Application of the Fractional Derivative Kelvin-Voigt Model for the Analysis of Impact Response of a Kirchhoff-Love Plate

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Abstract: In the present paper, we consider the problem on a transverse impact of an elastic sphere upon an elastic Kirchhoff-Love plate in a viscoelastic medium, the viscoelastic features of which are described by the fractional derivative Kelvin-Voigt model. Within the contact domain the contact force is defined by the Hertzian contact. The functional equation for determining the contact force is derived and its approximate analytical solution is obtained.

Key–Words: Fractional derivative Kelvin-Voigt model, Impact, Kirchhoff-Love Plate

1 Introduction

Nowadays fractional calculus is widely used in different fields of science and technology, including various dynamic problems of mechanics of solids and structures [1]. Usually in the papers relating to the dynamic response of viscoelastic plates the utilization of the Kelvin-Voigt model with fractional derivatives is carried out [2, 3]. As this takes place, it is supposed that Poisson's ratio is time-independent during the process of deformation and as a preassigned operator it is selected Young's operator

$$\tilde{E} = E_0 \left(1 + \tau_\sigma^\gamma D^\gamma\right), \tag{1}$$

where E_0 is relaxed Young's modulus, τ_σ is the retardation time, γ $(0 < \gamma \leq 1)$ is the fractional parameter, $D^\gamma$is the Riemann-Liouville fractional derivative [1]

$$D^\gamma x(t) = \frac{d}{dt}\int_0^t \frac{x(t')dt'}{\Gamma(1-\gamma)(t-t')^\gamma}, \tag{2}$$

$\Gamma(1-\gamma)$ is the Gamma function, and $x(t)$ is an arbitrary function.

However as experimental data have shown [4, 5], Poisson's ratio is always an operator $\tilde{\nu}$, and only the bulk extension-compression operator $\tilde{K}$ may be expressed as the time-independent value, which for the most viscoelastic materials weakly varies during deformation.

On the other hand, as it is shown in [6], the viscoelastic model (1) with a constant bulk extension-compression operator is completely inapplicable for description of the dynamic response of viscoelastic bodies, and the Kelvin-Voigt model itself is only acceptable for the description of the dynamic behaviour of elastic bodies in a viscoelastic medium.

By the way, when operator $\tilde{E}$ is defined by Eq. (1) and the Poisson's operator $\tilde{\nu}$ is considered as the time-independent value, then this case coincides with the case of the dynamic behaviour of elastic bodies in a viscoelastic medium. The authors of such papers, consciously or not, replace one problem with another, namely: a problem of the dynamic response of viscoelastic bodies in a conventional medium with a problem of dynamic response of elastic bodies in a viscoelastic medium.

In the present paper, we consider the problem on a transverse impact of an elastic sphere upon an elastic Kirchhoff-Love plate in a fractional derivative Kelvin-Voigt medium. Within the contact domain the contact force is defined by the Hertzian contact. The functional equation for determining the contact force will be obtained.

2 Problem Formulation

Let us consider the problem on a transverse impact of an elastic sphere upon a viscoelastic Kirchhoff-Love plate, when the viscoelastic features of the target are described by a fractional derivative Kelvin-Voigt model. In this case, the equations of motion of a spherical impactor of radius R and mass m and the viscoelastic rectangular plate with the dimensions a and b and of thickness h have, respectively, the form

$$m\ddot{w}_2 = -P(t), \tag{3}$$

$$\tilde{D}\nabla^2 w_1(x,y,t) + \rho h \ddot{w}_1(x,y,t) = P(t) \times \delta\left(x - \frac{a}{2}\right)\delta\left(y - \frac{b}{2}\right), \tag{4}$$

where h is the thickness, $w_1(x,y,t)$ is the plate deflection, ρ is its density, $P(t)$ is the contact force, overdots denote time-derivatives, $\nabla^2 = (\partial/\partial x + \partial/\partial y)^2$, x and y are Cartesian coordinates, $\delta(x - \frac{a}{2})$ is the Dirac delta-function, and $\tilde{D}$ is the viscoelastic operator, which at time-independent Poisson's ratio $\nu = \mathrm{const}$ could be represented as

$$\tilde{D} = \frac{\tilde{E}h^3}{12(1-\nu^2)} = \frac{E_0 h^3 (1 + \tau_\sigma^\gamma D^\gamma)}{12(1-\nu^2)} = D_0(1 + \tau_\sigma^\gamma D^\gamma), \tag{5}$$

what corresponds to the Kelvin-Voigt model with the fractional derivative, and $D^\gamma w_1$ is the Riemann-Liouville fractional derivative defined in (2).

Equations (3) and (4) are subjected to the following initial conditions:

$$w_1(x,y,0) = 0, \quad \dot{w}_1(x,y,0) = 0, \quad w_2(0) = 0, \quad \dot{w}_2(0) = V_0, \tag{6}$$

where V_0 is the initial velocity of the impactor at the moment of impact.

Integrating twice Eq. (3) yields

$$w_2(t) = -\frac{1}{m}\int_0^t P(t')(t-t')dt' + V_0 t. \tag{7}$$

Expanding displacement $w_1(x,y,t)$ for a simply-supported Kirchhoff-Love plate in terms of eigenfunctions

$$w_1(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} x_{mn}(t)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}, \tag{8}$$

and substituting (8) in (4) with due account for orthogonality of sines on the intervals $0 \le x \le a, \ \ 0 \le y \le b$, we have

$$\ddot{x}_{1mn}(t) + \Omega_{mn}^2(1 + \tau_\sigma^\gamma D^\gamma)x_{mn}(t) = F_{mn}(t)P(t), \tag{9}$$

where $x_{mn}(t)$ are generalized displacements, and

$$F_{mn}(t) = \frac{1}{\rho h}\sin\frac{n\pi}{2}\sin\frac{m\pi}{2},$$

$$\Omega_{mn}^2 = \frac{D_0}{\rho h}\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]^2.$$

Now let us show that Eq. (9) could be obtained as well, if we consider the case of vibrations of an elastic plate in a viscoelastic medium. Really, the equation describing vibrations of the elastic Kirchhoff-Love plate in the fractional derivative viscoelastic medium under the action of the contact force applied at the center of the plate has the form

$$\frac{D}{\rho h}\nabla^2 w + \frac{\mu}{\rho h}D^\gamma w + \ddot{w} = \frac{1}{\rho h}P(t) \times \delta\left(x - \frac{a}{2}\right)\delta\left(y - \frac{b}{2}\right), \tag{10}$$

where μ is the coefficient of viscosity, and $D = Eh^3/12(1-\nu^2)$ is the cylindrical rigidity.

Substituting (8) in (10), and considering the orthogonality condition for the eigenfunctions on the segments $0 \le x \le a, \quad 0 \le y \le b$, we are led to the infinite set of uncoupled equations

$$\ddot{x}_{mn}(t) + \frac{\mu_{mn}}{\rho h}D^\gamma x_{mn}(t) + \Omega_{mn}^2 x_{mn}(t) = F_{mn}(t)P(t), \quad (m,n = 1,2,...) \tag{11}$$

where μ_{mn} is the coefficient of viscosity of the harmonic with indices m and n.

Considering the Rayleigh hypothesis of proportionality between the elastic and viscous matrices, i.e.,

$$\frac{\mu_{mn}}{\rho h} = \Omega_{mn}^2 \tau_\sigma^\gamma, \tag{12}$$

where τ_σ^γ is the coefficient of proportionality, Eq. (11) is transformed in Eq. (9). Thus, our assumption is valid.

3 Green Function for the Fractional Derivative Kelvin-Voigt Model

In order to find the solution of Eq. (10), it is necessary to find the Green function $G_{mn}(t)$ for each oscillator from (9)

$$G_{mn}(t) = A_{0mn}(t) + A_{mn}e^{-\alpha_{mn}t}\sin(\omega_{mn}t - \varphi_{mn}), \tag{13}$$

where the indices mn indicate the ordinal number of the oscillator, and all values entering in (13) have the same structure and the same physical meaning as the corresponding values discussed in [1], i.e. A_{mn} is the amplitude, α_{mn} is the damping coefficient, and ω_{mn} and φ_{mn} are the frequency and phase, respectively.

Reference to Eq. (13) shows that the Green function possesses two terms, one of which, $A_{0mn}(t)$, describes the drift of the equilibrium position and is represented by the integral involving the distribution function of dynamic and rheological parameters, while the other term is the product of two time-dependent functions, exponent and sine, and it describes damped vibrations around the drifting equilibrium position.

Now let us write Eq. (9) in terms of the Green function $G_{mn}(t)$

$$\ddot{G}_{mn}(t) + \Omega_{mn}^2 \tau_\sigma^\gamma D^\gamma G_{mn}(t) + \Omega_{mn}^2 G_{mn}(t) = F_{mn}\delta(t) \quad (m,\, n = 1, 2, ...). \quad (14)$$

Applying the Laplace transform to Eq. (14) yields

$$\bar{G}_{mn} = \frac{F_{mn}}{p^2 + \kappa_{mn} p^\gamma + \Omega_{mn}^2}, \quad (15)$$

where an overbar denotes the Laplace transform of the corresponding function, p is the transform parameter, and $\kappa_{mn} = \Omega_{mn}^2 \tau_\sigma^\gamma$.

If we omit the numbers mn in (15), then it will coincide with formula (2.2.1) in Sect. 2.2 [7] devoted to the vibrations of the fractional derivative Kelvin-Voigt oscillator. All further formulas of this Section, (2.2.2)–(2.2.6), refer to the analysis of the roots of the characteristic equation

$$p^2 + \kappa_{mn} p^\gamma + \Omega_{mn}^2 = 0, \quad (16)$$

which at each pair of m and n possesses two complex conjugate roots $(p_{mn})_{1,2} = r_{mn} e^{\pm i\psi_{mn}} = -\alpha_{mn} \pm i\omega_{mn}$ (see the root locus at $m = 1,\ n = 1$ in Fig. 19 of [7]), and the inversion of the expression (15) on the first sheet of the Riemannian surface. If we insert the indices m and n in these formulas, then we obtain the desired relationship (13), where the function $A_{0mn}(t)$ describes the drift of the equilibrium position

$$A_{0mn}(t) = \int_0^\infty \tau^{-1} B_{mn}(\tau, \kappa_{mn}) e^{-t/\tau} d\tau, \quad (17)$$

the function $B_{mn}(\tau, \kappa_{mn})$

$$B_{mn}(\tau, \kappa_{mn}) = \frac{\sin \pi\gamma}{\pi}\, F_{mn} \tau [\theta_{mn}(\tau)]^{-1}$$
$$\times \left\{ [\theta_{mn}(\tau)]^{-1} \kappa_{mn}^{-1} \tau^{\gamma-2} + \theta_{mn}(\tau) \kappa_{mn} \tau^{2-\gamma} + 2\cos \pi\gamma \right\}$$

gives us the distribution of the creep (retardation) parameters of the dynamic system,

$$\theta_{mn}(\tau) = \tau^2 \Omega_{mn}^2 + 1,$$

and the amplitude A_{mn} and phase φ_{mn} of vibrations are defined, respectively, as

$$A_{mn} = 2F_{mn} \Big[4r_{mn}^2 + \gamma^2 \kappa_{mn}^2 r_{mn}^{2(\gamma-1)}$$
$$+ 4\gamma \kappa_{mn} r_{mn}^\gamma \cos(2-\gamma)\psi_{mn} \Big]^{-1/2},$$

$$\tan\varphi_{mn} = -\frac{2r_{mn}\cos\psi_{mn} + \gamma\kappa_{mn} r_{mn}^{\gamma-1} \cos(1-\gamma)\psi_{mn}}{2r_{mn}\sin\psi_{mn} - \gamma\kappa_{mn} r_{mn}^{\gamma-1} \sin(1-\gamma)\psi_{mn}}$$

4 Determination of the contact force

Knowing the Green functions, the solution of Eq. (2) takes the form

$$w_1(x, y, t) = \sum_{m=1}^\infty \sum_{n=1}^\infty \sin\left(\frac{m\pi}{a}\, x\right) \sin\left(\frac{n\pi}{b}\, y\right) \times \int_0^t G_{mn}(t - t') P(t') dt'. \quad (18)$$

Let us introduce the value characterizing the relative approach of the sphere and plate, i.e., penetration of the elastic plate by the elastic sphere, is

$$y(t) = w_2(t) - w_1\left(\frac{a}{2}, \frac{b}{2}, t\right), \quad (19)$$

which is connected with the contact force by the Hertzian law

$$P(t) = k y^{3/2}, \quad (20)$$

where

$$k = \frac{4}{3}\sqrt{R} E' \quad (21)$$

is the rigidity coefficient involving the geometry and elastic features of the impactor and the target,

$$\frac{1}{E'} = \frac{1 - \nu_1^2}{E_1} + \frac{1 - \nu_2^2}{E_2}, \quad (22)$$

ν_1, ν_2 and E_1, E_2 are Poisson's coefficients and Young's moduli, respectively, for the elastic target and impactor, indices 1 and 2 refer, respectively, to the target and impactor.

Now substituting (19) in the Hertzian contact law (20) with due account for Eqs. (7) and (9) , we are led to the functional integral equation for determining the contact force

$$k' P(t)^{2/3} = V_0 t - \frac{1}{m} \int_0^t P(t')(t - t') dt'$$
$$- \sum_{m=1}^\infty \sum_{n=1}^\infty \sin\left(\frac{m\pi}{2}\right) \sin\left(\frac{n\pi}{2}\right) \times \int_0^t G_{mn}(t - t') P(t') dt', \quad (23)$$

or the functional equation for defining the function $y(t)$

$$y(t) = V_0 t - \frac{k}{m}\int_0^t y^{3/2}(t')(t-t')dt' - k\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sin\left(\frac{m\pi}{2}\right)\sin\left(\frac{n\pi}{2}\right) \times \int_0^t G_{mn}(t-t')y^{3/2}(t')dt', \quad (24)$$

where $k' = k^{-3/2}$.

Since the impact process is of short duration and the Green function $G_{mn}(t)$, which vanishes to zero at $t = 0$ according to the limiting theorem

$$\lim_{p\to 0}\bar{G}_{mn}(p)p = G(0) = 0, \quad (25)$$

could be represented in the form

$$G_{mn}(t) \approx tA_{mn}\omega_{mn}\cos\varphi_{mn}, \quad (26)$$

then considering (25) and (26) Eqs (23) and (24) are reduced to

$$k'P(t)^{2/3} = V_0 t - k\Big[\frac{1}{m} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\omega_{mn} \times\cos\varphi_{mn}\sin\left(\frac{m\pi}{2}\right)\sin\left(\frac{n\pi}{2}\right)\Big] \times\int_0^t P(t')(t-t')dt', \quad (27)$$

$$y(t) = V_0 t - k\Big[\frac{1}{m} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\omega_{mn}\cos\varphi_{mn} \times\sin\left(\frac{m\pi}{2}\right)\sin\left(\frac{n\pi}{2}\right)\Big]\int_0^t y^{3/2}(t')(t-t')dt'. \quad (28)$$

Equations (27) and (28) could be solved numerically. But the short duration of the impact interaction process allows us to find an approximate analytical solution. Thus, as a first approximation for the function $y(t)$, the expression

$$y = V_0 t \quad (29)$$

could be utilized. Now substituting (29) in the right-hand side of (28) yields

$$y(t) = V_0 t - \frac{4}{35}\,\Delta_{\gamma mn}V_0^{3/2}t^{7/2}, \quad (30)$$

where $\Delta_{\gamma mn} = k\Big[\frac{1}{m}+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\omega_{mn}\cos\varphi_{mn} \times\sin\frac{m\pi}{2}\sin\frac{n\pi}{2}\Big]$.

5 Conclusion

In the present paper, the problem on transverse impact of an elastic spherical impactor upon an elastic Kirchhoff-Love plate in a viscoelastic medium has been formulated for the case, when the damping features of the surrounding medium are modelled by the fractional derivative Kelvin-Voigt model. The Green function for the target was constructed, what allows us to obtain the integral equation for the contact force and local indentation. An approximate analytical solution has been found.

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