The modes of some distributions in independent trials

JIGUANG SHAO
Beijing Jiaotong University
Department of Mathematics
Haidian, Beijing, 100044
CHINA
jgshao@bjtu.edu.cn

QIUYUAN WANG
Beijing Jiaotong University
Department of Mathematics
Haidian, Beijing, 100044
CHINA
qywang@bjtu.edu.cn

SHENG FU
University of Chinese Academy of Sciences
School of Mathematical Sciences
Shijingshan, Beijing, 100049
CHINA
fusheng13@mails.ucas.ac.cn

Abstract: By the methods of multivariate transition probability flow graphs, we define the multivariate geometric distribution and the multi-parameter geometric distribution in independent trials, derive their generating functions, probability distributions and the exact modes. For the multinomial distribution, we only get its mode in some cases. It is still an open question for all other cases. Following the multinomial distribution, we state the definition of the multivariate Poisson distribution, and then employ the joint probability generating function of multivariate Poisson distribution to discuss its modes. We also prove that k is the unique mode of the geometric distribution of order k. Finally, we propose some open questions to the interested readers as an objective and challenge for further study.

Key–Words: mode; multivariate geometric distribution; multivariate Poisson distribution; success run; MTPFG; probability generating function

1 Introduction

The mode is an important statistic of probability distribution. In the present paper, denote by m_ξ the mode of $P_n \hat{=} P(\xi = n), n = 0, 1, \cdots$, i.e. the value of n for which P_n attains its maximum. Similarly, denote by $\boldsymbol{m}_{\boldsymbol{\xi}_n}$ the mode of $P_{\boldsymbol{\xi}_n} \hat{=} P(\xi_1 = l_1, \cdots, \xi_n = l_n)$ is the value of vector $(l_1, \cdots, l_n)$ for which $P_{\boldsymbol{\xi}_n}$ attains its maximum, where $\boldsymbol{\xi}_n = (\xi_1, \cdots, \xi_n)$ is a n-dimensional random vector, $(l_1, \cdots, l_n) \in \mathbb{Z}^n$. It is well known that the modes of many usual distributions have been obtained, such as the mode of geometric distribution with parameter p is $m_{\xi,p} = 1$, the mode of Poisson distribution with parameter λ ($\lambda \in \mathbb{N}$) is $m_{\xi,\lambda} = \lambda$ or $\lambda - 1$. However, many modes of other distributions presented in the statistical and probability literature of recent decades are still awaiting discovery. For example, the binomial distribution of order k defined by Philippou and Makri [16], the geometric distribution of order k, the Poisson distribution of order k and the negative binomial distribution of order k defined by Philippou [15], etc. Only the mode of the Poisson distribution of order k was solved partially by Georghiou [9] and Philippou [17] in the above distributions.

As a continuation of Shao's work in [19, 20], we define the new multivariate geometric distribution denoted by $MG^n(p_1, \cdots, p_n)$ in independent trials, and then arrive at another new multi-parameter distribution denoted by $G^*(p_1, \cdots p_n)$. Besides this two distributions, the present paper discusses multinomial distribution $M^n(p_1, \cdots, p_n; N)$, multivariate Poisson distribution $MP^n(p_1, \cdots, p_n)$ and $G_k(p)$, i.e. the geometric distribution of order k. We investigate their generating functions, probability distributions and modes.

2 The modes of the multivariate geometric distribution and multi-parameter geometric distribution

In the present section, we define a new multivariate geometric distribution, and from it, we obtain another new generalized geometric distribution. Furthermore, we discuss their probability and statistics properties including the modes. We first introduce the theory of multivariate transition probability flow graphs, one of the important tools being applied widely in the sampling inspection field, to research the joint generating function of the multivariate distribution.

The methods of transition probability flow graphs are forceful in discussing some complicated discrete random variables and have a long development history. The methods appeared, for example, in Mason [13] and Koyama [12], and more recently in Fan [4, 5, 6] and Shao [19, 20]. Moreover, we conjecture that the methods should also be applied to finance statistics such as Shao [21] and Wang [22]. Based on decomposing the Markov chain formed by the variation of a nonnegative integer-valued random variable, ascertaining the states and routes, and setting probability functions to the routes, we can obtain a flow graph of the process being similar to the transition probability graph of the chain. Following the series-parallel operation rules, we can arrive at the probability generating function of the random variable from the flow graph. Note that if there are multiple random variables in the transition probability flow graphs, we give the different letters to the arguments of their transition probability functions and call it multivariate transition probability flow graphs (MTPFG). We provide a brief description for transition probability flow graphs as follows, for detail, the readers are referred to Fan [4, 5, 6] and Shao [19, 20].

Let τ be a nonnegative integer-valued random variable with probability space $(\Omega, \mathcal{F}, P)$, set $B_n = \{\tau = n\}$, then for any a fixed $B \in \mathcal{F}$, the transition probability function of τ is defined by

$$G_\tau(x; B) = \sum_{n=0}^{\infty} P(BB_n)x^n, \quad |x| \le 1.$$

Especially, $B = \Omega$ yields the probability generating function of τ as

$$G_\tau(x) = \sum_{n=0}^{\infty} P(\tau = n)x^n, \quad |x| \le 1.$$

Consider a Markov chain that takes on countable number of possible values. The transition process from state s_1 into s_2 denoted by $r : s_1 \Rightarrow s_2$ is called a route, and its transition time named step is a random variable. By the Markovian property, the steps of $s_1 \Rightarrow s_2$ and $s_2 \Rightarrow s_3$ are independent. The transition probability function of the route r is defined by

$$G_r(x) = \sum_{n=0}^{\infty} P_r(n)x^n, \quad |x| \le 1,$$

where $P_r(n)$ is the $n-$step transition probability of r. The route from s_1 into s_3 by way of s_2 denoted by $r_1 \cdot r_2$ is called a series route if the routes $r_1 : s_1 \Rightarrow s_2$ and $r_2 : s_2 \Rightarrow s_3$ are independent. The route denoted by $r_1 + r_2$ is called a parallel route of $r_1 : s_1 \Rightarrow s_2$ and $r_2 : s_1 \Rightarrow s_2$ if they are mutually exclusive. The conclusion of Lemma 1 is obvious.

Lemma 1 *[4, 20] Let $G_{r_1}(x)$ and $G_{r_2}(x)$ be respectively the transition probability functions of the routes r_1 and r_2, then $G_{r_1 \cdot r_2}(x) = G_{r_1}(x) \cdot G_{r_2}(x)$ and $G_{r_1+r_2}(x) = G_{r_1}(x) + G_{r_2}(x)$.*

The route $l : s_1 \Rightarrow s_2$ is called a straight route if no state is repeated in it. The route $o : s_1 \Rightarrow s_1$ is called a loop route on state s_1 if s_1 can be repeated infinitely, and all the repeated routes are independent identically distributed. For the straight route and loop route, we have

Lemma 2 *[20] Let $G_l(x)$ be the transition probability function of the straight routes $l : s_1 \Rightarrow s_2$, let $G_{o_1}(x)$ and $G_{o_2}(x)$ be respectively the transition probability functions of one repetition of the loop routes O_1 and O_2 on state s_1, then*

$$G_{l \cdot \boldsymbol{o_1}}(x) = G_l(x)/[1 - G_{o_1}(x)],$$

$$G_{l \cdot (\boldsymbol{o_1}+\boldsymbol{o_2})}(x) = G_l(x)/[1 - G_{o_1}(x) - G_{o_2}(x)].$$

Let $\tau_1, \cdots, \tau_n$ be n random variables with joint probability space $(\Omega, \mathcal{F}, P)$, for $B \in \mathcal{F}$, the joint transition probability function of $\boldsymbol{\tau}_n = (\tau_1, \tau_2, \cdots, \tau_n)$ is given by

$$\begin{aligned} &G_{\boldsymbol{\tau}_n}(x_1, x_2, \cdots, x_n; B) \\ &= \sum_{i_1, \cdots, i_n} P(\tau_1 = i_1, \cdots, \tau_n = i_n; B)x_1^{i_1} \cdots x_n^{i_n}, \end{aligned}$$

where $|x_k| \le 1, k = 1, \cdots, n$. When $B = \Omega$, $G_{\boldsymbol{\tau}_n}(x_1, \cdots, x_n; \Omega)$ is called the joint probability generating function of random vector $\boldsymbol{\tau}_n$, denoted by $G_{\boldsymbol{\tau}_n}(x_1, \cdots, x_n)$.

Lemma 3 *[20] Let $G_{\boldsymbol{\tau}}(x_1, x_2, \cdots, x_n)$ be the joint probability generating function of the random vector $\boldsymbol{\tau} \hat{=} (\tau_1, \tau_2, \cdots, \tau_n)$, then $G_{\boldsymbol{\tau}}(1, \cdots, 1, x_k, 1, \cdots, 1)$ is the probability generating function of τ_k, $k = 1, 2, \cdots, n$, and $G_{\boldsymbol{\tau}}(x, x, \cdots, x)$ is the probability generating function of $\tau = \sum_{j=1}^{n} \tau_j$, which does not depend on the independence of $\tau_1, \tau_2, \cdots, \tau_n$.*

Now we state the following definition of n-dimensional geometric distribution and discuss its properties.

Definition 4 *Assume that all possible outcomes of an independent trial are the events* $\omega_1, \cdots, \omega_n$*, which are mutually exclusive with respective success probabilities* $p_1, \cdots, p_n$ *satisfying* $\sum_{j=1}^{n} p_j = 1$*. The trial is continued until the occurrence of* $\omega_1 \cap \omega_2 \cap \cdots \cap \omega_n$*. Let* $\xi_1, \cdots, \xi_n$ *denote the number of occurrences of* $\omega_1, \cdots, \omega_n$ *respectively, then the random vector* $\boldsymbol{\xi}_n \hat{=} (\xi_1, \cdots, \xi_n)$ *is said to be the* n*-dimensional geometric distribution with parameter vector* $(p_1, \cdots, p_n)$*, denoted by* $MG^n(p_1, \cdots, p_n)$*.*

Theorem 5 *Let* $\boldsymbol{\xi}_n = (\xi_1, \cdots, \xi_n)$ *be a random vector distributed as* $MG^n(p_1, \cdots, p_n)$*, then its joint generating function is*

$$G_{\boldsymbol{\xi}_n}(x_1, \cdots, x_n) = \frac{\prod_{j=1}^{n} p_j x_j}{1 - \sum\limits_{j=1}^{n} p_j x_j + \prod\limits_{j=1}^{n} p_j x_j}. \qquad (1)$$

Proof. By the methods of MTPFG, for the parameter $n = 3$, the trial process starts at the beginning state B, if the event ω_1 occurs, then it enters state s_1 with the transition probability function p_1x_1, or lese it circles repeatedly in state B with the probability function $p_2x_2 + p_3x_3$. In state s_1, the process will enter state s_2 with function p_2x_2 if ω_2 occurs, if not, it will come back to state B with function p_3x_3 if ω_3 occurs or circles in state s_1 with function p_1x_1 if ω_1 occurs. In state s_2, the process will enter the ending state E with function p_3x_3 to finish if ω_3 occurs, otherwise, it will come back to state s_1 with function p_1x_1 if ω_1 occurs, or to state B with function p_2x_2 if ω_2 occurs. Hence, we get the multivariate transition probability flow graphs of $MG^3(p_1, p_2, p_3)$ in Figure 1.

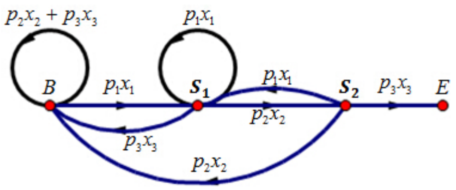

Figure 1: The MTPFG of $MG^3(p_1, p_2, p_3)$

By Lemmas 1 and 2, following Figure 1, we get the transition probability functions of the loop route in state B and state s_1 denoted respectively by $C_B(x)$ and $C_{s_1}(x)$ as follows

$$C_B(x) = p_2x_2 + p_3x_3 + \frac{p_1p_3x_1x_3 + p_1p_2^2x_1x_2^2}{1 - p_1x_1 - p_1p_2x_1x_2},$$

$$C_{s_1}(x) = p_1x_1 + p_1p_2x_1x_2,$$

and the transition probability function of the straight route from B to E is

$$L(x) = p_1p_2p_3x_1x_2x_3,$$

then the joint probability generating function of $\boldsymbol{\xi}_3 = (\xi_1, \xi_2, \xi_3)$ is given by

$$\begin{aligned} G_{\boldsymbol{\xi}_3}(x_1, x_2, x_3) &= \frac{L(x)}{(1 - C_B(x))(1 - C_{S_1}(x))} \\ &= \frac{p_1p_2p_3x_1x_2x_3}{1 - (p_1x_1 + p_2x_2 + p_3x_3) + p_1p_2p_3x_1x_2x_3}. \end{aligned}$$

Similarly, for $n = 4$, we have the MTPFG of the $MG^4(p_1, p_2, p_3, p_4)$ in Figure 2.

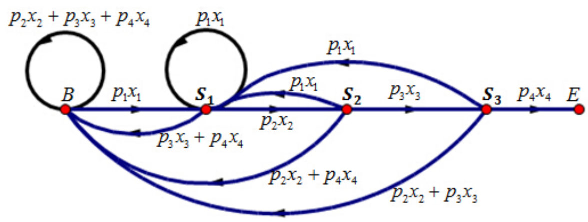

Figure 2: The MTPFG of $MG^4(p_1, p_2, p_3, p_4)$

By Figure 2, we get the joint generating function of $\boldsymbol{\xi}_4 = (\xi_1, \xi_2, \xi_3, \xi_4)$ as follows

$$\begin{aligned} &G_{\boldsymbol{\xi}_4}(x_1, x_2, x_3, x_4) \\ &= \frac{p_1p_2p_3p_4x_1x_2x_3x_4}{1 - \sum\limits_{j=1}^{4} p_jx_j + p_1p_2p_3p_4x_1x_2x_3x_4}. \end{aligned}$$

Furthermore, generalizing the above from $n = 3$ and $n = 4$, we shall come to the joint generating function of $\boldsymbol{\xi}_n$

$$\begin{aligned} &G_{\boldsymbol{\xi}_n}(x_1, x_2, \cdots, x_n) \\ &= \frac{p_1p_2 \cdots p_nx_1x_2 \cdots x_n}{1 - (p_1x_1 + \cdots + p_nx_n) + p_1 \cdots p_nx_1 \cdots x_n} \\ &= \frac{\prod\limits_{j=1}^{n} p_jx_j}{1 - \sum\limits_{j=1}^{n} p_jx_j + \prod\limits_{j=1}^{n} p_jx_j}. \end{aligned}$$

Theorem 5 has been proved. □

Remark 6 *For the random vector* $\boldsymbol{\xi}_n = (\xi_1, \cdots, \xi_n)$ *distributed as* $MG^n(p_1, \cdots, p_n)$*, we have*

$$\begin{aligned} &E\boldsymbol{\xi}_n = (E\xi_1, \cdots, E\xi_n) \\ &= \left(\frac{p_1}{p_1p_2 \cdots p_n}, \cdots, \frac{p_n}{p_1p_2 \cdots p_n}\right) \end{aligned}$$

and

$$Var\boldsymbol{\xi}_n = (Var\xi_1, \cdots, Var\xi_n) = \left(\frac{p_1(p_1 - p_1\cdots p_n)}{(p_1p_2\cdots p_n)^2}, \cdots, \frac{p_n(p_n - p_1\cdots p_n)}{(p_1p_2\cdots p_n)^2}\right).$$

Theorem 7 *Let $\boldsymbol{\xi}_n = (\xi_1, \cdots, \xi_n)$ be a random vector distributed as $MG^n(p_1, \cdots, p_n)$, then the probability distribution is given by*

$$P(\xi_1 = m_1+1, \cdots, \xi_n = m_n+1) = \sum_{s=0}^{\min\{m_1,\cdots,m_n\}} \binom{m_1+\cdots+m_n-(n-1)s}{s, m_1-s, \cdots, m_n-s} \times (-1)^s p_1^{m_1+1}\cdots p_n^{m_n+1}, \quad (2)$$

where $m_1, \cdots, m_n \in \{0, 1, \cdots\}$.

Proof. Following formula (1), we come to

$$G_{\boldsymbol{\xi}_n}(x_1, x_2, \cdots, x_n) = \frac{\prod_{j=1}^n p_jx_j}{1 - \sum_{j=1}^n p_jx_j + \prod_{j=1}^n p_jx_j}$$
$$= \prod_{j=1}^n p_jx_j \sum_{N=0}^{\infty}\left(\sum_{j=1}^n p_jx_j - \prod_{j=1}^n p_jx_j\right)^N$$
$$= \sum_{N=0}^{\infty} \sum_{\substack{l_1,\cdots,l_{n+1}\ni \\ l_1+\cdots+l_{n+1}=N}} \binom{N}{l_1, \cdots, l_{n+1}} (-1)^{l_{n+1}} \times p_1^{l_1+l_{n+1}+1}\cdots p_n^{l_n+l_{n+1}+1} x_1^{l_1+l_{n+1}+1}\cdots x_n^{l_n+l_{n+1}+1}$$
$$= \sum_{l_1,\cdots,l_{n+1}} \binom{l_1+\cdots+l_{n+1}}{l_1, \cdots, l_{n+1}} (-1)^{l_{n+1}} \times p_1^{l_1+l_{n+1}+1}\cdots p_n^{l_n+l_{n+1}+1} x_1^{l_1+l_{n+1}+1}\cdots x_n^{l_n+l_{n+1}+1}$$
$$= \sum_{m_1,\cdots,m_n} \sum_{s=0}^{\min\{m_1,\cdots,m_n\}} \binom{\sum_{j=1}^n m_j - (n-1)s}{s, m_1-s, \cdots, m_n-s} (-1)^s \prod_{j=1}^n p_j^{m_j+1} \prod_{j=1}^n x_j^{m_j+1}.$$

Therefore

$$P(\xi_1 = m_1+1, \cdots, \xi_n = m_n+1) = \sum_{s=0}^{\min\{m_1,\cdots,m_n\}} \binom{m_1+\cdots+m_n-(n-1)s}{s, m_1-s, \cdots, m_n-s} \times(-1)^s p_1^{m_1+1}\cdots p_n^{m_n+1},$$

where $m_1, \cdots, m_n \in \{0, 1, \cdots\}$. This completes the proof of Theorem 7. □

Theorem 8 *Let $\boldsymbol{m}_{\boldsymbol{\xi}_n}$ denote the mode of $\boldsymbol{\xi}_n$ distributed as $MG^n(p_1, p_2, \cdots p_n)$, then*

$$\boldsymbol{m}_{\boldsymbol{\xi}_n} = (1, 1, \cdots, 1).$$

Proof. We consider different cases:
Case I: When $\min\{m_1, \cdots, m_n\} = 0$, by formula (2), we have

$$\begin{aligned} &P(\xi_1 = m_1+1, \cdots, \xi_n = m_n+1) = \\ &= \binom{m_1+\cdots+m_n}{0, m_1, \cdots, m_n} p_1^{m_1+1}\cdots p_n^{m_n+1} \\ &\leq (p_1+\cdots+p_n)^{m_1+\cdots+m_n} \cdot p_1\cdots p_n \\ &= p_1\cdots p_n = P(\xi_1 = 1, \cdots, \xi_n = 1). \end{aligned}$$

Case II: When $\min\{m_1, \cdots, m_n\} = 1$, by formula (2), we get

$$\begin{aligned} &P(\xi_1 = m_1+1, \cdots, \xi_n = m_n+1) = \\ &= \binom{m_1+\cdots+m_n}{0, m_1, \cdots, m_n} p_1^{m_1+1}\cdots p_n^{m_n+1} \\ &- \binom{m_1+\cdots+m_n-n+1}{1, m_1-1, \cdots, m_n-1} p_1^{m_1+1}\cdots p_n^{m_n+1} \\ &< \binom{m_1+\cdots+m_n}{0, m_1, \cdots, m_n} p_1^{m_1+1}\cdots p_n^{m_n+1} \\ &< p_1\cdots p_n = P(\xi_1 = 1, \cdots, \xi_n = 1). \end{aligned}$$

Case III: When $\min\{m_1, \cdots, m_n\} \geq 2$, the similar proof method of Case II in (2) yields

$$\begin{aligned} &P(\xi_1 = m_1+1, \cdots, \xi_n = m_n+1) \\ &< P(\xi_1 = 1, \cdots, \xi_n = 1). \end{aligned}$$

Together with Cases I, II and III, we conclude that $(1, \cdots, 1)$ is the unique mode of $MG^n(p_1, \cdots p_n)$. □

Definition 9 *If $\boldsymbol{\xi}_n = (\xi_1, \xi_2, \cdots \xi_n)$ is distributed as $MG^n(p_1, p_2, \cdots p_n)$, let $\xi_n^* = \sum_{j=1}^n \xi_j$, then we say that ξ_n^* is a multi-parameter random variable distributed as $G^*(p_1, p_2, \cdots p_n)$. The probability generating function of ξ_n^* is given by*

$$G_{\xi_n^*}(x) = \frac{p_1p_2\cdots p_nx^n}{1 - x + p_1p_2\cdots p_nx^n}. \quad (3)$$

Remark 10 *1) Definition 9 is based on Lemma 3 and Theorem 5.*

2) When $n = 1$, formula (3) yields $G_{\xi_1^*}(x) = p_1x/(1-x+p_1x)$, which is the exact probability generating function of usual geometric distribution with parameter p_1. It means that $G^*(p_1, \cdots p_n)$ is a generalized geometric distribution.

Theorem 11 *Let ξ_n^* be a random variable distributed as $G^*(p_1, p_2, \cdots p_n)$, then its probability distribution is given by*

$$P(\xi_n^* = n+m) = \sum_{s=0}^{[\frac{m}{n}]} \binom{m-s(n-1)}{s} (-1)^s \rho^{s+1},$$

where $\rho = p_1 p_2 \cdots p_n$, $m = 0, 1, \cdots$. Note that $[x]$ denotes the greatest integer not exceeding $x \in \mathbb{R}$.

Proof. The symbol $\rho = p_1 p_2 \cdots p_n$ in (3) yields

$$\begin{aligned} G_{\xi_n^*}(x) &= \frac{\rho x^n}{1-x+\rho x^n} = \rho x^n \sum_{N=0}^{\infty} (x-\rho x^n)^N \\ &= \rho x^n \sum_{N=0}^{\infty} \sum_{m=0}^{N} \binom{N}{m} x^{N-m} (-\rho x^n)^m \\ &= \rho x^n \sum_{N=0}^{\infty} \sum_{m=0}^{N} \binom{N}{m} (-\rho)^m x^{m(n-1)+N} \\ &= \rho x^n \sum_{m=0}^{\infty} \sum_{s=0}^{[\frac{m}{n}]} \binom{m-s(n-1)}{s} (-\rho)^s x^m \\ &= \sum_{m=0}^{\infty} \sum_{s=0}^{[\frac{m}{n}]} \binom{m-s(n-1)}{s} (-\rho)^s \rho x^{m+n} \\ &= \sum_{m=0}^{\infty} \sum_{s=0}^{[\frac{m}{n}]} \binom{m-s(n-1)}{s} (-1)^s \rho^{s+1} x^{m+n} \\ &= \sum_{m=0}^{\infty} P(\xi_n^* = m+n) x^{m+n}. \end{aligned}$$

Obviously, the last equation contains the probability distribution of ξ_n^*. □

Theorem 12 *Let ξ_n^* be a random variable distributed as $G^*(p_1, p_2, \cdots p_n)$, then the modes of ξ_n^* are*

$$m_{\xi_n^*} = n, n+1, \cdots, 2n-1.$$

Proof. When $m \le n$, for $P_m = P(\xi_n^* = m)$, by formula (3), we can find that

$$P_0 = P_1 = \cdots = P_{n-1} = 0, P_n = p_1 p_2 \cdots p_n.$$

When $m > n$, from (3), we have

$$G_{\xi_n^*}(x) \cdot (1 - x + \rho x^n) = \rho x^n,$$

where $\rho = p_1 p_2 \cdots p_n$. Differentiating m times both sides of the above with respect to x, then setting $x = 0$ yields

$$G_{\xi_n^*}^{(m)}(0) - m G_{\xi_n^*}^{(m-1)}(0) + \binom{m}{n} G_{\xi_n^*}^{(m-n)}(0) n! \rho = 0.$$

By $P_m = G_{\xi_n^*}^{(m)}(0)/m!$, we get the recurrence relation

$$P_m - P_{m-1} + \rho \cdot P_{m-n} = 0, m > n.$$

Therefore,

$$P_n = P_{n+1} = \cdots = P_{2n-1} > P_{2n} > P_{2n+1} > \cdots.$$

Theorem 12 has been proven. □

3 The modes of the multinomial distribution and multivariate Poisson distribution

Suppose that all the possible outcomes of an independent trial are the events $e_1, e_2, \cdots, e_n, n = 1, 2, \cdots$, which are mutually exclusive with respective success probabilities $p_1, p_2, \cdots, p_n$, $\sum_{l=1}^{n} p_l = 1$. Let $\eta_1, \eta_2, \cdots, \eta_n$ denote the number of occurrences of $e_1, e_2, \cdots, e_n$ in N independent trials respectively, then the random vector $\boldsymbol{\eta}_n = (\eta_1, \cdots, \eta_n)$ is said to be the multinomial distribution with parameter vector $(p_1, \cdots, p_n)$, denoted by $M^n(p_1, \cdots, p_n; N)$.

Lemma 13 *[19] Suppose that $\boldsymbol{\eta}_n = (\eta_1, \eta_2, \cdots, \eta_n)$ is a n-dimensional random vector distributed as $M^n(p_1, \cdots, p_n; N)$, then its joint probability generating function is*

$$G_{\boldsymbol{\eta}_n}(x_1, \cdots, x_n) = (p_1 x_1 + \cdots + p_n x_n)^N. \quad (4)$$

And its joint probability distribution is

$$\begin{aligned} &P(\eta_1 = k_1, \eta_2 = k_2, \cdots, \eta_n = k_n) \\ &= \binom{N}{k_1, k_2, \cdots, k_n} p_1^{k_1} p_2^{k_2} \cdots p_n^{k_n}, \end{aligned} \quad (5)$$

where $k_1 + k_2 + \cdots + k_n = N$.

Theorem 14 *If the random vector $\boldsymbol{\eta}_n = (\eta_1, \cdots, \eta_n)$ is distributed as $M^n(1/n, \cdots, 1/n; N)$, then its mode(s) denoted by $\boldsymbol{m}_{\boldsymbol{\eta}_n}$ is given by*

$$\boldsymbol{m}_{\boldsymbol{\eta}_n} = (N/n, \cdots, N/n)$$

if $N/n \in \mathbb{N}$, and

$$\boldsymbol{m}_{\boldsymbol{\eta}_n} = ([N/n] + \delta_1, \cdots, [N/n] + \delta_n)$$

if $N/n \notin \mathbb{N}$, where $\delta_k = 0$ or 1 $(k = 1, \cdots, n)$ and $\sum_{k=1}^{n} \delta_k = N - n[N/n]$.

Proof. For $n = 3$, by formula (5), the probability distribution of $\boldsymbol{\eta}_3 = (\eta_1, \eta_2, \eta_3)$ distributed as $M^3(1/3, 1/3, 1/3; N)$ is presented as follows

$$P(\eta_1 = k_1, \eta_2 = k_2, \eta_3 = k_3) = \frac{N!}{k_1!k_2!k_3!}\left(\frac{1}{3}\right)^N.$$

Case I: If the number of trials $N = 3K, K \in \mathbb{N}$, let r be the total sum of the positive number from $k_1 - K, k_2 - K$ and $k_3 - K$, and by the fact that $k_1 + k_2 + k_3 = 3K$, we know that the total sum of the negative number from $k_1 - K, k_2 - K$ and $k_3 - K$ is $-r$. Let Δ_r^+ and Δ_r^- be respectively the increase factor and decrease factor from $K!K!K!$ to $k_1!k_2!k_3!$, we find:

$$\Delta_r^+ \geq (K+1)^r, \Delta_r^- < K^r.$$

Hence,

$$k_1!k_2!k_3! = K!K!K!\frac{\Delta_r^+}{\Delta_r^-} > K!K!K!,$$

the above yields

$$K!K!K! = \min_{k_1+k_2+k_3=N}\{k_1!k_2!k_3!\}.$$

Therefore

$$\begin{aligned}
&P(\eta_1 = K, \eta_2 = K, \eta_3 = K) = \frac{N!}{K!K!K!}\left(\frac{1}{3}\right)^N \\
&= \max_{k_1+k_2+k_3=N}\left\{\frac{N!}{k_1!k_2!k_3!}\left(\frac{1}{3}\right)^N\right\} \\
&= \max\{P(\eta_1 = k_1, \eta_2 = k_2, \eta_3 = k_3)\}.
\end{aligned}$$

Case II: If the number of trials $N = 3K + 1, K \in \mathbb{N}$, by the similar methods in Case I, noting that Δ_r^+ and Δ_r^- are respectively the increase factor and decrease factor from $(K + 1)!K!K!$ to $k_1!k_2!k_3!$, we obtain

$$\Delta_r^+ \geq (K+1)^r, \Delta_r^- \leq (K+1)^r,$$

hence we come to

$$k_1!k_2!k_3! = (K+1)!K!K!\frac{\Delta_r^+}{\Delta_r^-} \geq (K+1)!K!K!,$$

then

$$\begin{aligned}
&P(\eta_1 = K+1, \eta_2 = K, \eta_3 = K) = \\
&P(\eta_1 = K, \eta_2 = K+1, \eta_3 = K) = \\
&P(\eta_1 = K, \eta_2 = K, \eta_3 = K+1) = \\
&= \frac{N!}{(K+1)!K!K!}\left(\frac{1}{3}\right)^N \\
&= \max_{k_1+k_2+k_3=N}\left\{\frac{N!}{k_1!k_2!k_3!}\left(\frac{1}{3}\right)^N\right\} \\
&= \max\{P(\eta_1 = k_1, \eta_2 = k_2, \eta_3 = k_3)\}.
\end{aligned}$$

Case III: If the number of trials $N = 3K + 2, K \in \mathbb{N}$, it is not difficult to arrive at

$$\begin{aligned}
&P(\eta_1 = K+1, \eta_2 = K+1, \eta_3 = K) = \\
&P(\eta_1 = K+1, \eta_2 = K, \eta_3 = K+1) = \\
&P(\eta_1 = K, \eta_2 = K+1, \eta_3 = K+1) = \\
&= \frac{N!}{(K+1)!(K+1)!K!}\left(\frac{1}{3}\right)^N \\
&= \max_{k_1+k_2+k_3=N}\left\{\frac{N!}{k_1!k_2!k_3!}\left(\frac{1}{3}\right)^N\right\} \\
&= \max\{P(\eta_1 = k_1, \eta_2 = k_2, \eta_3 = k_3)\}.
\end{aligned}$$

Combining Cases I, II with III, we conclude that when $n = 3$, Theorem 14 holds. One can easily generalize the conclusion to any integer $n \geq 1$. $\square$

Theorem 15 *If the random vector $\boldsymbol{\eta}_n = (\eta_1, \cdots, \eta_n)$ is distributed as $M^n(p_1, \cdots, p_n; N)$ satisfying $Np_l \in \mathbb{N}, l = 1, \cdots, n$, then the mode of $\boldsymbol{\eta}_n$ is given by*

$$\boldsymbol{m}_{\boldsymbol{\eta}_n} = (Np_1, Np_2, \cdots, Np_n).$$

Proof. We calculate

$$\begin{aligned}
&\frac{N!}{(Np_1+1)!(Np_2-1)!(Np_3)!}p_1^{Np_1+1}p_2^{Np_2-1}p_3^{Np_3} \\
&= \frac{N! \cdot p_1^{Np_1}p_2^{Np_2}p_3^{Np_3}}{(Np_1)!(Np_2)!(Np_3)!} \cdot \frac{Np_2}{Np_1+1} \cdot \frac{p_1}{p_2} \\
&< \frac{N!}{(Np_1)!(Np_2)!(Np_3)!} \cdot p_1^{Np_1}p_2^{Np_2}p_3^{Np_3}.
\end{aligned}$$

Hence

$$\begin{aligned}
&P(\eta_1 = Np_1 + 1, \eta_2 = Np_2 - 1, \eta_3 = Np_3) \\
&< P(\eta_1 = Np_1, \eta_2 = Np_2, \eta_3 = Np_3).
\end{aligned}$$

Similarly, for $0 \leq Np_k + l_k \leq N, l_k \in \mathbb{Z}, k = 1, 2, 3$, and $l_1 + l_2 + l_3 = 0$, we have

$$\begin{aligned}
&P(\eta_1 = Np_1 + l_1, \eta_2 = Np_2 + l_2, \eta_3 = Np_3 + l_3) \\
&\leq P(\eta_1 = Np_1, \eta_2 = Np_2, \eta_3 = Np_3),
\end{aligned}$$

with equality if and only if $l_1, l_2, l_3 = 0$. Hence, (Np_1, Np_2, Np_3) is the unique mode of multinomial distribution $M^3(p_1, p_2, p_3; N)$.

For a fixed $n \in \mathbb{N}$, the similar discussion can conclude that the unique mode of $M^n(p_1, \cdots, p_n; N)$ is $(Np_1, \cdots, Np_n)$ if $Np_l \in \mathbb{N}, l = 1, \cdots, n$. $\square$

Remark 16 *For any reals $p_1 > 0, \cdots, p_n > 0$ satisfying $p_1 + \cdots + p_n = 1$, the mode of multinomial distribution $M^n(p_1, \cdots, p_n; N)$ cannot be equal to $(Np_1, \cdots, Np_n)$ if $Np_l, l = 1, \cdots, n$ are not all integers. But we conjecture that the mode is pretty close to $(Np_1, \cdots, Np_n)$ and we propose it as one of the open questions in section 5.*

Suppose that $\boldsymbol{\eta}_{n+1} = (\eta_1, \cdots, \eta_{n+1})$ is a $(n+1)$-dimensional random vector distributed as multinomial distribution $M^{n+1}(p_1, \cdots, p_{n+1}; N)$. Another random we are interest in is $\boldsymbol{\zeta}_n = (\eta_1, \cdots, \eta_n)$. Let $G^*_{\boldsymbol{\zeta}_n}(x_1, \cdots, x_n)$ denote its joint probability generating function, then we have

Theorem 17 *Assume* $(p_1, \cdots, p_n) \to (0, \cdots, 0)$ *and* $(Np_1, \cdots, Np_n) \to (\lambda_1, \cdots, \lambda_n)$ *as* $N \to \infty$, *where* $\lambda_1 > 0, \cdots, \lambda_n > 0$, *then*

$$G^*_{\boldsymbol{\zeta}_n}(x_1, \cdots, x_n) \xrightarrow{N\to\infty} e^{\lambda_1(x_1-1)+\cdots+\lambda_n(x_n-1)}.$$

Proof. By Lemma 3 and formula (4),

$$\begin{aligned} & G^*_{\boldsymbol{\zeta}_n}(x_1, \cdots, x_n) = G_{\boldsymbol{\eta}_{n+1}}(x_1, \cdots, x_n, 1) \\ & = (p_1x_1 + \cdots + p_nx_n + p_{n+1})^N \\ & = (p_1x_1 + \cdots + p_nx_n + 1 - p_1 - \cdots - p_n)^N \\ & = (1 + p_1(x_1-1) + \cdots + p_n(x_n-1))^N \\ & = \left(1 + \sum_{j=1}^n p_j(x_j-1)\right)^{\frac{\sum_{j=1}^n N\cdot p_j(x_j-1)}{\sum_{j=1}^n p_j(x_j-1)}}. \end{aligned}$$

Therefore,

$$G^*_{\boldsymbol{\zeta}_n}(x_1, \cdots, x_n) \xrightarrow{N\to\infty} e^{\lambda_1(x_1-1)+\cdots+\lambda_n(x_n-1)}.$$

This completes the proof. □

Remark 18 *Theorem 17 is inspired by Dai [3].*

Definition 19 *The random vector* $\boldsymbol{\zeta}_n = (\zeta_1, \cdots, \zeta_n)$ *is said to have the* n*-dimensional Poisson distribution with parameter vector* $(\lambda_1, \cdots, \lambda_n)$ *and to be denoted by* $MP^n(\lambda_1, \cdots, \lambda_n)$, *if its joint probability generating function is given by*

$$G_{\boldsymbol{\zeta}_n}(x_1, \cdots, x_n) = e^{\lambda_1(x_1-1)+\cdots+\lambda_n(x_n-1)}.$$

Remark 20 *The mean and variance vector of the random vector* $\boldsymbol{\zeta}_n$ *in Definition 19 are* $E\boldsymbol{\zeta}_n = Var\boldsymbol{\zeta}_n = (\lambda_1, \lambda_2, \cdots, \lambda_n)$.

Theorem 21 *Let* $(\zeta_1, \cdots, \zeta_n)$ *be a random vector distributed as* $MP^n(\lambda_1, \cdots, \lambda_n)$, *then*

$$\begin{aligned} & P(\zeta_1 = m_1, \zeta_2 = m_2, \cdots, \zeta_n = m_n) \\ & = \frac{\lambda_1^{m_1}\lambda_2^{m_2}\cdots\lambda_n^{m_n}}{m_1!m_2!\cdots m_n!} e^{-(\lambda_1+\lambda_2+\cdots+\lambda_n)}, \end{aligned}$$

where $m_1, m_2, \cdots, m_n \in \{0, 1, \cdots\}$.

Proof. Following the joint probability generating function of $MP^n(\lambda_1, \cdots, \lambda_n)$, we arrive at

$$\begin{aligned} & G_{\boldsymbol{\zeta}_n}(x_1, \cdots, x_n) \\ & = e^{-\sum_{j=1}^n \lambda_j} \sum_{m=0}^{\infty} \frac{(\lambda_1x_1 + \lambda_2x_2 + \cdots + \lambda_nx_n)^m}{m!} \\ & = e^{-\sum_{j=1}^n \lambda_j} \sum_{m=0}^{\infty} \sum_{\substack{m_1,\cdots,m_n \ni \\ m_1+\cdots+m_n=m}} \frac{1}{m!}\binom{m}{m_1, \cdots, m_n} \times \\ & \lambda_1^{m_1}\lambda_2^{m_2}\cdots\lambda_n^{m_n} x_1^{m_1}x_2^{m_2}\cdots x_n^{m_n} \\ & = e^{-(\lambda_1+\cdots+\lambda_n)} \sum_{m=0}^{\infty} \sum_{\substack{m_1,m_2,\cdots,m_n \ni \\ m_1+m_2+\cdots+m_n=m}} \\ & \frac{\lambda_1^{m_1}\lambda_2^{m_2}\cdots\lambda_n^{m_n}}{m_1!m_2!\cdots m_n!} x_1^{m_1}x_2^{m_2}\cdots x_n^{m_n} \\ & = \sum_{m=0}^{\infty} \sum_{\substack{m_1,m_2,\cdots,m_n \ni \\ m_1+m_2+\cdots+m_n=m}} \frac{\lambda_1^{m_1}\lambda_2^{m_2}\cdots\lambda_n^{m_n}}{m_1!m_2!\cdots m_n!} \times \\ & e^{-(\lambda_1+\lambda_2+\cdots+\lambda_n)} x_1^{m_1}x_2^{m_2}\cdots x_n^{m_n} \\ & = \sum_{m_1,\cdots,m_n} \frac{\lambda_1^{m_1}\cdots\lambda_n^{m_n}}{m_1!\cdots m_n!} e^{-(\lambda_1+\cdots+\lambda_n)} x_1^{m_1}\cdots x_n^{m_n}, \end{aligned}$$

following which we can derive the joint probability distribution. Theorem 21 has been proved. □

Theorem 22 *Let* $\boldsymbol{\zeta}_n = (\zeta_1, \cdots, \zeta_n)$ *be a random vector distributed as* $MP^n(\lambda_1, \cdots, \lambda_n)$, *then the modes of* $\boldsymbol{\zeta}_n$ *denoted by* $\boldsymbol{m}_{\boldsymbol{\zeta}_n}$ *are*

$$\boldsymbol{m}_{\boldsymbol{\zeta}_n} = (m_1^*, m_2^*, \cdots, m_n^*),$$

where $m_s^* = \lambda_s$ *or* $\lambda_s - 1$ *if* $\lambda_s \in \mathbb{N}$; $m_s^* = [\lambda_s]$ *if* $\lambda_s \notin \mathbb{N}, s = 1, 2, \cdots, n$.

Proof. By Theorem 21, we have

$$\begin{aligned} & P(\zeta_1 = m_1, \zeta_2 = m_2, \cdots, \zeta_n = m_n) \\ & = \left(\frac{\lambda_1^{m_1}}{m_1!}\right)\left(\frac{\lambda_2^{m_2}}{m_2!}\right)\cdots\left(\frac{\lambda_n^{m_n}}{m_n!}\right) e^{-\sum_{j=1}^n \lambda_j}. \end{aligned}$$

For any fixed reals $\lambda_1, \lambda_2, \cdots, \lambda_n > 0$, note that $e^{-(\lambda_1+\lambda_2+\cdots+\lambda_n)}$ is a constant, hence, $P(\zeta_1 = m_1, \cdots, \zeta_n = m_n)$ attains its maximum if and only if $\left(\frac{\lambda_1^{m_1}}{m_1!}\right), \left(\frac{\lambda_2^{m_2}}{m_2!}\right) \cdots, \left(\frac{\lambda_n^{m_n}}{m_n!}\right)$ attain their maxima respectively.

Case I: If $\lambda_s \in \mathbb{N}, s = 1, \cdots, n$, then

$$\begin{aligned} & \frac{\lambda_s^{m_s}}{m_s!} = \frac{\lambda_s}{1}\cdot\frac{\lambda_s}{2}\cdots\frac{\lambda_s}{m_s} \le \frac{\lambda_s}{1}\cdot\frac{\lambda_s}{2}\cdots\frac{\lambda_s}{\lambda_s - 1} \\ & = \frac{\lambda_s}{1}\cdot\frac{\lambda_s}{2}\cdots\frac{\lambda_s}{\lambda_s - 1}\cdot\frac{\lambda_s}{\lambda_s}. \end{aligned}$$

Hence, when $m_s = \lambda_s - 1$ or λ_s,

$$\frac{\lambda_s^{m_s}}{m_s!}, s = 1, \cdots, n$$

attains its maximum.
Case II: If some $\lambda_s \notin \mathbb{N}, s = 1, \cdots, n$, then

$$\frac{\lambda_s^{m_s}}{m_s!} = \frac{\lambda_s}{1} \cdot \frac{\lambda_s}{2} \cdots \frac{\lambda_s}{m_s} \leq \frac{\lambda_s}{1} \cdot \frac{\lambda_s}{2} \cdots \frac{\lambda_s}{[\lambda_s]}.$$

Hence, when $m_s = [\lambda_s]$,

$$\frac{\lambda_s^{m_s}}{m_s!}, s = 1, \cdots, n$$

attains its maximum. Thus the proof is complete. □

4 The mode of the geometric distribution of order k

In this section, we shall discuss the mode of the geometric distribution of order k. There are many papers in the literature dealing with how to get its generating function, moment generating function and probability distribution etc, such as Philippou [15], Barry [2] and Shao [20]. Note that this probability distribution is defined by success run, which has been the basic concept in Bernoulli trial. It is a specified sequence of k consecutive success that may occur at some point in the series of Bernoulli trials, where k is the length of it. The run theory has been investigated by many authors. For more detail about it, the readers are referred to Feller [7], Fu [8], Han [11], Muselli [14] and Schwager [18], $et\ al$.

Definition 23 *[1, 20] Let $\xi_{(k)}$ be the number of trials until the occurrence of the success run of length k in Bernoulli trials with success probability p, then $\xi_{(k)}$ is a random variable distributed as the geometric distribution of order k with parameter p denoted by $G_k(p)$.*

Lemma 24 *[20] If $\xi_{(k)}$ is distributed as $G_k(p)$, then its probability generating function is given by*

$$G_{\xi_{(k)}}(x) = \frac{p^k x^k - p^{k+1} x^{k+1}}{1 - x + qp^k x^{k+1}}.$$

Theorem 25 *The probability distribution of the random variable distributed as $G_k(p)$ is*

$$P(\xi_{(k)} = m + k) = \sum_{\substack{m_1, \cdots, m_k \ni \\ \sum_{l=1}^{k} l \cdot m_l = m}} \binom{\sum_{l=1}^{k} m_l}{m_1, \cdots, m_k} \left(\frac{q}{p}\right)^{\sum_{l=1}^{k} m_l} p^{m+k},$$

where $m = 0, 1, 2, \cdots$.

Proof. By Lemma 24, we get

$$\begin{aligned} G_{\xi_{(k)}}(x) &= \frac{p^k x^k}{1 - qx(1 + px + \cdots + p^{k-1}x^{k-1})} \\ &= \frac{p^k x^k}{1 - \frac{q}{p}(px + \cdots + p^k x^k)} \\ &= p^k x^k \sum_{n=0}^{\infty} \left(\frac{q}{p}\right)^n (px + \cdots + p^k x^k)^n \\ &= \sum_{n=0}^{\infty} \left(\frac{q}{p}\right)^n \sum_{\substack{l_1, \cdots, l_k \ni \\ \sum_{i=1}^{k} l_i = n}} \binom{n}{l_1, \cdots, l_k} (px)^{k + \sum_{i=1}^{k} i \cdot l_i} \\ &= \sum_{n=0}^{\infty} \sum_{\substack{l_1, \cdots, l_k \ni \\ \sum_{i=1}^{k} l_i = n}} \left(\frac{q}{p}\right)^n \binom{n}{l_1, \cdots, l_k} (px)^{k + \sum_{i=1}^{k} i \cdot l_i} \\ &= \sum_{m=0}^{\infty} \sum_{\substack{m_1, \cdots, m_k \ni \\ \sum_{l=1}^{k} l \cdot m_l = m}} \binom{\sum_{l=1}^{k} m_l}{m_1, \cdots, m_k} \left(\frac{q}{p}\right)^{\sum_{l=1}^{k} m_l} (px)^{m+k}, \end{aligned}$$

then, from the above we can easy come to the probability distribution of $\xi_{(k)}$. □

Theorem 26 *For any a fixed k ($k \in \mathbb{N}$), the geometric distribution of order k has a unique mode $m_{\xi_{(k)}, p} = k$.*

Proof. We consider different cases:
Case I: When $k = 2$, by Lemma 24, we get the probability generating function of $\xi_{(2)}$ as follows

$$G_{\xi_{(2)}}(x) = \frac{p^2 x^2 - p^3 x^3}{1 - x + qp^2 x^3}. \tag{6}$$

Let $P_n = P(\xi_{(2)} = n), n = 0, 1, \cdots$ be the probability distribution of $\xi_{(2)}$, then we have

$$\begin{aligned} G_{\xi_{(2)}}(x) &= \sum_{n=0}^{\infty} P_n x^n, \\ P_n &= G^{(n)}_{\xi_{(2)}}(0)/n!, \ n = 0, 1, 2 \cdots, \end{aligned} \tag{7}$$

where $G^{(n)}_{\xi_{(2)}}(0)$ is the n-th derivative of $G_{\xi_{(2)}}(x)$ at $x = 0$, especially, $G^{(0)}_{\xi_{(2)}}(0) = G_{\xi_{(2)}}(0)$.

On the other hand, by equations (6) and (7), we shall get the probability values $P_0 = P_1 = 0, P_2 = p^2, P_3 = qp^2$. When $n \geq 4$, we consider the following equation

$$G_{\xi_{(2)}}(x)\cdot(1-x+qp^2x^3)=p^2x^2-p^3x^3.$$

Differentiating n times $G_{\xi_{(2)}}(x)\cdot(1-x+qp^2x^3)$ and $p^2x^2-p^3x^3$ in the above respectively, then setting $x=0$, we have

$$\binom{n}{0}G^{(n)}_{\xi_{(2)}}(0)-\binom{n}{1}G^{(n-1)}_{\xi_{(2)}}(0)+\binom{n}{3}G^{(n-3)}_{\xi_{(2)}}(0)\cdot 3!qp^2=0. \quad (8)$$

Combining (7) with (8), we are able to arrive at $P_n=P_{n-1}-qp^2\cdot P_{n-3},\ \ n\geq 4$, from which we can derive $P_3=P_4>P_5>P_6>\cdots$. Note that $P_0=P_1=0, P_2=p^2, P_3=qp^2$, then we have $P_2=\max\{P_n,\ \ n=0,1,2,\cdots\}$. Therefore, $m_{\xi_{(2)},p}=2$.

Case II: When $k=3$, following Lemma 24, the probability generating function of $\xi_{(3)}$ distributed as $G_3(p)$ is represented as follows

$$G_{\xi_{(3)}}(x)=\frac{p^3x^3-p^4x^4}{1-x+qp^3x^4}.$$

By the formula

$$P_n=P(\xi_{(3)}=n)=G^{(n)}_{\xi_3}(0)/n!, n=0,1,\cdots, \quad (9)$$

we shall get

$$P_0=P_1=P_2=0, P_3=p^3, P_4=qp^3.$$

When $n\geq 5$, following the probability generating function $G_{\xi_{(3)}}(x)$, we have

$$G_{\xi_{(3)}}(x)\cdot(1-x+qp^3x^4)=p^3x^3-p^4x^4,$$

Differentiating n times both sides of the above, we arrive at

$$\sum_{r=0}^{n}\binom{n}{r}G^{(n-r)}_{\xi_{(3)}}(x)\cdot(1-x+qp^3x^4)^{(r)}=0.$$

Let $x=0$, we get

$$\binom{n}{0}G^{(n)}_{\xi_{(3)}}(0)-\binom{n}{1}G^{(n-1)}_{\xi_{(3)}}(0)+\binom{n}{4}G^{(n-4)}_{\xi_{(3)}}(0)\cdot 4!qp^3=0. \quad (10)$$

By equations (9) and (10), we obtain the recurrence relation

$$P_n=P_{n-1}-qp^3\cdot P_{n-4}, n=5,6,\cdots,$$

obviously, by the recurrence, we shall get

$$P_4=P_5=P_6>P_7>P_8>\cdots,$$

note that $P_0=P_1=P_2=0, P_3=p^3, P_4=qp^3$, then we obtain $P_3=\max\{P_n, n=0,1,2,\cdots\}$. Hence, $m_{\xi_{(3)},p}=3$.

By the similar proofs of Case I and Case II, we can conclude that $m_{\xi_{(k)},p}=k$. □

5 Conclusion

In section 2, we introduce the multivariate transition probability flow graphs methods to study the new distributions $MG^n(p_1,\cdots,p_n)$ and $G^*(p_1,\cdots,p_n)$, obtain their modes $\boldsymbol{m}_{\boldsymbol{\xi}_n}=(1,1,\cdots,1)$ and $m_{\xi^*}=n,n+1,\cdots,2n-1$. In section 3, for the multinomial distribution $M^n(p_1,\cdots,p_n;N)$, we only get its mode in some cases. It is still an open question for all other cases. From the multinomial distribution, we state the definition of the multivariate Poisson distribution $MP^n(\lambda_1,\cdots,\lambda_n)$, and then consider its joint probability distribution and modes. In section 4 we show that $m_{\xi_{(k)},p}=k$ is the unique mode of the geometric distribution of order k.

Finally, we offer the following open questions to the interested readers as an objective and challenge for further study.

Question 1 *[10] Let $N_n^{(k)}$ be the number of the success run of length k in n Bernoulli trials with success probability p. The probability distribution of it denoted by $B_k(n,p)$ is called the binomial distribution of order k with parameter (n,p). We have*

$$P(N_n^{(k)}=r)=\sum_{s=0}^{k-1}\sum_{\substack{m_1,\cdots,m_k\ni\\ \sum_{l=1}^{k}l\cdot m_l=n-s-kr}}\binom{\sum_{l=1}^{k}m_l+r}{m_1,\cdots,m_k,r}\left(\frac{q}{p}\right)^{\sum_{l=1}^{k}m_l}p^n,$$

where $r=0,1,\cdots,[n/k]$.
What is the mode of $B_k(n,p)$?

Question 2 *[20] Let $\xi_{(k,r)}$ be the number of trials until the r^{th} occurrence of the success run of length k in Bernoulli trials with success probability p. Then $\xi_{(k,r)}$ is a random variable distributed as the negative binomial distribution of order k with parameter (r,p) denoted by $NB_k(r,p)$. Its probability generating function is presented as follows*

$$G_{\xi_{(k,r)}}(x)=\left(\frac{p^kx^k-p^{k+1}x^{k+1}}{1-x+qp^kx^{k+1}}\right)^r.$$

What is the mode of $NB_k(r,p)$?

Question 3 *The random variable $\zeta_{(k)}$ is said to have the Poisson distribution of order k with parameter λ denoted by $P_k(\lambda)$ if its probability generating function is given by*

$$G_{\zeta_{(k)}}(x)=e^{\lambda(x+x^2+\cdots+x^k-k)}.$$

In addition, its probability distribution is

$$P(\zeta_{(k)} = m) = \sum_{m_1, m_2, \cdots, m_k} \frac{\lambda^{m_1+m_2+\cdots+m_k}}{m_1! m_2! \cdots m_k!} e^{-\lambda k},$$

where $m_1 + 2m_2 + \cdots + km_k = m, m = 0, 1, \cdots$. *What is the mode of* $P_k(\lambda)$*?*

Question 4 *What is the mode of multinomial distribution* $M^n(p_1, \cdots, p_n; N)$ *if* $Np_l, l = 1, \cdots, n$ *are not all integers?*

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