhere in the transport lines. Since the parameters α_j, $j = 1,2,3$ are the probabilities of passengers going out of the system at site a_j, we can assume that $0 < \alpha_j < 1$, $j = 1,2,3$ from the practice point of view. This means that there always exists the output of the system. The parameters $\beta_i, j = 1,2,3$ have similar property, but they only describe the rate that the passenger can go for different directions. The $u_j(t), j = 1,2,3$, are the input of the system, and are the number of passengers at site a_j from outside, so they are nonnegative.

We define number

$$N_1(t) = \int_0^1 X_{12}(s,t)ds, \quad N_2(t) = \int_0^1 X_{21}(r,t)dr,$$

$$N_3(t) = \int_0^1 X_{23}(s,t)ds, \quad N_4(t) = \int_0^1 X_{32}(r,t)dr,$$

$$N_5(t) = \int_0^1 X_{31}(s,t)ds, \quad N_6(t) = \int_0^1 X_{13}(r,t)dr.$$

Clearly, $\sum_{j=1}^{6} N_j$ is the total number of passengers in the transport network at the moment t.

The model (1) describes the dynamic behavior of the transport network without input. Its behavior is determined mainly by the quantities $\int_0^1(\nu_{ij}(s) - \mu_{ij}(s))ds$. In this paper, we mainly discuss this case. The model (2) gives the effect of input on the transport network, including the effect of the quantities $\int_0^1(\nu_{ij}(s) - \mu_{ij}(s))ds$.

3 Well-posedness of the transport system

In the present paper we discuss model (1). For simplicity of notations, we set

$$\begin{array}{llll} x_1(s,t) = y_{12}(s,t), & c_1 = c_{12}, & \mu_1(s) = \mu_{12}(s), & \nu_1(s) = \nu_{12}(s), \\ x_2(s,t) = y_{21}(s,t), & c_2 = c_{21}, & \mu_2(s) = \mu_{21}(s), & \nu_2(s) = \nu_{21}(s), \\ x_3(s,t) = y_{23}(s,t), & c_3 = c_{23}, & \mu_3(s) = \mu_{23}(s), & \nu_3(s) = \nu_{23}(s), \\ x_4(s,t) = y_{32}(s,t), & c_4 = c_{32}, & \mu_4(s) = \mu_{32}(s), & \nu_4(s) = \nu_{32}(s), \\ x_5(s,t) = y_{31}(s,t), & c_5 = c_{31}, & \mu_5(s) = \mu_{31}(s), & \nu_5(s) = \nu_{31}(s), \\ x_6(s,t) = y_{13}(s,t), & c_6 = c_{13}, & \mu_6(s) = \mu_{13}(s), & \nu_6(s) = \nu_{13}(s) \end{array}$$

and

$$\begin{array}{lll} \omega_{11} = \beta_1, & \omega_{22} = \beta_2, & \omega_{23} = (1-\beta_2), \\ \omega_{16} = (1-\beta_1), & \omega_{34} = (1-\beta_3), & \omega_{35} = \beta_3, \\ k_1 = \omega_{11}(1-\alpha_1), & k_2 = \omega_{22}(1-\alpha_2), & k_3 = \omega_{23}(1-\alpha_2), \\ k_4 = \omega_{34}(1-\alpha_3), & k_5 = \omega_{35}(1-\alpha_3), & k_6 = \omega_{16}(1-\alpha_1). \end{array}$$

Then we have

$$\begin{cases} \frac{\partial x_j(s,t)}{\partial t} + c_j\frac{\partial x_j(s,t)}{\partial s} = (\nu_j(s) - \mu_j(s))x_j(s,t), \\ x_1(0,t) = k_1(x_2(1,t) + x_5(1,t)), \\ x_2(0,t) = k_2(x_1(1,t) + x_4(1,t)) \\ x_3(0,t) = k_3(x_1(1,t) + x_4(1,t)), \\ x_4(0,t) = k_4(x_3(1,t) + x_6(1,t)), \\ x_5(0,t) = k_5(x_3(1,t) + x_6(1,t)), \\ x_6(0,t) = k_6(x_2(1,t) + x_5(1,t)), \\ x_j(s,0) = x_{j,0}(s), \\ s \in (0,1), \quad t > 0, \quad j = 1,2\ldots,6. \end{cases} \tag{3}$$

We introduce the vector-valued function

$$X(s,t) = (x_j(s,t))^T$$

and define diagonal matrices

$$C = \operatorname{diag}(c_j),$$

$$V(s) = \operatorname{diag}(\nu_j(s)),$$
$$U(s) = \operatorname{diag}(\mu_j(s)).$$

with $j = 1,2,\ldots,6$ and a transmission matrix

$$\Gamma = \begin{pmatrix} 0 & k_1 & 0 & 0 & k_1 & 0 \\ k_2 & 0 & 0 & k_2 & 0 & 0 \\ k_3 & 0 & 0 & k_3 & 0 & 0 \\ 0 & 0 & k_4 & 0 & 0 & k_4 \\ 0 & 0 & k_5 & 0 & 0 & k_5 \\ 0 & k_6 & 0 & 0 & k_6 & 0 \end{pmatrix}$$

Then the differential equations in (3) can be rewritten into

$$\frac{\partial X(s,t)}{\partial t} + C\frac{\partial X(s,t)}{\partial s} = (V(s) - U(s))X(s,t)$$

and the boundary conditions in (3) can be written as

$$X(0,t) = \Gamma X(1,t).$$

Thus (3) can be rewritten as a vector-valued partial differential equations

$$\begin{cases} \frac{\partial X(s,t)}{\partial t} + C\frac{\partial X(s,t)}{\partial s} = [V(s) - U(s)]X(s,t) \\ X(0,t) = \Gamma X(1,t) \\ X(s,0) = X_0(s) \end{cases} \tag{4}$$

In what follows, we shall discuss the well-posedness of the system (4). Based on the physical meaning of the problem, we take the state space

$$\mathbb{X} = (L^1[0,1])^6,$$

For each $F = (f_1, f_2, f_3, f_4, f_5, f_6) \in \mathbb{X}$, we define the weighted norm

$$\|F\|_{\mathbb{X}} = \sum_{j=1}^{6} \frac{1}{c_j} \|f_j\|_{L^1[0,1]}.$$

Obviously, $\mathbb{X}$ is a Banach space.

We define an operator A in space $\mathbb{X}$ by

$$A = -C\frac{d}{ds} = \mathrm{diag}(-c_j\frac{d}{ds}), \quad j = 1, 2, \ldots, 6$$

with domain

$$D(A) = \{F = (f_j)_{j=1,2\ldots 6} \in (W^{1,1}[0,1])^6 \mid F(0) = \Gamma F(1)\}.$$

Define an operator $B : \mathbb{X} \to \mathbb{X}$ by

$$B = V(s) - U(s) = \mathrm{diag}(\nu_j(s) - \mu_j(s)), \quad j = 1, 2, \ldots, 6$$

with domain $D(B) = \mathbb{X}$.

With the help of these notations, we can write the equation (4) into an evolutionary equation in $\mathbb{X}$

$$\begin{cases} \frac{d}{dt}X(t) = (A + B)X(t), t \geq 0, \\ X(0) = X_0 = (x_{10}(s), x_{20}(s), \ldots, x_{60}(s)). \end{cases} \tag{5}$$

First we have the following result.

Theorem 1. *Let A be defined as before. Then the system operator A is a closed and densely defined linear operator in $\mathbb{X}$.*

This is a simple verification, we omit the detail.

Theorem 2. *Let A be defined as before. Then the following assertions hold*

(I) A is a dissipative operator in $\mathbb{X}$.

(II) It holds that $\mathbb{C}_+ = \left\{\gamma \in \mathbb{C} \mid \Re\gamma \geq 0\right\} \subset \rho(A)$.

Proof. A direct verification shows that the dual space of $\mathbb{X}$ is $\mathbb{X}^* = (L^\infty[0,1])^6$ with the norm for $F \in \mathbb{X}^*$

$$\|F\| = \max\left\{\|f_1\|_\infty, \|f_2\|_\infty, \cdots, \|f_6\|_\infty\right\}$$

and for any $P \in \mathbb{X}$ and $F \in \mathbb{X}^*$, the dual product is defined as

$$\langle P, F\rangle_{\mathbb{X},\mathbb{X}^*} = \langle P, F\rangle_c = \sum_{j=1}^{6} \frac{1}{c_j} \int_0^1 p_j(s) f_j(s) ds.$$

Step 1. A is dissipative operator in $\mathbb{X}$.

For any real $P = (p_1, \cdots, p_6) \in D(A)$, we define $Q = (q_1, \cdots, q_6)$ where $q_j = \|P\|\mathrm{sign}(p_j), j = 1, 2, \cdots, 6$. Obviously, $Q \in \mathbb{X}^*$ and

$$Q \in \mathcal{F}(P) = \left\{Q \in \mathbb{X}^* \mid \langle P, Q\rangle = ||P||^2 = ||Q||^2\right\}.$$

In addition,

$$\begin{aligned} \frac{\langle AP,Q\rangle_c}{\|P\|} &= -\sum_{j=1}^{6} \int_0^1 \frac{dp_j(s)}{ds}\mathrm{sign}(p_j(s))ds \\ &= \sum_{j=1}^{6}(|p_j(0)| - |p_j(1)|). \end{aligned}$$

Using the boundary conditions

$$\begin{aligned} p_1(0) &= k_1(p_2(1) + p_5(1)), \\ p_2(0) &= k_2(p_1(1) + p_4(1)), \\ p_3(0) &= k_3(p_1(1) + p_4(1)), \\ p_4(0) &= k_4(p_3(1) + p_6(1)), \\ p_5(0) &= k_5(p_3(1) + p_6(1)), \\ p_6(0) &= k_6(p_2(1) + p_5(1)), \end{aligned} \tag{6}$$

we have

$$\begin{aligned} &|p_1(0)| + |p_6(0)| \leq (1 - \alpha_1)(|p_2(1)| + |p_5(1)|), \\ &|p_2(0)| + |p_3(0)| \leq (1 - \alpha_2)(|p_1(1)| + |p_4(1)|), \\ &|p_4(0)| + |p_5(0)| \leq (1 - \alpha_3)(|p_3(1)| + |p_6(1)|). \end{aligned}$$

So we have

$$\begin{aligned} & \sum_{j=1}^{6}(|p_j(0)| - |p_j(1)|) \\ \leq\ & -\alpha_1(|p_2(1)| + |p_5(1)|) - \alpha_2(|p_1(1)| \\ & +|p_4(1)|) - \alpha_3(|p_3(1)| + |p_6(1)|) \\ <\ & 0. \end{aligned}$$

Therefore, we have $\frac{\langle AP,Q\rangle}{\|P\|} < 0$. So, A is dissipative.

Step 2. Set $\mathbb{C}_+ = \left\{\gamma \in \mathbb{C} \mid \Re\gamma \geq 0\right\}$, then $\mathbb{C}_+ \subset \rho(A)$.

For any $F \in \mathbb{X}$ and $\gamma \in \mathbb{C}$, we consider the resolvent equation $(\gamma I - A)P = F$, that is

$$(\gamma + c_i\frac{d}{ds})p_i(s) = f_i(s), \quad i = 1, 2 \ldots, 6$$

Obviously, the ordinary differential equations have a general solution

$$p_i(s) = p_i(0)e^{-\frac{\gamma}{c_i}s} + \frac{1}{c_i}\int_0^s f_i(t)e^{-\frac{\gamma}{c_i}(s-t)}dt, \quad (7)$$

with $i = 1, 2\ldots, 6$, substituting (7) into (6) leads to algebraic equations

$$\begin{aligned}
& p_1(0) - k_1[p_2(0)e^{-\frac{\gamma}{c_2}} + p_5(0)e^{-\frac{\gamma}{c_5}}] \\
= \ & k_1[\tfrac{1}{c_2}\textstyle\int_0^1 f_2(t)e^{-\frac{\gamma}{c_2}(1-t)}dt + \tfrac{1}{c_5}\int_0^1 f_5(t)e^{-\frac{\gamma}{c_5}(1-t)}dt] \\
& p_2(0) - k_2[p_1(0)e^{-\frac{\gamma}{c_1}} + p_4(0)e^{-\frac{\gamma}{c_4}}] \\
= \ & k_2[\tfrac{1}{c_1}\textstyle\int_0^1 f_1(t)e^{-\frac{\gamma}{c_1}(1-t)}dt + \tfrac{1}{c_4}\int_0^1 f_4(t)e^{-\frac{\gamma}{c_4}(1-t)}dt] \\
& p_3(0) - k_3[p_1(0)e^{-\frac{\gamma}{c_1}} + p_4(0)e^{-\frac{\gamma}{c_4}}] \\
= \ & k_3[\tfrac{1}{c_1}\textstyle\int_0^1 f_1(t)e^{-\frac{\gamma}{c_1}(1-t)}dt + \tfrac{1}{c_4}\int_0^1 f_4(t)e^{-\frac{\gamma}{c_4}(1-t)}dt] \\
& p_4(0) - k_4[p_3(0)e^{-\frac{\gamma}{c_3}} + p_6(0)e^{-\frac{\gamma}{c_6}}] \\
= \ & k_4[\tfrac{1}{c_3}\textstyle\int_0^1 f_3(t)e^{-\frac{\gamma}{c_3}(1-t)}dt + \tfrac{1}{c_6}\int_0^1 f_6(t)e^{-\frac{\gamma}{c_6}(1-t)}dt] \\
& p_5(0) - k_5[p_3(0)e^{-\frac{\gamma}{c_3}} + p_6(0)e^{-\frac{\gamma}{c_6}}] \\
= \ & k_5[\tfrac{1}{c_3}\textstyle\int_0^1 f_3(t)e^{-\frac{\gamma}{c_3}(1-t)}dt + \tfrac{1}{c_6}\int_0^1 f_6(t)e^{-\frac{\gamma}{c_6}(1-t)}dt] \\
& p_6(0) - k_6[p_2(0)e^{-\frac{\gamma}{c_2}} + p_5(0)e^{-\frac{\gamma}{c_5}}] \\
= \ & k_6[\tfrac{1}{c_2}\textstyle\int_0^1 f_2(t)e^{-\frac{\gamma}{c_2}(1-t)}dt + \tfrac{1}{c_5}\int_0^1 f_5(t)e^{-\frac{\gamma}{c_5}(1-t)}dt]
\end{aligned}$$

where $(p_1(0), p_2(0), \ldots p_6(0))$ will be determined later. Set $D(\gamma)$ denote matrix

$$\begin{pmatrix}
1 & -k_1e^{-\frac{\gamma}{c_2}} & 0 & 0 & -k_1e^{-\frac{\gamma}{c_5}} & 0 \\
-k_2e^{-\frac{\gamma}{c_1}} & 1 & 0 & -k_2e^{-\frac{\gamma}{c_4}} & 0 & 0 \\
-k_3e^{-\frac{\gamma}{c_1}} & 0 & 1 & -k_3e^{-\frac{\gamma}{c_4}} & 0 & 0 \\
0 & 0 & -k_4e^{-\frac{\gamma}{c_3}} & 1 & 0 & -k_4e^{-\frac{\gamma}{c_6}} \\
0 & 0 & -k_5e^{-\frac{\gamma}{c_3}} & 0 & 1 & -k_5e^{-\frac{\gamma}{c_6}} \\
0 & -k_6e^{-\frac{\gamma}{c_2}} & 0 & 0 & -k_6e^{-\frac{\gamma}{c_5}} & 1
\end{pmatrix}$$

Clearly, the algebraic equations have a unique solution if and only if $\det D(\gamma) \neq 0$. When $\det D(\gamma) \neq 0$, the algebraic equations have a unique solution $(p_1(0), p_2(0),$
$\cdots, p_6(0))$ that implies that the resolvent equation has a unique solution, and hence $\gamma \in \rho(A)$. Therefore, we have

$$\sigma(A) = \{\gamma \in \mathbb{C} \mid \det D(\gamma) = 0\}.$$

A direct calculation gives

$$\begin{aligned}
& \det D(\gamma) \\
= \ & 1 - k_1k_2e^{-(\frac{1}{c_1}+\frac{1}{c_2})\gamma} - k_3k_4e^{-(\frac{1}{c_3}+\frac{1}{c_4})\gamma} \\
& -k_5k_6e^{-(\frac{1}{c_5}+\frac{1}{c_6})\gamma} - k_1k_3k_5e^{-(\frac{1}{c_1}+\frac{1}{c_3}+\frac{1}{c_5})\gamma} \\
& -k_2k_4k_6e^{-(\frac{1}{c_2}+\frac{1}{c_4}+\frac{1}{c_6})\gamma},
\end{aligned} \quad (8)$$

When $\Re\gamma \geq 0$,

$$\begin{aligned}
& |\det D(\gamma)| \\
\geq \ & 1 - k_1k_2|e^{-(\frac{1}{c_1}+\frac{1}{c_2})\gamma}| - k_3k_4|e^{-(\frac{1}{c_3}+\frac{1}{c_4})\gamma}| \\
& -k_5k_6|e^{-(\frac{1}{c_5}+\frac{1}{c_6})\gamma}| - k_1k_3k_5|e^{-(\frac{1}{c_1}+\frac{1}{c_3}+\frac{1}{c_5})\gamma}| \\
& -k_2k_4k_6|e^{-(\frac{1}{c_2}+\frac{1}{c_4}+\frac{1}{c_6})\gamma}| \\
\geq \ & 1 - [k_1k_2 + k_3k_4 + k_5k_6 + k_1k_3k_5 + k_2k_4k_6]H(\gamma)
\end{aligned}$$

where

$$\begin{aligned}
& H(\gamma) \\
= \ & \max\{|e^{-(\frac{1}{c_1}+\frac{1}{c_2})\gamma}|, |e^{-(\frac{1}{c_3}+\frac{1}{c_4})\gamma}|, |e^{-(\frac{1}{c_5}+\frac{1}{c_6})\gamma}|, \\
& |e^{-(\frac{1}{c_1}+\frac{1}{c_3}+\frac{1}{c_5})\gamma}|, |e^{-(\frac{1}{c_2}+\frac{1}{c_4}+\frac{1}{c_6})\gamma}|\}.
\end{aligned}$$

Since $0 < \alpha_j < 1, j = 1, 2, 3$, and $\omega_{11}\omega_{22} + \omega_{23}\omega_{34} + \omega_{16}\omega_{35} + \omega_{11}\omega_{23}\omega_{35} + \omega_{16}\omega_{22}\omega_{34} = 1$, and $H(\gamma) \leq 1$ as $\Re\gamma \geq 0$, so we have $|\det D(\gamma)| > 0$. Therefore, $\{\gamma \in \mathbb{C} \mid \Re\gamma \geq 0\} \subset \rho(A)$. □

Theorem 3. *Let A be defined as before, then A generates a C_0 semigroup $T(t)$ on $\mathbb{X}$.*

Proof. The semigroup theory of bounded linear operators (see [22]) asserts that A generates a C_0 semigroup $T(t)_{t\in\mathbb{R}}$ on $\mathbb{X}$. □

Theorem 4. *Let A and B be defined as before, then $A+B$ generates a C_0 semigroup on $\mathbb{X}$, and hence the abstract Cauchy problem (5) has a unique solution.*

Proof. It is easy to prove that B is a bounded operator, since $B : \mathbb{X} \to \mathbb{X}$, $\|B\| \leq \max_{1\leq j\leq 6}(|\nu_j - \mu_j|) = 1$. The perturbation theory of semigroup asserts that $A + B$ also generates a C_0 semigroup. Hence, the abstract Cauchy problem (5) has a unique solution. □

Note that model (3) describes a practice problem, the solution is the passenger number. So we have to prove that the equation has a positive solution. To this end, we set

$$\mathbb{X}_+ = \{F = (f_j) \in \mathbb{X} \mid f_j(s) \geq 0, \forall 1 \leq j \leq 6\},$$

Clearly, $\mathbb{X}_+$ is a positive cone, and $(\mathbb{X}, \mathbb{X}_+)$ is a Banach lattice (see [24]).

A bounded linear operator T on Banach lattice $\mathbb{X}$ is said to be a positive operator if $T\mathbb{X}_+ \subset \mathbb{X}_+$. A C_0 semigroup $T(t)$ is said to be a positive semigroup if for all $t \geq 0$, $T(t)\mathbb{X}_+ \subset \mathbb{X}_+$ (see [23]).

The following theorem ensures the existence of positive solution to (5).

Theorem 5. *Let $\mathbb{X}$, A and B be defined as before, and $T(t)$ be the C_0 semigroup generated by $A + B$. Then $T(t)$ is a positive semigroup on $\mathbb{X}$. And hence the equation has a unique positive solution for $X_0(s) \in \mathbb{X}_+$.*

Proof. Let $T(t)$ be the C_0 semigroup generated by $A+B$. According to the positive semigroup theory (see [23]), we only need to show $A+B-bI$ is a dispersive operator in $\mathbb{X}$ and $R(I-(A+B-bI)) = \mathbb{X}$, where $b = \max_j \max_s |\nu_j(s) - \mu_j(s)|$. Since $A+B$ generates a C_0 semigroup that implies $R(I-(A+B-bI)) = \mathbb{X}$, we only need to show $A+B-bI$ is a dispersive operator in $\mathbb{X}$.

For any real $P = (p_1, p_2, \cdots, p_6) \in D(A+B) = D(A)$, we set $Q = \|P\|(q_1, \cdots, q_6)$, where $q_j = \text{sign}_+(p_j), j = 1, 2, \cdots, 6$, and

$$\text{sign}_+(p_j) = \begin{cases} 1, & p_j > 0, \\ 0, & p_j \le 0. \end{cases}$$

Similarly, as in the proof of Theorem 2, we calculate the value $\langle (A+B)P, Q\rangle$ as follows

$$\begin{aligned}
&\frac{\langle (A+B)P,Q\rangle}{\|P\|} \\
= \; & -\sum_{j=1}^{6} \int_0^1 \frac{dp_j(s)}{ds}\text{sign}_+(p_j(s))ds \\
& + \sum_{j=1}^{6} \frac{1}{c_j} \int_0^1 (\nu_j(s) - \mu_j(s)) p_j^+(s) ds \\
\le \; & -\alpha_1(|p_2(1)| + |p_5(1)|) - \alpha_2(|p_1(1)| \\
& + |p_4(1)|) - \alpha_3(|p_3(1)| + |p_6(1)|) \\
& + \max_j \max_s |\nu_j(s) - \mu_j(s)| \sum_{j=1}^{6} \frac{\int_0^1 p_j^+(s)ds}{c_j} \\
< \; & \max_j \max_s |\nu_j(s) - \mu_j(s)| \sum_{j=1}^{6} \frac{\int_0^1 p_j^+(s)ds}{c_j} \\
= \; & b\frac{\langle P,Q\rangle}{\|P\|}.
\end{aligned}$$

So, $\langle (A+B-bI)P, Q\rangle < 0$, this means that $A+B$ is dispersive. According to [22], $A+B-bI$ generates a C_0 positive semigroup $e^{-bt}T(t)$. So $A+B$ generates a C_0 positive semigroup $T(t)$. □

4 Spectral analysis of $A+B$

In this section, we shall carry out a complete spectral analysis for operator $A+B$. We observe from Step 2 in the proof of Theorem 2 that for $\Re\gamma > 0$, $R(\gamma, A)$ is a compact operator on $\mathbb{X}$, and B is a bounded linear operator on $\mathbb{X}$. So for $\lambda \in \rho(A+B)$, $R(\lambda, A+B)$ also is a compact operator on $\mathbb{X}$. Therefore, we have the following result.

Theorem 6. *Let A and B be defined as before. Then $A+B$ is a resolvent compact operator, and hence its spectrum of $A+B$ consists of all isolated eigenvalues of finite multiplicity, i.e., $\sigma(A+B) = \sigma_p(A+B)$.*

Based on the above result, we only need to discuss the eigenvalue problem of $A+B$.

4.1 Eigenvalue problem

In this subsection, we study the eigenvalues of $A+B$ and its distribution. For simplicity, we denote $v_j(s) = -[\nu_j(s) - \mu_j(s)], j = 1, 2, \cdots, 6$.

For $\lambda \in \mathbb{C}$, we consider the eigenvalue problem of $A+B$, i.e., $(\lambda I - A - B)P = 0$, whose analytical expression is given by

$$\begin{cases}
\lambda p_j(s) + c_j p_j'(s) + v_j(s)p_j(s) = 0, \; s \in (0,1), \\
p_1(0) = k_1(p_2(1) + p_5(1)), \\
p_2(0) = k_2(p_1(1) + p_4(1)), \\
p_3(0) = k_3(p_1(1) + p_4(1)), \\
p_4(0) = k_4(p_3(1) + p_6(1)), \\
p_5(0) = k_5(p_3(1) + p_6(1)), \\
p_6(0) = k_6(p_2(1) + p_5(1)).
\end{cases} \tag{9}$$

with $j = 1, 2, \cdots, 6$, Obviously, the differential equations in (9) have a general solution

$$p_j(s) = p_j(0)e^{-\frac{1}{c_j}\int_0^s [\lambda + v_j(t)]dt}, \quad j = 1, 2, \cdots, 6. \tag{10}$$

Substituting (10) in the boundary conditions leads to the following algebraic equations

$$G(\lambda)\begin{pmatrix} p_1(0) \\ p_2(0) \\ p_3(0) \\ p_4(0) \\ p_5(0) \\ p_6(0) \end{pmatrix} = 0 \tag{11}$$

where $G(\lambda)$ denote matrix

$$\left(\begin{array}{cccccc}
1 & -k_1E_2(\lambda) & 0 & 0 & -k_1E_5(\lambda) & 0 \\
-k_2E_1(\lambda) & 1 & 0 & -k_2E_4(\lambda) & 0 & 0 \\
-k_3E_1(\lambda) & 0 & 1 & -k_3E_4(\lambda) & 0 & 0 \\
0 & 0 & -k_4E_3(\lambda) & 1 & 0 & -k_4E_6(\lambda) \\
0 & 0 & -k_5E_3(\lambda) & 0 & 1 & -k_5E_6(\lambda) \\
0 & -k_6E_2(\lambda) & 0 & 0 & -k_6E_5(\lambda) & 1
\end{array}\right)$$

with $E_j(\lambda) = e^{-\frac{1}{c_j}\int_0^1 [\lambda + v_j(s)]ds}$(j=1,2..., 6). Clearly, the algebraic equation (11) has a nonzero solution if and only if the determinant of the coefficients matrix vanishes, i.e.,$|G(\lambda)| = 0$. For simplicity, we set

$$\widehat{v_j} = \int_0^1 v_j(s)ds, \quad j = 1, 2 \ldots, 6.$$

Then we have

$$E_j(\lambda) = e^{-\frac{1}{c_j}\int_0^1 [\lambda + v_j(s)]ds} = e^{-\frac{1}{c_j}(\lambda + \widehat{v_j})}.$$

with $j = 1, 2 \dots, 6$, a direct calculation gives

$$\begin{aligned} & M(\lambda) \\ = & |G(\lambda)| \\ = & 1 - k_1k_2E_1(\lambda)E_2(\lambda) \\ & -k_3k_4E_3(\lambda)E_4(\lambda) - k_5k_6E_5(\lambda)E_6(\lambda) \\ & -k_1k_3k_5E_1(\lambda)E_3(\lambda)E_5(\lambda) \\ & -k_2k_4k_6E_2(\lambda)E_4(\lambda)E_6(\lambda) \end{aligned} \tag{12}$$

Obviously, if λ is an zero of $M(\lambda)$, then (11) has a nonzero solution. Hence the nonzero functions given in (10) forms a solution to (9). So λ also is an eigenvalue of $A + B$. Therefore, we only need to discuss the zeros of $M(\lambda)$.

Theorem 7. *Let A and B be defined as before. Then the following statements hold*

1) The point spectrum of $A + B$, $\sigma_p(A + B)$, is given by

$$\sigma_p(A + B) = \{\lambda \in \mathbb{C} \mid M(\lambda) = 0\}.$$

2) There exists a positive constant $h > 0$ such that

$$\sigma_p(A + B) \subset \{\lambda \in \mathbb{C} \mid -h \le \Re\lambda \le h\}.$$

Proof. The first assertion is obvious, we only prove the second assertion.

For $\Re\lambda > 0$, when $\Re\lambda > -\max\{\widehat{v}_j, j = 1, 2, \cdots, 6\}$, we have $\Re\lambda + \widehat{v}_j > 0$, which implies $\lim_{\Re\lambda \to \infty} E_j(\lambda) = 0$. Therefore,

$$\lim_{\Re\lambda \to +\infty} M(\lambda) = 1.$$

Set

$$\begin{aligned} m = & \max\{\frac{1}{c_1} + \frac{1}{c_2}, \frac{1}{c_3} + \frac{1}{c_4}, \frac{1}{c_5} + \frac{1}{c_6}, \\ & \frac{1}{c_1} + \frac{1}{c_3} + \frac{1}{c_5}, \frac{1}{c_2} + \frac{1}{c_4} + \frac{1}{c_6}\}. \end{aligned}$$

For $\Re\lambda < 0$, we have

$$\begin{aligned} & e^{m\lambda}M(\lambda) \\ = & e^{m\lambda} - k_1k_2e^{-(\frac{\widehat{v_1}}{c_1} + \frac{\widehat{v_2}}{c_2})}e^{(m - \frac{1}{c_1} - \frac{1}{c_2})\lambda} \\ & -k_3k_4e^{-(\frac{\widehat{v_3}}{c_3} + \frac{\widehat{v_4}}{c_4})}e^{(m - \frac{1}{c_3} - \frac{1}{c_4})\lambda} \\ & -k_5k_6e^{-(\frac{\widehat{v_5}}{c_5} + \frac{\widehat{v_6}}{c_6})}e^{(m - \frac{1}{c_5} - \frac{1}{c_6})\lambda} \\ & -k_1k_3k_5e^{-(\frac{\widehat{v_1}}{c_1} + \frac{\widehat{v_3}}{c_3} + \frac{\widehat{v_5}}{c_5})}e^{(m - \frac{1}{c_1} - \frac{1}{c_3} - \frac{1}{c_5})\lambda} \\ & -k_2k_4k_6e^{-(\frac{\widehat{v_2}}{c_2} + \frac{\widehat{v_4}}{c_4} + \frac{\widehat{v_6}}{c_6})}e^{(m - \frac{1}{c_2} - \frac{1}{c_4} - \frac{1}{c_6})\lambda}, \end{aligned}$$

and hence

$$\lim_{\Re\lambda \to -\infty} e^{m\lambda}M(\lambda) \neq 0$$

So we can find a positive constant h and positive constants d_1 and d_2 such that when $|\Re\lambda| > h$, it holds

$$d_1e^{m\Re\lambda} \le |M(\lambda)| \le d_2.$$

The desired result follows from above inequality. □

Corollary 8. *Let A and B be defined as before. Then the spectrum of $A + B$ distributes symmetrically with respect to the real axis.*

Note that $M(\lambda) = M(\bar{\lambda})$ for all $\lambda \in C$. So the result of Corollary 8 is obvious.

Corollary 9. *Let A and B be defined as before. If all $\widehat{v_j} = 0$, then there exists a positive constant $h > 0$ such that*

$$\sigma_p(A + B) \subset \{\lambda \in \mathbb{C} \mid -h \le \Re\lambda < 0\}.$$

Proof. If all $\widehat{v_j} = 0$, then (12) precisely has the form (8), the estimation in Theorem 7 gives $|M(\lambda)| > 0$ for all $\Re\lambda \ge 0$. The desired result follows. □

4.2 The geometric and algebraic multiplicity of eigenvalue of $A + B$

In this subsection, we discuss the geometric and algebraic multiplicity of eigenvalue of $A + B$. Let $\lambda \in \mathbb{C}$ such that $M(\lambda) = 0$, i.e., If there is at least one five order sub-matrix of the coefficient matrix of (11), whose the determinant is nonzero, for example,

$$\begin{vmatrix} 1 & -k_1E_2(\lambda) & 0 & 0 & -k_1E_5(\lambda) \\ -k_2E_1(\lambda) & 1 & 0 & -k_2E_4(\lambda) & 0 \\ -k_3E_1(\lambda) & 0 & 1 & -k_3E_4(\lambda) & 0 \\ 0 & 0 & -k_4E_3(\lambda) & 1 & 0 \\ 0 & 0 & -k_5E_3(\lambda) & 0 & 1 \end{vmatrix} \neq 0$$

then the geometric multiplicity of eigenvalue λ is one. In what follows we always assume that the geometric multiplicity of the eigenvalue of $A + B$ is one. To obtain the algebraic multiplicity of λ, we prove a general result.

Theorem 10. *Let $b_j, j = 1, 2, \cdots, m$ be a scaler group of distinct nonzero complex number, $a_j \neq 0, j = 1, 2, \cdots, m$. Then the zeros of the exponential polynomial*

$$p(z) = \sum_{j=1}^{m} a_je^{b_jz} - a_0$$

are at most of multiplicity m.

Proof. Let $\lambda \in \mathbb{C}$ such that $p(\lambda) = 0$. We consider the Taylor expansion of $p(z)$ at λ

$$\begin{aligned} & p(z) \\ = & p(\lambda) + \sum_{k=1}^{m-1} \frac{p^{(k)}(\lambda)}{k!}(z - \lambda)^k + \frac{p^{(m)}(\lambda)}{m!}(z - \lambda)^m + R(z) \end{aligned}$$

where

$$p^{(k)}(z) = \sum_{j=1}^{m} a_jb_j^ke^{b_jz}, \quad k = 1, 2, \cdots.$$

If $p^{(k)}(\lambda) = 0, k = 0, 1, 2, \cdots, m-1$, since the matrix

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ b_1 & b_2 & b_3 & \cdots & b_m \\ b_1^2 & b_2^2 & b_3^2 & \cdots & b_m^2 \\ \vdots & \ddots & \cdots & \cdots & \vdots \\ b_1^{m-1} & b_2^{m-1} & b_3^{m-1} & \cdots & b_m^{m-1} \end{pmatrix}_{m\times m}$$

is regular, the algebraic equations

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ b_1 & b_2 & \cdots & b_m \\ b_1^2 & b_2^2 & \cdots & b_m^2 \\ \vdots & \ddots & \cdots & \vdots \\ b_1^{m-1} & b_2^{m-1} & \cdots & b_m^{m-1} \end{pmatrix}_{m\times m} \begin{pmatrix} a_1 e^{b_1\lambda} \\ a_2 e^{b_2\lambda} \\ a_3 e^{b_3\lambda} \\ \vdots \\ a_m e^{b_m\lambda} \end{pmatrix} = \begin{pmatrix} a_0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

might have the solution

$$a_k e^{b_k\lambda} = \frac{a_0 \prod\limits_{j\neq k,}^{m} b_j \prod\limits_{i\neq j, i, j\neq k} (b_i - b_j)}{\prod\limits_{i\neq j}^{m} (b_i - b_j)}, j = 1, 2, \cdots, m.$$

This means that λ might be a m-order zero of $p(z)$.

But for any $\lambda \in \mathbb{C}$, it cannot make

$$p^{(k)}(\lambda) = 0, k = 1, 2, \cdots, m.$$

This is because the equations implies $a_j e^{b_j\lambda} = 0$ for all j. Obviously, it is impossible. □

Remark 11. *Theorem 10 shows that the zeros of $p(\lambda)$ are at most of m-order. If the scalar groups $\{b_j, j = 1, 2, \cdots, m\}$ and $\{a_j, j = 1, 2, \cdots, m\}$ satisfy certain conditions, for example, there exists an η such that ηb_j are integers, then the zeros of $p(z)$ are simple.*

Applying Theorem 10 to our model, we have the following result.

Theorem 12. *Let A and B be defined as before. For $\lambda \in \sigma(A+B)$, we denote by $m_a(\lambda)$ the algebraic multiplicity of λ. Then the following assertions are true.*

1) $\sup_{\lambda\in\sigma(A+B)} m_a(\lambda) \leq 5$;

2) If $c_j, j = 1, 2, 3\cdots, 6$, satisfy the conditions

$$\frac{1}{c_1} + \frac{1}{c_2} = \frac{1}{c_3} + \frac{1}{c_4} = \frac{1}{c_5} + \frac{1}{c_6} = b_1 \tag{13}$$

and

$$\frac{1}{c_1} + \frac{1}{c_3} + \frac{1}{c_5} = \frac{1}{c_2} + \frac{1}{c_4} + \frac{1}{c_6} = b_2 \tag{14}$$

then, all eigenvalues of $A+B$ are simple.

Proof. Note that

$$\begin{aligned} & M(\lambda) \\ = \ & 1 - k_1k_2 e^{-(\frac{\widehat{v_1}}{c_1} + \frac{\widehat{v_2}}{c_2})} e^{-(\frac{1}{c_1} + \frac{1}{c_2})\lambda} \\ & -k_3k_4 e^{-(\frac{\widehat{v_3}}{c_3} + \frac{\widehat{v_4}}{c_4})} e^{-(\frac{1}{c_3} + \frac{1}{c_4})\lambda} \\ & -k_5k_6 e^{-(\frac{\widehat{v_5}}{c_5} + \frac{\widehat{v_6}}{c_6})} e^{-(\frac{1}{c_5} + \frac{1}{c_6})\lambda} \\ & -k_1k_3k_5 e^{-(\frac{\widehat{v_1}}{c_1} + \frac{\widehat{v_3}}{c_3} + \frac{\widehat{v_5}}{c_5})} e^{-(\frac{1}{c_1} + \frac{1}{c_3} + \frac{1}{c_5})\lambda} \\ & -k_2k_4k_6 e^{-(\frac{\widehat{v_2}}{c_2} + \frac{\widehat{v_4}}{c_4} + \frac{\widehat{v_6}}{c_6})} e^{-(\frac{1}{c_2} + \frac{1}{c_4} + \frac{1}{c_6})\lambda}, \end{aligned}$$

it has the form as described in Theorem 10. Since we have shown that the geometric multiplicity of λ is one, so its algebraic multiplicity is equal to the order of zero of $M(\lambda)$. Applying Theorem 10 here $m = 5$, we have $m_a(\lambda) \leq 5$. The first assertion follows.

If the conditions (13) and (14) hold, then $2b_2 = 3b_1$, and $M(\lambda)$ has the form

$$M(\lambda) = 1 - a_1 e^{-b_1\lambda} - a_2 e^{-\frac{3}{2}b_1\lambda}$$

where

$$\begin{aligned} a_1 \ &= \ k_1k_2 e^{-(\frac{\widehat{v_1}}{c_1} + \frac{\widehat{v_2}}{c_2})} + k_3k_4 e^{-(\frac{\widehat{v_3}}{c_3} + \frac{\widehat{v_4}}{c_4})} \\ & \quad + k_5k_6 e^{-(\frac{\widehat{v_5}}{c_5} + \frac{\widehat{v_6}}{c_6})} \end{aligned}$$

and

$$\begin{aligned} a_2 \ &= \ k_1k_3k_5 e^{-(\frac{\widehat{v_1}}{c_1} + \frac{\widehat{v_3}}{c_3} + \frac{\widehat{v_5}}{c_5})} \\ & \quad + k_2k_4k_6 e^{-(\frac{\widehat{v_2}}{c_2} + \frac{\widehat{v_4}}{c_4} + \frac{\widehat{v_6}}{c_6})}. \end{aligned}$$

According to Theorem 10, the order of zeros of $M(\lambda)$ is at most two.

Set $z = e^{-\frac{b_1}{2}\lambda}$, then $M(\lambda) = 0$ is equivalent to

$$a_2 z^3 + a_1 z^2 - 1 = 0.$$

Note that $a_1 > 0$ and $a_2 > 0$. The algebraic equation has three distinct zeros z_1, z_2, z_3. Thus the zeros of $M(\lambda)$ are given by

$$\lambda_{j,n} = -\frac{2}{b_1} \ln |z_j| - \frac{2\varphi_j}{b_1} i - \frac{4n\pi}{b_1} i, \quad j = 1, 2, 3,$$

for $\forall n \in \mathbb{Z}$, where $\varphi_j = \arg(z_j)$. Therefore, all eigenvalues of $A+B$ are simple. □

4.3 Incompleteness of eigenvectors of $A+B$

In this section, we shall discuss the completeness problem of the spectrum of $A+B$. We say the spectrum of operator $A+B$ is complete in $\mathbb{X}$ if the span of its root vectors is dense in $\mathbb{X}$. Otherwise, it is said to be incomplete. In this subsection, we assume that all eigenvalues of $A+B$ are simple. Thus the completeness problem of the spectrum of $A+B$ becomes the dense problem of the span of its eigenvectors.

First, we find out the eigenvector of $A+B$ corresponding to $\lambda\in\sigma(A+B)$.

Theorem 13. *Let A and B be defined as before. Assume that all eigenvalues of $A+B$ are simple. Then for any $\lambda\in\sigma(A+B)$, then an eigenvector is $\Phi(\lambda)=(p_j(s,\lambda))_{j=1,2,\cdots,6}$ where*

$$\begin{cases} p_1(s)=k_1e^{-\int_0^s\frac{\lambda+v_1(r)}{c_1}dr}\\ p_2(s)=\frac{k_1k_2E_1(\lambda)+k_2k_4k_6E_4(\lambda)E_6(\lambda)}{1-k_3k_4E_3(\lambda)E_4(\lambda)}e^{-\int_0^s\frac{\lambda+v_2(r)}{c_2}dr}\\ p_3(s)=\frac{k_1k_3E_1(\lambda)+k_3k_4k_6E_4(\lambda)E_6(\lambda)}{1-k_3k_4E_3(\lambda)E_4(\lambda)}e^{-\int_0^s\frac{\lambda+v_3(r)}{c_3}dr}\\ p_4(s)=\frac{k_4k_6E_6(\lambda)+k_1k_3k_4E_1(\lambda)E_3(\lambda)}{1-k_3k_4E_3(\lambda)E_4(\lambda)}e^{-\int_0^s\frac{\lambda+v_4(r)}{c_4}dr}\\ p_5(s)=\frac{k_5k_6E_6(\lambda)+k_1k_3k_5E_1(\lambda)E_3(\lambda)}{1-k_3k_4E_3(\lambda)E_4(\lambda)}e^{-\int_0^s\frac{\lambda+v_5(r)}{c_5}dr}\\ p_6(s)=k_6e^{-\int_0^s\frac{\lambda+v_6(r)}{c_6}dr}\end{cases}\tag{15}$$

Proof. We consider the eigenvalue problem of $A+B$, i.e., $(A+B)\Phi(\lambda)=\lambda\Phi(\lambda)$, $\lambda\in\sigma(A+B)$. That is equivalent to the following equations have nonzero solution:

$$\begin{cases} -c_jp_j'(s)-v_j(s)p_j(s)=\lambda p_j(s),\ s\in(0,1),\\ p_1(0)=k_1(p_2(1)+p_5(1)),\\ p_2(0)=k_2(p_1(1)+p_4(1)),\\ p_3(0)=k_3(p_1(1)+p_4(1)),\\ p_4(0)=k_4(p_3(1)+p_6(1)),\\ p_5(0)=k_5(p_3(1)+p_6(1)),\\ p_6(0)=k_6(p_2(1)+p_5(1)),\end{cases}\tag{16}$$

with $j=1,2,\cdots,6$, obviously,

$$p_j(s)=p_j(0)e^{-\int_0^s\frac{\lambda+v_j(r)}{c_j}dr},j=1,2,\cdots,6$$

and $(p_1(0),p_2(0),\cdots,p_6(0))$ satisfies the following algebraic equations

$$\begin{cases} p_1(0)-k_1E_2(\lambda)p_2(0)-k_1E_5(\lambda)p_5(0)=0,\\ -k_2E_1(\lambda)p_1(0)+p_2(0)-k_2E_4(\lambda)p_4(0)=0,\\ -k_3E_1(\lambda)p_1(0)+p_3(0)-k_3E_4(\lambda)p_4(0)=0,\\ -k_4E_3(\lambda)p_3(0)+p_4(0)-k_4E_6(\lambda)p_6(0)=0,\\ -k_5E_3(\lambda)p_3(0)+p_5(0)-k_5E_6(\lambda)p_6(0)=0,\\ -k_6E_2(\lambda)p_2(0)-k_6E_5(\lambda)p_5(0)+p_6(0)=0.\end{cases}\tag{17}$$

Note that

$$\begin{aligned} p_1(0)+p_6(0)&=(1-\alpha_1)(p_2(0)E_2(\lambda)+p_5(0)E_5(\lambda)),\\ p_2(0)+p_3(0)&=(1-\alpha_2)(p_1(0)E_1(\lambda)+p_4(0)E_4(\lambda)),\\ p_4(0)+p_5(0)&=(1-\alpha_3)(p_3(0)E_3(\lambda)+p_6(0)E_6(\lambda)).\end{aligned}$$

Set

$$\begin{aligned} &p_1(0)=k_1\xi,p_6(0)=k_6\xi,p_2(0)=k_2\eta,\\ &p_3(0)=k_3\eta,p_4(0)=k_4\mu,p_5(0)=k_5\mu.\end{aligned}$$

Then we have

$$k_2E_2(\lambda)\eta+k_5E_5(\lambda)\mu=\xi,\tag{18}$$
$$k_1E_1(\lambda)\xi+k_4E_4(\lambda)\mu=\eta,\tag{19}$$
$$k_3E_3(\lambda)\eta+k_6E_6(\lambda)\xi=\mu.\tag{20}$$

From (19) and (20) we get that

$$\begin{aligned}\mu&=\frac{k_6E_6(\lambda)+k_1k_3E_1(\lambda)E_3(\lambda)}{1-k_3k_4E_3(\lambda)E_4(\lambda)}\xi,\\ \eta&=\frac{k_1E_1(\lambda)+k_4k_6E_4(\lambda)E_6(\lambda)}{1-k_3k_4E_3(\lambda)E_4(\lambda)}\xi.\end{aligned}$$

Since $\lambda\in\sigma(A+B)$, that is, $M(\lambda)=0$, substituting above into (18) verifies the equality. Therefore,

$$\begin{cases} p_1(0)=k_1\xi\\ p_2(0)=\frac{k_1k_2E_1(\lambda)+k_2k_4k_6E_4(\lambda)E_6(\lambda)}{1-k_3k_4E_3(\lambda)E_4(\lambda)}\xi\\ p_3(0)=\frac{k_1k_3E_1(\lambda)+k_3k_4k_6E_4(\lambda)E_6(\lambda)}{1-k_3k_4E_3(\lambda)E_4(\lambda)}\xi\\ p_4(0)=\frac{k_4k_6E_6(\lambda)+k_1k_3k_4E_1(\lambda)E_3(\lambda)}{1-k_3k_4E_3(\lambda)E_4(\lambda)}\xi\\ p_5(0)=\frac{k_5k_6E_6(\lambda)+k_1k_3k_5E_1(\lambda)E_3(\lambda)}{1-k_3k_4E_3(\lambda)E_4(\lambda)}\xi\\ p_6(0)=k_6\xi\end{cases}$$

Thus, Taking $\xi=1$, we got the formula (15). □

A direct computation shows that the dual operator $(A+B)^*$ of $A+B$ is of the form

$$(A+B)^*=A^*+B^*=\mathrm{diag}(c_j\frac{d}{ds}-v_j(s))\tag{21}$$

with domain

$$D((A+B)^*)=\left\{\begin{array}{l} Q=(q_1,q_2,\cdots,q_6),\\ q_j,q_j'\in(L^\infty[0,1])^6,j=1,2\dots,6,\\ q_1(1)=q_4(1)=k_2q_2(0)+k_3q_3(0),\\ q_2(1)=q_5(1)=k_1q_1(0)+k_6q_6(0),\\ q_3(1)=q_6(1)=k_4q_4(0)+k_5q_5(0).\end{array}\right\}\tag{22}$$

Theorem 14. *Let A and B be defined as before. Then $\sigma(A+B)=\sigma((A+B)^*)$ and for each $\lambda\in\sigma((A+B)^*)$, the corresponding eigenvector $\Psi(\lambda)=(q_j(s,\lambda))_{j=1,2,\ldots,6}$ satisfying $\langle\Phi(\lambda),\Psi(\lambda)\rangle_{\mathbb{X},\mathbb{X}^*}=1$ is of the components*

$$\begin{cases} q_1(s)=\frac{k_2E_1(\lambda)E_2(\lambda)+k_3k_5E_1(\lambda)E_3(\lambda)E_5(\lambda)}{1-k_3k_4E_3(\lambda)E_4(\lambda)}\widehat{\eta}e^{\frac{1}{c_1}\int_0^s[\lambda+v_1(t)]dt}\\ q_2(s)=E_2(\lambda)\widehat{\eta}e^{\frac{1}{c_2}\int_0^s[\lambda+v_2(t)]dt}\\ q_3(s)=\frac{k_2k_4E_3(\lambda)E_2(\lambda)E_4(\lambda)+k_5E_3(\lambda)E_5(\lambda)}{1-k_3k_4E_3(\lambda)E_4(\lambda)}\widehat{\eta}e^{\frac{1}{c_3}\int_0^s[\lambda+v_3(t)]dt}\\ q_4(s)=\frac{k_2E_2(\lambda)E_4(\lambda)+k_3k_5E_3(\lambda)E_4(\lambda)E_5(\lambda)}{1-k_3k_4E_3(\lambda)E_4(\lambda)}\widehat{\eta}e^{\frac{1}{c_4}\int_0^s[\lambda+v_4(t)]dt}\\ q_5(s)=E_5(\lambda)\widehat{\eta}e^{\frac{1}{c_5}\int_0^s[\lambda+v_5(t)]dt}\\ q_6(s)=\frac{k_2k_4E_2(\lambda)E_4(\lambda)E_6(\lambda)+k_5E_5(\lambda)E_6(\lambda)}{1-k_3k_4E_3(\lambda)E_4(\lambda)}\widehat{\eta}e^{\frac{1}{c_6}\int_0^s[\lambda+v_6(t)]dt}\end{cases}\tag{23}$$

where

$$\widehat{\eta}=\frac{1-k_3k_4E_3(\lambda)E_4(\lambda)}{M'(\lambda)}$$

Proof. For $\lambda \in \sigma((A+B)^*)$, the analytical expression of the eigenvalue problem $(A+B)^*q = \lambda q$ is given by

$$\begin{cases} c_j q_j'(s) - v_j(s)q_j(s) = \lambda q_j(s), \ s \in (0,1), \\ q_1(1) = q_4(1) = k_2q_2(0) + k_3q_3(0), \\ q_2(1) = q_5(1) = k_1q_1(0) + k_6q_6(0), \\ q_3(1) = q_6(1) = k_4q_4(0) + k_5q_5(0). \end{cases}$$

with $j = 1, 2, \cdots, 6$, obviously,

$$q_j(s) = q_j(0)e^{\frac{1}{c_j}\int_0^s [\lambda + v_j(t)]dt}, \quad j = 1, 2, \cdots, 6.$$

and $(q_1(0), q_2(0), \ldots, q_6(0))$ satisfies the following algebraic equations

$$\begin{aligned} q_1(1) = q_4(1) = k_2q_2(0) + k_3q_3(0), \\ q_2(1) = q_5(1) = k_1q_1(0) + k_6q_6(0), \\ q_3(1) = p_6(1) = k_4q_4(0) + k_5q_5(0). \end{aligned}$$

Set $q_1(1) = q_4(1) = \widehat{\xi}$, $q_2(1) = q_5(1) = \widehat{\eta}$ and $q_3(1) = q_6(1) = \widehat{\mu}$, then

$$\begin{aligned} q_1(0) = E_1(\lambda)\widehat{\xi}, \quad q_4(0) = E_4(\lambda)\widehat{\xi}, \\ q_2(0) = E_2(\lambda)\widehat{\eta}, \quad q_5(0) = E_5(\lambda)\widehat{\eta}, \\ q_3(0) = E_3(\lambda)\widehat{\mu}, \quad q_6(0) = E_6(\lambda)\widehat{\mu}, \end{aligned}$$

and

$$\widehat{\xi} - k_2E_2(\lambda)\widehat{\eta} - k_3E_3(\lambda)\widehat{\mu} = 0 \tag{24}$$

$$\widehat{\eta} - k_1E_1(\lambda)\widehat{\xi} - k_6E_6(\lambda)\widehat{\mu} = 0 \tag{25}$$

$$\widehat{\mu} - k_4E_4(\lambda)\widehat{\xi} - k_5E_5(\lambda)\widehat{\eta} = 0$$

Solving(24) and (25) yield

$$\widehat{\mu} = \frac{k_5E_5(\lambda) + k_2k_4E_2(\lambda)E_4(\lambda)}{1 - k_3k_4E_3(\lambda)E_4(\lambda)}\widehat{\eta}$$

$$\widehat{\xi} = \frac{k_2E_2(\lambda) + k_3k_5E_3(\lambda)E_5(\lambda)}{1 - k_3k_4E_3(\lambda)E_4(\lambda)}\widehat{\eta}$$

Therefore,

$$\begin{cases} q_1(0) = \frac{k_2E_1(\lambda)E_2(\lambda) + k_3k_5E_1(\lambda)E_3(\lambda)E_5(\lambda)}{1 - k_3k_4E_3(\lambda)E_4(\lambda)}\widehat{\eta} \\ q_2(0) = E_2(\lambda)\widehat{\eta} \\ q_3(0) = \frac{k_2k_4E_3(\lambda)E_2(\lambda)E_4(\lambda) + k_5E_3(\lambda)E_5(\lambda)}{1 - k_3k_4E_3(\lambda)E_4(\lambda)}\widehat{\eta} \\ q_4(0) = \frac{k_2E_2(\lambda)E_4(\lambda) + k_3k_5E_3(\lambda)E_4(\lambda)E_5(\lambda)}{1 - k_3k_4E_3(\lambda)E_4(\lambda)}\widehat{\eta} \\ q_5(0) = E_5(\lambda)\widehat{\eta} \\ q_6(0) = \frac{k_2k_4E_2(\lambda)E_4(\lambda)E_6(\lambda) + k_5E_5(\lambda)E_6(\lambda)}{1 - k_3k_4E_3(\lambda)E_4(\lambda)}\widehat{\eta} \end{cases} \tag{26}$$

To determine the coefficient $\widehat{\eta}$, using $M(\lambda) = 0$, formula (15) and (23), we calculate the dual product $\langle \Phi(\lambda), \Psi(\lambda) \rangle_c$ as follows

$$\begin{aligned} & \langle \Phi(\lambda), \Psi(\lambda) \rangle_c \\ = & \sum_{j=1}^{6} \frac{1}{c_j} \int_0^1 p_j(s)q_j(s)ds \\ = & \sum_{j=1}^{6} \frac{1}{c_j} p_j(0)q_j(0) \\ = & \frac{1}{c_1}k_1 \frac{k_2E_1(\lambda)E_2(\lambda) + k_3k_5E_1(\lambda)E_3(\lambda)E_5(\lambda)}{1 - k_3k_4E_3(\lambda)E_4(\lambda)}\widehat{\eta} \\ & + \frac{1}{c_2} \frac{k_1k_2E_1(\lambda) + k_2k_4k_6E_4(\lambda)E_6(\lambda)}{1 - k_3k_4E_3(\lambda)E_4(\lambda)} E_2(\lambda)\widehat{\eta} \\ & + \frac{1}{c_3} \frac{k_1k_3E_1(\lambda) + k_3k_4k_6E_4(\lambda)E_6(\lambda)}{1 - k_3k_4E_3(\lambda)E_4(\lambda)} \\ & \times \frac{k_2k_4E_3(\lambda)E_2(\lambda)E_4(\lambda) + k_5E_3(\lambda)E_5(\lambda)}{1 - k_3k_4E_3(\lambda)E_4(\lambda)}\widehat{\eta} \\ & + \frac{1}{c_4} \frac{k_4k_6E_6(\lambda) + k_1k_3k_4E_1(\lambda)E_3(\lambda)}{1 - k_3k_4E_3(\lambda)E_4(\lambda)} \\ & \times \frac{k_2E_2(\lambda)E_4(\lambda) + k_3k_5E_3(\lambda)E_4(\lambda)E_5(\lambda)}{1 - k_3k_4E_3(\lambda)E_4(\lambda)}\widehat{\eta} \\ & + \frac{1}{c_5} \frac{k_5k_6E_6(\lambda) + k_1k_3k_5E_1(\lambda)E_3(\lambda)}{1 - k_3k_4E_3(\lambda)E_4(\lambda)} E_5(\lambda)\widehat{\eta} \\ & + \frac{1}{c_6}k_6 \frac{k_2k_4E_2(\lambda)E_4(\lambda)E_6(\lambda) + k_5E_5(\lambda)E_6(\lambda)}{1 - k_3k_4E_3(\lambda)E_4(\lambda)}\widehat{\eta} \\ = & \frac{\widehat{\eta}M'(\lambda)}{1 - k_3k_4E_3(\lambda)E_4(\lambda)} \end{aligned}$$

where

$$\begin{aligned} & M'(\lambda) \\ = & \left(\tfrac{1}{c_1} + \tfrac{1}{c_2}\right) k_1k_2E_1(\lambda)E_2(\lambda) \\ & + \left(\tfrac{1}{c_3} + \tfrac{1}{c_4}\right) k_3k_4E_3(\lambda)E_4(\lambda) \\ & + \left(\tfrac{1}{c_5} + \tfrac{1}{c_6}\right) k_5k_6E_5(\lambda)E_6(\lambda) \\ & + \left(\tfrac{1}{c_1} + \tfrac{1}{c_3} + \tfrac{1}{c_5}\right) k_1k_3k_5E_1(\lambda)E_3(\lambda)E_5(\lambda) \\ & + \left(\tfrac{1}{c_2} + \tfrac{1}{c_4} + \tfrac{1}{c_6}\right) k_2k_4k_6E_2(\lambda)E_4(\lambda)E_6(\lambda) \end{aligned}$$

Since $\lambda \in \sigma(A+B)$ is a simple eigenvalue, it holds that $M'(\lambda) \neq 0$. Therefore, taking

$$\widehat{\eta} = \frac{1 - k_3k_4E_3(\lambda)E_4(\lambda)}{M'(\lambda)}$$

we have $\langle \Phi(\lambda), \Psi(\lambda) \rangle_c = 1$. □

Let $\sigma(A+B) = \{\lambda_n, n \in \mathbb{Z}\}$ and let $\Phi(\lambda)$ and $\Psi(\lambda)$ be given as (15) and (23). Then it holds that

$$\langle \Phi(\lambda_n), \Psi(\lambda_m) \rangle_c = \delta_{nm}, \quad \forall m, n \in \mathbb{Z}.$$

Note that $(1 - k_3k_4E_3(\lambda_n)E_4(\lambda_n)) = 0$ will leads to $M(\lambda_n) \neq 0$. Without loss of generality we can assume that

$$\inf_{\lambda \in \sigma(A+B)} |1 - k_3k_4E_3(\lambda)E_4(\lambda)| > 0$$

and

$$\inf_{\lambda\in\sigma(A+B)} |M'(\lambda)| > 0.$$

A direct calculation gives

$$\sup_n ||\Phi(\lambda_n)|| < \infty, \sup_n ||\Psi(\lambda_n)|| < \infty.$$

Since

$$E(\lambda_n, A+B)F = \langle F, \widehat{\Psi}_n\rangle \Phi_n, \quad \forall F \in \mathbb{X},$$

so

$$||E(\lambda_n, A+B)|| \le ||\widehat{\Psi}_n|| ||\Phi_n|| < \infty, \forall n \in \mathbb{Z}.$$

This property makes us wish to expand the solution of the equation (5) according to its eigenvectors. However, the following result shows that it is impossible.

Theorem 15. *Let A and B be defined as before. Then the eigenvectors of $A+B$ are not complete in $\mathbb{X}$.*

Proof. Set

$$Sp(A+B) = \overline{\mathrm{span}\{\Phi(\lambda_n), n \in \mathbb{Z}\}}.$$

In order to prove $Sp(A+B) \neq \mathbb{X}$, we only need to prove that there exists $\widehat{F} = (f_1, f_2 \dots f_6) \neq 0$ such that $\widehat{F}(\Phi(\lambda_n)) = 0, \forall n \in \mathbb{Z}$.

We take a vector $\widehat{F}$ as follows

$$\begin{cases} \hat{f}_1(s) = k_6 e^{\frac{1}{c_1}\int_0^s v_1(t)dt}\chi_{[0,\frac{c_1}{c_1+c_6}]} \\ \hat{f}_2(s) = 0 \\ \hat{f}_3(s) = 0 \\ \hat{f}_4(s) = 0 \\ \hat{f}_5(s) = 0 \\ \hat{f}_6(s) = -k_1 e^{\frac{1}{c_6}\int_0^s v_6(t)dt}\chi_{[0,\frac{c_6}{c_1+c_6}]} \end{cases} \tag{27}$$

Obviously, $\widehat{F} = (f_1, f_2 \dots f_6) \in \mathbb{X}^*$, and

$$\begin{aligned} & \langle \Phi(\lambda_n), \widehat{F}\rangle_c \\ = \; & \tfrac{1}{c_1}\int_0^1 k_1 e^{-\frac{1}{c_1}\int_0^s[\lambda_n+v_1(t)]dt} k_6 e^{\frac{1}{c_1}\int_0^s v_1(t)dt}\chi_{[0,\frac{c_1}{c_1+c_6}]}ds \\ & -\tfrac{1}{c_6}\int_0^1 k_6 e^{-\frac{1}{c_6}\int_0^s[\lambda_n+v_6(t)]dt} k_1 e^{\frac{1}{c_6}\int_0^s v_6(t)dt}\chi_{[0,\frac{c_6}{c_1+c_6}]})ds \\ = \; & \omega_{11}k_6^2\left[\tfrac{1}{c_1}\int_0^{\frac{c_1}{c_1+c_6}} e^{-\frac{\lambda_n}{c_1}s}ds - \tfrac{1}{c_6}\int_0^{\frac{c_6}{c_1+c_6}} e^{-\frac{\lambda_n}{c_6}s}ds\right] \\ = \; & \omega_{11}k_6^2\left[\int_0^{\frac{1}{c_1+c_6}} e^{-\lambda_n r}dr - \int_0^{\frac{1}{c_1+c_6}} e^{-\lambda_n r}dr\right] \\ = \; & 0 \end{aligned}$$

Therefore, $Sp(A+B) \neq \mathbb{X}$. So the spectrum of $A+B$ is incomplete in $\mathbb{X}$. ☐

Remark 16. *Since the spectrum of $A+B$ is incomplete, we cannot expand the solution of (5) according to its eigenvectors. Maybe it is possible that we can write the solution in to the following form*

$$X(t) = \sum_{n=-\infty}^{\infty} \langle X_0, \Psi(\lambda_n)\rangle_c e^{\lambda_n t}\Phi(\lambda_n) + R(t)$$

where $R(t)$ is a remainder term. However, the first term is a series, we need to study its convergence. Here we do not discuss this problem.

5 A simple analysis for operation strategy

In this section, we simply discuss the operational strategies for the transport networks.

5.1 Dominant eigenvalue of the system

First we discuss real eigenvalue of $A+B$.

Theorem 17. *Let A and B be defined as before. Then there exists unique a real eigenvalue λ_0 of $A+B$ that satisfying the following property*

1) for any $\lambda \in \sigma(A+B)$, $\Re\lambda \le \lambda_0$;

2) there is a positive eigenvector $\Phi(\lambda_0)$.

Proof. For $\lambda \in \mathbb{R}$, we consider the function $M(\lambda) = 0$. Since

$$\lim_{\lambda\to\infty} M(\lambda) = 1, \quad \lim_{\lambda\to-\infty} M(\lambda) = -\infty$$

and $M'(\lambda) > 0, \forall\lambda \in \mathbb{R}$, so there is unique a real zero of $M(\lambda)$, denote it λ_0.

Note that

$$\begin{aligned} M(\lambda_0) \;=\; & 1 - k_1k_2 e^{-(\frac{\widehat{v_1}}{c_1}+\frac{\widehat{v_2}}{c_2})}e^{-(\frac{1}{c_1}+\frac{1}{c_2})\lambda_0} \\ & -k_3k_4 e^{-(\frac{\widehat{v_3}}{c_3}+\frac{\widehat{v_4}}{c_4})}e^{-(\frac{1}{c_3}+\frac{1}{c_4})\lambda_0} \\ & -k_5k_6 e^{-(\frac{\widehat{v_5}}{c_5}+\frac{\widehat{v_6}}{c_6})}e^{-(\frac{1}{c_5}+\frac{1}{c_6})\lambda_0} \\ & -k_1k_3k_5 e^{-(\frac{\widehat{v_1}}{c_1}+\frac{\widehat{v_3}}{c_3}+\frac{\widehat{v_5}}{c_5})}e^{-(\frac{1}{c_1}+\frac{1}{c_3}+\frac{1}{c_5})\lambda_0} \\ & -k_2k_4k_6 e^{-(\frac{\widehat{v_2}}{c_2}+\frac{\widehat{v_4}}{c_4}+\frac{\widehat{v_6}}{c_6})}e^{-(\frac{1}{c_2}+\frac{1}{c_4}+\frac{1}{c_6})\lambda_0}. \end{aligned}$$

For any $\lambda \in \mathbb{C}$, $\Re\lambda > \lambda_0$, it holds that

$$\begin{aligned} |M(\lambda)| \;\ge\; & 1 - k_1k_2 e^{-(\frac{\widehat{v_1}}{c_1}+\frac{\widehat{v_2}}{c_2})}e^{-(\frac{1}{c_1}+\frac{1}{c_2})\Re\lambda} \\ & -k_3k_4 e^{-(\frac{\widehat{v_3}}{c_3}+\frac{\widehat{v_4}}{c_4})}e^{-(\frac{1}{c_3}+\frac{1}{c_4})\Re\lambda} \\ & -k_5k_6 e^{-(\frac{\widehat{v_5}}{c_5}+\frac{\widehat{v_6}}{c_6})}e^{-(\frac{1}{c_5}+\frac{1}{c_6})\Re\lambda} \\ & -k_1k_3k_5 e^{-(\frac{\widehat{v_1}}{c_1}+\frac{\widehat{v_3}}{c_3}+\frac{\widehat{v_5}}{c_5})}e^{-(\frac{1}{c_1}+\frac{1}{c_3}+\frac{1}{c_5})\Re\lambda} \\ & -k_2k_4k_6 e^{-(\frac{\widehat{v_2}}{c_2}+\frac{\widehat{v_4}}{c_4}+\frac{\widehat{v_6}}{c_6})}e^{-(\frac{1}{c_2}+\frac{1}{c_4}+\frac{1}{c_6})\Re\lambda} \\ >\; & 0. \end{aligned}$$

So for any $\lambda \in \sigma(A+B)$, we have $\Re\lambda \le \lambda_0$.

Since λ_0 is real, and the term

$$1 - k_3k_4E_3(\lambda_0)E_4(\lambda_0) > 0$$

so the functions $p_j(s), j = 1, 2, \cdots, 6$, given in (15) are positive. The second assertion follows. ☐

Theorem 18. *Let A and B be defined as before, and λ_0 be the real eigenvalue of $A+B$. If the set*

$$N=\{\frac{1}{c_1}+\frac{1}{c_2},\frac{1}{c_3}+\frac{1}{c_4},\frac{1}{c_5}+\frac{1}{c_6},\frac{1}{c_1}+\frac{1}{c_3}+\frac{1}{c_5},\frac{1}{c_2}+\frac{1}{c_4}+\frac{1}{c_6}\}$$

has greatest common divisor, denote it $\gcd(N)$, then there exist eigenvalues of $A+B$ on the line $\Re z=\lambda_0$. If N has not the great common divisor, there is no other eigenvalue of $A+B$ on the line $\Re\lambda_0$.

Proof. We consider the points on the line $\Re z=\lambda_0$, i.e., $z=\lambda_0+bi$. If there exists a z such that $M(z)=0$, then taking $\Re M(z)$, we get

$$\begin{aligned}&k_1k_2E_1(\lambda_0)E_2(\lambda_0)\cos b(\tfrac{1}{c_1}+\tfrac{1}{c_2})\\&+k_3k_4E_3(\lambda_0)E_4(\lambda_0)\cos b(\tfrac{1}{c_3}+\tfrac{1}{c_4})\\&+k_5k_6E_5(\lambda_0)E_6(\lambda_0)\cos b(\tfrac{1}{c_5}+\tfrac{1}{c_6})\\&+k_1k_3k_5E_1(\lambda_0)E_3(\lambda_0)E_5(\lambda_0)\cos b(\tfrac{1}{c_1}+\tfrac{1}{c_3}+\tfrac{1}{c_5})\\&+k_2k_4k_6E_2(\lambda_0)E_4(\lambda_0)E_6(\lambda_0)\cos b(\tfrac{1}{c_2}+\tfrac{1}{c_4}+\tfrac{1}{c_6})=1,\end{aligned}$$

Note that $M(\lambda_0)=0$, so above equality holds if and only if

$$\begin{aligned}&\cos b(\tfrac{1}{c_1}+\tfrac{1}{c_2})=\cos b(\tfrac{1}{c_3}+\tfrac{1}{c_4})=\cos b(\tfrac{1}{c_5}+\tfrac{1}{c_6})\\&=\cos b(\tfrac{1}{c_1}+\tfrac{1}{c_3}+\tfrac{1}{c_5})=\cos b(\tfrac{1}{c_2}+\tfrac{1}{c_4}+\tfrac{1}{c_6})=1\end{aligned}$$

Therefore, the set N has the greatest common divisor $\gcd(N)$. In this case, the eigenvalues of $A+B$ take the form $\lambda=\lambda_0+i\frac{2\pi}{\gcd(N)}$. If there is no the greatest common divisor of N, then there is no other spectrum point of $A+B$ on the line $\Re z=\lambda_0$. $\square$

5.2 Analysis of operation strategy

Now let us return to our model. We learn the practice meaning for some qualities. First, the notation

$$\widehat{v_j}=\int_0^1(\mu_j(s)-\nu_j(s))ds$$

is the mean absorption rate, which indicates the rate of passenger departure the line. If $\widehat{v_j}>0$, this means that on this line the number of passengers getting on is less than that getting off. If $\widehat{v_j}<0$, this means that the number of passengers getting on is larger than that getting off.

The notation $\frac{1}{c_j}$ is the time of the vehicle running on the line. Since we have normalized the distance to 1 between site a_i and a_j, the c_j is the speed of vehicle running.

The real eigenvalue λ_0 satisfies the equality

$$\begin{aligned}1\ =\ &k_1k_2e^{-(\frac{\widehat{v_1}}{c_1}+\frac{\widehat{v_2}}{c_2})}e^{-(\frac{1}{c_1}+\frac{1}{c_2})\lambda_0}\\&+k_3k_4e^{-(\frac{\widehat{v_3}}{c_3}+\frac{\widehat{v_4}}{c_4})}e^{-(\frac{1}{c_3}+\frac{1}{c_4})\lambda_0}\\&+k_5k_6e^{-(\frac{\widehat{v_5}}{c_5}+\frac{\widehat{v_6}}{c_6})}e^{-(\frac{1}{c_5}+\frac{1}{c_6})\lambda_0}\\&+k_1k_3k_5e^{-(\frac{\widehat{v_1}}{c_1}+\frac{\widehat{v_3}}{c_3}+\frac{\widehat{v_5}}{c_5})}e^{-(\frac{1}{c_1}+\frac{1}{c_3}+\frac{1}{c_5})\lambda_0}\\&+k_2k_4k_6e^{-(\frac{\widehat{v_2}}{c_2}+\frac{\widehat{v_4}}{c_4}+\frac{\widehat{v_6}}{c_6})}e^{-(\frac{1}{c_2}+\frac{1}{c_4}+\frac{1}{c_6})\lambda_0}.\end{aligned}\tag{28}$$

Noting that

$$\omega_{11}\omega_{22}+\omega_{23}\omega_{34}+\omega_{16}\omega_{35}+\omega_{11}\omega_{23}\omega_{35}+\omega_{16}\omega_{22}\omega_{34}=1,$$

we can rewrite (28) into

$$\begin{aligned}&\omega_{11}\omega_{22}[1-(1-\alpha_1)(1-\alpha_2)e^{-(\frac{\widehat{v_1}}{c_1}+\frac{\widehat{v_2}}{c_2})}e^{-(\frac{1}{c_1}+\frac{1}{c_2})\lambda_0}]\\&+\omega_{23}\omega_{34}[1-(1-\alpha_2)(1-\alpha_3)e^{-(\frac{\widehat{v_3}}{c_3}+\frac{\widehat{v_4}}{c_4})}e^{-(\frac{1}{c_3}+\frac{1}{c_4})\lambda_0}]\\&+\omega_{16}\omega_{35}[1-(1-\alpha_1)(1-\alpha_3)e^{-(\frac{\widehat{v_5}}{c_5}+\frac{\widehat{v_6}}{c_6})}e^{-(\frac{1}{c_5}+\frac{1}{c_6})\lambda_0}]\\&+\omega_{11}\omega_{23}\omega_{35}\\&\times[1-(1-\alpha_1)(1-\alpha_2)(1-\alpha_3)e^{-(\frac{\widehat{v_1}}{c_1}+\frac{\widehat{v_3}}{c_3}+\frac{\widehat{v_5}}{c_5})}e^{-(\frac{1}{c_1}+\frac{1}{c_3}+\frac{1}{c_5})\lambda_0}]\\&+\omega_{16}\omega_{22}\omega_{34}\\&\times[1-(1-\alpha_1)(1-\alpha_2)(1-\alpha_3)e^{-(\frac{\widehat{v_2}}{c_2}+\frac{\widehat{v_4}}{c_4}+\frac{\widehat{v_6}}{c_6})}e^{-(\frac{1}{c_2}+\frac{1}{c_4}+\frac{1}{c_6})\lambda_0}]\\&=0\end{aligned}\tag{29}$$

Since we have assumed that $\alpha_j\neq 0, j=1,2,3$, this means that there is a system output at each site, so when

$$\begin{cases}(1-\alpha_1)(1-\alpha_2)e^{-(\frac{\widehat{v_1}}{c_1}+\frac{\widehat{v_2}}{c_2})}<1,\\(1-\alpha_2)(1-\alpha_3)e^{-(\frac{\widehat{v_3}}{c_3}+\frac{\widehat{v_4}}{c_4})}<1,\\(1-\alpha_1)(1-\alpha_3)e^{-(\frac{\widehat{v_5}}{c_5}+\frac{\widehat{v_6}}{c_6})}<1,\\(1-\alpha_1)(1-\alpha_2)(1-\alpha_3)e^{-(\frac{\widehat{v_2}}{c_2}+\frac{\widehat{v_4}}{c_4}+\frac{\widehat{v_6}}{c_6})}<1,\\(1-\alpha_1)(1-\alpha_2)(1-\alpha_3)e^{-(\frac{\widehat{v_1}}{c_1}+\frac{\widehat{v_3}}{c_3}+\frac{\widehat{v_5}}{c_5})}<1\end{cases}$$

then $\lambda_0<0$ that denotes the defect rate. In this case, the system decays exponentially.

For λ_0, we discuss the distribution of passengers in the system. To this end we calculate the eigenvector given in (15). Since

$$\sum_{j=1}^6||p_j||_{L^1[0,1]}=\sum_{j=1}^6 p_j(0)\widetilde{p_j}(\lambda_0)$$

is the total number of passengers in the system, where $\widetilde{p_j}(\lambda_0)=\int_0^1 e^{-\int_0^s\frac{\lambda_0+v_j(r)}{c_j}dr}ds$. Set

$$\frac{P(\lambda_0)}{1-k_3k_4E_3(\lambda_0)E_4(\lambda_0)}=\sum_{j=1}^6 p_j(0)\widetilde{p_j}(\lambda_0).$$

Then the distribution is $\widetilde{\Phi}(\lambda_0)$ where

$$\begin{cases}p_1(s)=\frac{k_1(1-k_3k_4E_3(\lambda_0)E_4(\lambda_0))}{P(\lambda_0)}e^{-\int_0^s\frac{\lambda_0+v_1(r)}{c_1}dr}\\p_2(s)=\frac{k_1k_2E_1(\lambda_0)+k_2k_4k_6E_4(\lambda_0)E_6(\lambda_0))}{P(\lambda_0)}e^{-\int_0^s\frac{\lambda_0+v_2(r)}{c_2}dr}\\p_3(s)=\frac{k_1k_3E_1(\lambda_0)+k_3k_4k_6E_4(\lambda_0)E_6(\lambda_0)}{P(\lambda_0)}e^{-\int_0^s\frac{\lambda_0+v_3(r)}{c_3}dr}\\p_4(s)=\frac{k_4k_6E_6(\lambda_0)+k_1k_3k_4E_1(\lambda_0)E_3(\lambda_0)}{P(\lambda_0)}e^{-\int_0^s\frac{\lambda_0+v_4(r)}{c_4}dr}\\p_5(s)=\frac{k_5k_6E_6(\lambda_0)+k_1k_3k_5E_1(\lambda_0)E_3(\lambda_0)}{P(\lambda_0)}e^{-\int_0^s\frac{\lambda_0+v_5(r)}{c_5}dr}\\p_6(s)=\frac{k_6(1-k_3k_4E_3(\lambda_0)E_4(\lambda_0))}{P(\lambda_0)}e^{-\int_0^s\frac{\lambda_0+v_6(r)}{c_6}dr}\end{cases}$$

Set $\Psi(\lambda_0)$ is the eigenvector of $(A+B)^*$ satisfying $\langle\Phi(\lambda_0),\Psi(\lambda_0)\rangle_c=1$.

On the other hand, we can assume that $\widehat{v}_j < 0$, i.e., the number of passengers getting on is larger than that getting off, then there is a certain possibility that there are some spectral points located in the right-half plane, which means the number of passengers increase. In this case, we can adjust the c_j to to make the number of passengers steady. For example, let c_j be raised to c'_j, then we have

$$\begin{aligned}
&k_1k_2e^{-(\frac{\widehat{v}_1}{c'_1}+\frac{\widehat{v}_2}{c'_2})}e^{-(\frac{1}{c'_1}+\frac{1}{c'_2})\lambda_0}\\
&+k_3k_4e^{-(\frac{\widehat{v}_3}{c'_3}+\frac{\widehat{v}_4}{c'_4})}e^{-(\frac{1}{c'_3}+\frac{1}{c'_4})\lambda_0}\\
&+k_5k_6e^{-(\frac{\widehat{v}_5}{c'_5}+\frac{\widehat{v}_6}{c'_6})}e^{-(\frac{1}{c'_5}+\frac{1}{c'_6})\lambda_0}\\
&+k_1k_3k_5e^{-(\frac{\widehat{v}_1}{c'_1}+\frac{\widehat{v}_3}{c'_3}+\frac{\widehat{v}_5}{c'_5})}e^{-(\frac{1}{c'_1}+\frac{1}{c'_3}+\frac{1}{c'_5})\lambda_0}\\
&+k_2k_4k_6e^{-(\frac{\widehat{v}_2}{c'_2}+\frac{\widehat{v}_4}{c'_4}+\frac{\widehat{v}_6}{c'_6})}e^{-(\frac{1}{c'_2}+\frac{1}{c'_4}+\frac{1}{c'_6})\lambda_0}=1.
\end{aligned} \tag{30}$$

Compared with (28), we have changed the spectrum distribution of the operator because of the change of the c_j.

Remark 19. *The above theorems give spectral distribution of $A + B$ under a very special situation. Based on the difference of the spectral distribution, we need the different operation strategies:*

(1) When all spectrum points are located on the imaginary axis, which means that the system has an almost period solution, there are certain numbers of the passengers in the system. Therefore, the transport company need not to adjust the numbers of vehicles.

(2) If there are some spectral points located in the right-half plane, which means the numbers of the passengers are asymptotically increasing, then the transport company should increase the numbers of buses to balance the transport networks.

(3) If all spectral points except zero are located in the left-half plane, which means the numbers of the passengers are asymptotically decreasing, in this case the system has a steady state P, then the transport company should decrease suitably the numbers of buses to ensure the effects of the transport networks.

6 Conclusion

In the present paper, we study a regular triangle bi-directed transport network by the approach of the operator semigroup theory and linear operator spectral theory. First, we give a description for transport networks by using the partial differential equations, and prove the well-posedness of the system. Then we prove that the spectrum of the system operator is composed of isolated eigenvalue of finite multiplicity, all root vectors are incomplete in the state space. Finally, we discuss some operation strategies for the transport networks based on the spectral distribution. In this paper, we only discussed some simple situations. Due to the variety of parameters, we need to discuss much more complex cases, that will be our next work.

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