Potential Eventual Positivity of One Specific Tree Sign Pattern

BER-LIN YU
Huaiyin Institute of Technology
Faculty of Mathematics and Physics
Huia'an, Jiangsu 223003
P.R. China
berlinyu@163.com

JIE CUI
Huaiyin Institute of Technology
School of Economics and Management
Huia'an, Jiangsu 223003
P.R. China
cuijie@hyit.edu.cn

Abstract: A sign pattern is a matrix whose entries belong to the set $\{+, -, 0\}$. An n-by-n sign pattern $\mathcal{A}$ is said to allow an eventually positive matrix if there exist some real matrices A with the same sign pattern as $\mathcal{A}$ and a positive integer k_0 such that $A^k > 0$ for all $k \geq k_0$. Identifying and classifying the n-by-n sign patterns that allow an eventually positive matrix were posed as two open problems by Berman, Catral, Dealba, et al. [Sign patterns that allow eventual positivity, *ELA*, 19(2010): 108-120]. In this article, a tree sign pattern $\mathcal{A}$ obtained from the double star sign pattern $S_{3,2}$ by adding an pendent edge is investigated. Some necessary or sufficient conditions for the sign pattern $\mathcal{A}$ to allow an eventually positive matrix are established first. Then all the minimal tree sign patterns that allow an eventually positive matrix are identified as five specific tree sign patterns. Finally, all the tree sign patterns that allow an eventually positive matrix is classified.

Key–Words: Sign pattern, Tree, Eventually positive matrix; Checkerboard block matrix

1 Introduction

A sign pattern is a matrix $\mathcal{A} = [\alpha_{ij}]$ with entries in the set $\{+, -, 0\}$. An n-by-n real matrix A with the same sign pattern as $\mathcal{A}$ is called a realization of $\mathcal{A}$. The set of all realizations of sign pattern $\mathcal{A}$ is called the qualitative class of $\mathcal{A}$ and is denoted by $Q(\mathcal{A})$. A subpattern of $\mathcal{A} = [\alpha_{ij}]$ is an n-by-n sign pattern $\mathcal{B} = [\beta_{ij}]$ such that $\beta_{ij} = 0$ whenever $\alpha_{ij} = 0$. If $\mathcal{B} \neq \mathcal{A}$, then $\mathcal{B}$ is a proper subpattern of $\mathcal{A}$. If $\mathcal{B}$ is a subpattern of $\mathcal{A}$, then $\mathcal{A}$ is said to be a superpattern of $\mathcal{B}$. A pattern $\mathcal{A}$ is reducible if there is a permutation matrix $\mathcal{P}$ such that

$$\mathcal{P}^T \mathcal{A} \mathcal{P} = \begin{bmatrix} \mathcal{A}_{11} & 0 \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{bmatrix},$$

where $\mathcal{A}_{11}$ and $\mathcal{A}_{22}$ are square matrices of order at least one. A pattern is irreducible if it is not reducible; see, e.g. [1].

A sign pattern matrix $\mathcal{A}$ is said to require a certain property P referring to real matrices if every real matrix $A \in Q(\mathcal{A})$ has the property P and allow P or be potentially P if there is some $A \in Q(\mathcal{A})$ that has property P.

Recall that an n-by-n real matrix A is said to be eventually positive if there exists a positive integer k_0 such that $A^k > 0$ for all $k \geq k_0$; see, e.g., [2] and [3]. Eventually positive matrices have applications to dynamical systems in situations where it is of interest to determine whether an initial trajectory reaches positivity at a certain time and remains positive thereafter; see e.g., [4]. An n-by-n sign pattern $\mathcal{A}$ is said to allow an eventually positive matrix or be potentially eventually positive (PEP), if there exists some $A \in Q(\mathcal{A})$ such that A is eventually positive; see, e.g., [5] and the references therein. An n-by-n sign pattern $\mathcal{A}$ is said to be a minimal potentially eventually positive sign pattern (MPEP sign pattern) if $\mathcal{A}$ is PEP and no proper subpattern of $\mathcal{A}$ is PEP; see, e.g. [6] for more details. Sign patterns that allow an eventually positive matrix have been studied first in [5], where a sufficient condition and some necessary conditions for a sign pattern to be potentially eventually positive have been established. However, the identification of necessary and sufficient conditions for an n-by-n sign pattern ($n \geq 4$) to be potentially eventually positive remains open. Also open is the classification of sign patterns that are potentially eventually positive.

Recall that an n-by-n real matrix A is said to be power-positive if there exists a nonnegative integer k such that $A^k > 0$. An n-by-n sign pattern $\mathcal{A}$ is said to allow a power-positive matrix or be potentially power-positive (PPP), if there exists some $A \in Q(\mathcal{A})$ such that A is power-positive; see, e.g., [7]. A relation between potentially eventually positive sign patterns and potentially power-positive sign patterns has been established in [7]. An n-by-n sign pattern $\mathcal{A}$ is said to a minimal potentially power-positive (MPPP) sign pattern, if $\mathcal{A}$ is potentially power-positive and no proper subpattern of $\mathcal{A}$ is potentially power-positive; see, [8] for example. A relation between the minimal potentially eventually positive sign patterns and the minimal potentially power-positive sign patterns has been investigate in [8]. At present, there are a few literatures on the potential eventual positivity of some specific sign pattern matrices; see e.g., [6], [7], [8], [9] and [10]. Especially, the potentially eventually positive double star sign patterns have been identified and classified in [6]. More recently, the minimal potentially eventually positive tridiagonal sign patterns have been identified and all potentially eventually positive tridiagonal sign patterns have been classified in [10].

In this article, we focus on the eventual positivity of a specific tree sign pattern $\mathcal{A}$ obtained from the double star sign pattern $S_{3,2}$ by adding an pendent edge. Our work is organized as follows. In Section 2, some preliminary results for sign pattern $\mathcal{A}$ to allow an eventually positive matrix are established. In Section 3, all the minimal potentially eventually positive tree sign patterns are identified as the five specific tree sign patterns, and hence the potentially eventually positive tree sign patterns are classified. Some remarks and future work are concluded in Section 4.

2 Preliminary Results for Sign Pattern $\mathcal{A}$ to be Potentially Eventually Positive

We begin this section with introducing some necessary graph theoretical concepts which can be seen from [2] and the references therein.

A square sign pattern $\mathcal{A} = [\alpha_{ij}]$ is combinatorially symmetric if $\alpha_{ij} \neq 0$ whenever $\alpha_{ji} \neq 0$. Let $G(\mathcal{A})$ be the graph of order n with vertices 1, 2, ..., n and an edge $\{i, j\}$ joining vertices i and j if and only if $i \neq j$ and $\alpha_{ij} \neq 0$. We call $G(\mathcal{A})$ the graph of the pattern $\mathcal{A}$. A combinatorially symmetric sign pattern matrix $\mathcal{A}$ is called a tree sign pattern if $G(\mathcal{A})$ is a tree. Similarly, path (or tridiagonal) and double star sign patterns can be defined.

A sign pattern $\mathcal{A} = [\alpha_{ij}]$ has signed digraph $\Gamma(\mathcal{A})$ with vertex set $\{1, 2, \cdots, n\}$ and a positive (respectively, negative) arc from i to j if and only if α_{ij} is positive (respectively, negative). A (directed) simple cycle of length k is a sequence of k arcs $(i_1, i_2), (i_2, i_3), \cdots, (i_k, i_1)$ such that the vertices $i_1, \cdots, i_k$ are distinct. Recall that a digraph $D = (V, E)$ is primitive if it is strongly connected and the greatest common divisor of the lengths of its cycles is 1. It is well known that a digraph D is primitive if and only if there exists a natural number k such that for all $V_i \in V$, $V_j \in V$, there is a walk of length k from V_i to V_j. A nonnegative sign pattern $\mathcal{A}$ is primitive if its signed digraph $\Gamma(\mathcal{A})$ is primitive; see, e.g. [5] for more details.

For a sign pattern $\mathcal{A} = [\alpha_{ij}]$, we define the positive part of $\mathcal{A}$ to be $\mathcal{A}^+ = [\alpha_{ij}^+]$, where $\alpha_{ij}^+ = +$ for $\alpha_{ij} = +$, otherwise $\alpha_{ij}^+ = 0$. The negative part of $\mathcal{A}$ can be defined similarly. In [5], it has been shown that if sign pattern $\mathcal{A}^+$ is primitive, then $\mathcal{A}$ is PEP. Here, we cite some necessary conditions for an n-by-n sign pattern to be potentially eventually positive in [5] as Lemmas 1 to 5 in order to state our work clearly.

Lemma 1. *If the n-by-n sign pattern $\mathcal{A}$ is PEP, then every superpattern of $\mathcal{A}$ is PEP.*

Lemma 2. *If the n-by-n sign pattern $\mathcal{A}$ is PEP, then the sign pattern $\hat{\mathcal{A}}$ obtained from sign pattern $\mathcal{A}$ by changing all 0 and $-$ diagonal entries to $+$ is also PEP.*

Lemma 3. *If the n-by-n sign pattern $\mathcal{A}$ is PEP, then there is an eventually positive matrix $A \in Q(\mathcal{A})$ such that*

(1) $\rho(A) = 1$.

(2) $A\mathbf{1} = \mathbf{1}$, where $\mathbf{1}$ is the $n \times 1$ all ones vector.

If $n \geq 2$, the sum of all the off-diagonal entries of A is positive.

We denote a sign pattern consisting entirely of positive (respectively, negative) entries by $[+]$ (respectively, $[-]$). Let $[+]_{i\times i}$ be a square block sign pattern of order i consisting entirely of positive entries. For block sign patterns, we have the following two lemmas.

Lemma 4. *If $\mathcal{A}$ is the checkerboard block sign pattern*

$$\begin{bmatrix} [+] & [-] & [+] & \cdots \\ [-] & [+] & [-] & \cdots \\ [+] & [-] & [+] & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

with square diagonal blocks. Then $-\mathcal{A}$ is not PEP, and if $\mathcal{A}$ has a negative entry, then $\mathcal{A}$ is not PEP.

Lemma 5. *Let the n-by-n sign pattern*

$$\mathcal{A} = \begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{bmatrix},$$

where $\mathcal{A}_{11}$, $\mathcal{A}_{22}$ are square. If $\mathcal{A}_{12} = \mathcal{A}_{12}^{-}$, $\mathcal{A}_{21} = \mathcal{A}_{21}^{+}$, then $\mathcal{A}$ and $\mathcal{A}^T$ are not PEP.

Recall that a vertex with degree 1 in a graph is called a leaf vertex or end vertex, and the edge incident with that vertex is called a pendant edge; see [1] for example. Now we turn to the specific sign pattern which are obtained from a double star sign pattern by adding a pendent edge. Since sign patterns $\mathcal{A}$ is potentially eventually positive if and only if $\mathcal{A}^T$ or $\mathcal{P}^T\mathcal{A}\mathcal{P}$ is potentially eventually positive, for any permutation pattern $\mathcal{P}$. Thus, without loss of generality, let sign pattern

$$\mathcal{A} = [\alpha_{ij}] = \begin{bmatrix} ? & * & * & * & 0 & 0 \\ * & ? & 0 & 0 & 0 & 0 \\ * & 0 & ? & 0 & 0 & 0 \\ * & 0 & 0 & ? & * & 0 \\ * & 0 & 0 & * & ? & * \\ 0 & 0 & 0 & 0 & * & ? \end{bmatrix},$$

where $?$ denotes an entry from $\{+,-,0\}$ and $*$ denotes a nonzero entry. Note that $\mathcal{A}$ is a combinatorially symmetric sign pattern and the graph $G(\mathcal{A})$ is a tree. It is clear that $\mathcal{A}$ is not a doublet star sign pattern which has been investigated in [6].

The following propositions are necessary for tree sign pattern $\mathcal{A}$ to be potentially eventually positive.

Proposition 6. *If tree sign pattern $\mathcal{A}$ is potentially eventually positive, then there exists some $i \in \{1,2,3,4,5,6\}$ such that $\alpha_{ii} = +$.*

Proof: By a way of contradiction, assume that $\alpha_{ii} = -$ or 0 for all $i = 1,2,\ldots,6$. Since sign pattern $\mathcal{A}$ is potentially eventually positive, it follows that $\alpha_{12} = \alpha_{21} = +$, $\alpha_{13} = \alpha_{31} = +$, and $\alpha_{56} = \alpha_{65} = +$. What is more, by Lemma 5, we have $\alpha_{14} = \alpha_{41}$, and $\alpha_{45} = \alpha_{54}$. Up to equivalence, the potentially eventually positive sign pattern $\mathcal{A}$ must be one of the following three sign patterns:

$$\mathcal{A}' = \left[\begin{array}{c|cc|c|c|c} \ominus & + & + & - & 0 & 0 \\ \hline + & \ominus & 0 & 0 & 0 & 0 \\ + & 0 & \ominus & 0 & 0 & 0 \\ \hline - & 0 & 0 & \ominus & + & 0 \\ \hline 0 & 0 & 0 & + & \ominus & + \\ \hline 0 & 0 & 0 & 0 & + & \ominus \end{array}\right],$$

$$\mathcal{A}'' = \left[\begin{array}{c|ccc|c|c} \ominus & + & + & + & 0 & 0 \\ \hline + & \ominus & 0 & 0 & 0 & 0 \\ + & 0 & \ominus & 0 & 0 & 0 \\ + & 0 & 0 & \ominus & + & 0 \\ \hline 0 & 0 & 0 & + & \ominus & + \\ \hline 0 & 0 & 0 & 0 & + & \ominus \end{array}\right],$$

and

$$\mathcal{A}''' = \left[\begin{array}{c|cccc|c} \ominus & + & + & + & 0 & 0 \\ \hline + & \ominus & 0 & 0 & 0 & 0 \\ + & 0 & \ominus & 0 & 0 & 0 \\ + & 0 & 0 & \ominus & - & 0 \\ 0 & 0 & 0 & - & \ominus & + \\ \hline 0 & 0 & 0 & 0 & + & \ominus \end{array}\right],$$

where $\ominus$ denotes an entry from the set $\{0,-\}$. It is clear that sign patterns $\mathcal{A}'$, $\mathcal{A}''$ and $\mathcal{A}'''$ are a proper subpattern of some checkerboard block sign patterns, respectively. By Lemmas 4 and 1, $\mathcal{A}'$, $\mathcal{A}''$ and $\mathcal{A}'''$ are not potentially eventually positive; a contradiction. Hence, there exists some $i \in \{1,2,3,4,5,6\}$ such that $\alpha_{ii} = +$. $\square$

Proposition 7. *If tree sign pattern $\mathcal{A}$ is potentially eventually positive, then $\mathcal{A}$ is symmetric.*

Proof: Since sign pattern $\mathcal{A}$ is potentially eventually positive, let real matrix $A \in Q(\mathcal{A})$ be eventually positive. By Lemma 3, let $a_{22} = 1 - a_{21}$, $a_{33} = 1 - a_{31}$, $a_{44} = 1 - a_{41} - a_{45}$, $a_{55} = 1 - a_{54} - a_{56}$ and $a_{66} = 1 - a_{65}$. To complete the proof, it suffices to show that $a_{21}a_{12} > 0$, $a_{31}a_{13} > 0$, $a_{41}a_{14} > 0$, $a_{45}a_{54} > 0$ and $a_{56}a_{65} > 0$. Suppose the positive left eigenvector of A is $w = (w_1, w_2, \ldots, w_6)^T$. Then by $w^TA = w^T$, we have the following equalities:

$$w_5 a_{56} + w_6(1 - a_{65}) = w_6, \tag{1}$$

$$w_4 a_{45} + w_5(1 - a_{54} - a_{56}) + w_6 a_{65} = w_5, \tag{2}$$

$$w_1 a_{14} + w_4(1 - a_{41} - a_{45}) + w_5 a_{54} = w_4, \tag{3}$$

$$w_1 a_{13} + w_3(1 - a_{31}) = w_3, \tag{4}$$

and

$$w_1 a_{12} + w_2(1 - a_{21}) = w_2. \tag{5}$$

By Equalities (1), (4) and (5), we have $a_{21}a_{12} > 0$, $a_{31}a_{13} > 0$ and $a_{56}a_{65} > 0$. By Equalities (1), (2) and (3), we have $a_{41}a_{14} > 0$ and $a_{45}a_{54} > 0$. It follows that $\mathcal{A}$ is symmetric. $\square$

Lemma 8. *The following five tree sign patterns*

$$\mathcal{A}_{12} = \begin{bmatrix} ? & - & + & + & 0 & 0 \\ - & ? & 0 & 0 & 0 & 0 \\ + & 0 & ? & 0 & 0 & 0 \\ + & 0 & 0 & ? & + & 0 \\ 0 & 0 & 0 & + & ? & + \\ 0 & 0 & 0 & 0 & + & ? \end{bmatrix},$$

$$\mathcal{A}_{13} = \begin{bmatrix} ? & + & - & + & 0 & 0 \\ + & ? & 0 & 0 & 0 & 0 \\ - & 0 & ? & 0 & 0 & 0 \\ + & 0 & 0 & ? & + & 0 \\ 0 & 0 & 0 & + & ? & + \\ 0 & 0 & 0 & 0 & + & ? \end{bmatrix},$$

$$\mathcal{A}_{14} = \begin{bmatrix} ? & + & + & - & 0 & 0 \\ + & ? & 0 & 0 & 0 & 0 \\ + & 0 & ? & 0 & 0 & 0 \\ - & 0 & 0 & ? & + & 0 \\ 0 & 0 & 0 & + & ? & + \\ 0 & 0 & 0 & 0 & + & ? \end{bmatrix},$$

$$\mathcal{A}_{15} = \begin{bmatrix} ? & + & + & + & 0 & 0 \\ + & ? & 0 & 0 & 0 & 0 \\ + & 0 & ? & 0 & 0 & 0 \\ + & 0 & 0 & ? & - & 0 \\ 0 & 0 & 0 & - & ? & + \\ 0 & 0 & 0 & 0 & + & ? \end{bmatrix},$$

and

$$\mathcal{A}_{16} = \begin{bmatrix} ? & + & + & + & 0 & 0 \\ + & ? & 0 & 0 & 0 & 0 \\ + & 0 & ? & 0 & 0 & 0 \\ + & 0 & 0 & ? & + & 0 \\ 0 & 0 & 0 & + & ? & - \\ 0 & 0 & 0 & 0 & - & ? \end{bmatrix},$$

are not potentially eventually positive.

Proof: By a way of contradiction, assume that sign patterns $\mathcal{A}_{12}$, $\mathcal{A}_{13}$, $\mathcal{A}_{14}$, $\mathcal{A}_{15}$ and $\mathcal{A}_{16}$ are potentially eventually positive. Let $\hat{\mathcal{A}_{12}}$, $\hat{\mathcal{A}_{13}}$, $\hat{\mathcal{A}_{14}}$, $\hat{\mathcal{A}_{15}}$ and $\hat{\mathcal{A}_{16}}$ be sign patterns obtained by changing all diagonal entries of sign patterns $\mathcal{A}_{12}$, $\mathcal{A}_{13}$, $\mathcal{A}_{14}$, $\mathcal{A}_{15}$ and $\mathcal{A}_{16}$ to $+$, respectively. By Lemma 2, sign patterns $\hat{\mathcal{A}_{12}}$, $\hat{\mathcal{A}_{13}}$, $\hat{\mathcal{A}_{14}}$, $\hat{\mathcal{A}_{15}}$ and $\hat{\mathcal{A}_{16}}$ are also potentially eventually positive. But sign patterns $\hat{\mathcal{A}_{12}}$, $\hat{\mathcal{A}_{13}}$, $\hat{\mathcal{A}_{14}}$, $\hat{\mathcal{A}_{15}}$ and $\hat{\mathcal{A}_{16}}$ are a proper subpattern of the checkerboard block sign patterns

$$\begin{bmatrix} [+]_{1\times1} & [-] & [+] \\ [-] & [+]_{1\times1} & [-] \\ [+] & [-] & [+]_{4\times4} \end{bmatrix},$$

$$\begin{bmatrix} [+]_{2\times2} & [-] & [+] \\ [-] & [+]_{1\times1} & [-] \\ [+] & [-] & [+]_{3\times3} \end{bmatrix},$$

$$\begin{bmatrix} [+]_{3\times3} & [-] \\ [-] & [+]_{3\times3} \end{bmatrix},$$

$$\begin{bmatrix} [+]_{4\times4} & [-] \\ [-] & [+]_{2\times2} \end{bmatrix},$$

and

$$\begin{bmatrix} [+]_{5\times5} & [-] \\ [-] & [+]_{1\times1} \end{bmatrix},$$

respectively. It follows that sign patterns $\hat{\mathcal{A}_{12}}$, $\hat{\mathcal{A}_{13}}$, $\hat{\mathcal{A}_{14}}$, $\hat{\mathcal{A}_{15}}$ and $\hat{\mathcal{A}_{16}}$ are not potentially eventually positive. Consequently, sign patterns $\mathcal{A}_{12}$, $\mathcal{A}_{13}$, $\mathcal{A}_{14}$, $\mathcal{A}_{15}$ and $\mathcal{A}_{16}$ are not potentially eventually positive; a contradiction. □

Lemma 9. *The following ten tree sign patterns are not potentially eventually positive:*

$$\mathcal{A}_{23} = \begin{bmatrix} ? & - & - & + & 0 & 0 \\ - & ? & 0 & 0 & 0 & 0 \\ - & 0 & ? & 0 & 0 & 0 \\ + & 0 & 0 & ? & + & 0 \\ 0 & 0 & 0 & + & ? & + \\ 0 & 0 & 0 & 0 & + & ? \end{bmatrix},$$

$$\mathcal{A}_{24} = \begin{bmatrix} ? & - & + & - & 0 & 0 \\ - & ? & 0 & 0 & 0 & 0 \\ + & 0 & ? & 0 & 0 & 0 \\ - & 0 & 0 & ? & + & 0 \\ 0 & 0 & 0 & + & ? & + \\ 0 & 0 & 0 & 0 & + & ? \end{bmatrix},$$

$$\mathcal{A}_{25} = \begin{bmatrix} ? & - & + & + & 0 & 0 \\ - & ? & 0 & 0 & 0 & 0 \\ + & 0 & ? & 0 & 0 & 0 \\ + & 0 & 0 & ? & - & 0 \\ 0 & 0 & 0 & - & ? & + \\ 0 & 0 & 0 & 0 & + & ? \end{bmatrix},$$

$$\mathcal{A}_{26} = \begin{bmatrix} ? & - & + & + & 0 & 0 \\ - & ? & 0 & 0 & 0 & 0 \\ + & 0 & ? & 0 & 0 & 0 \\ + & 0 & 0 & ? & + & 0 \\ 0 & 0 & 0 & + & ? & - \\ 0 & 0 & 0 & 0 & - & ? \end{bmatrix},$$

$$\mathcal{A}_{34} = \begin{bmatrix} ? & + & - & - & 0 & 0 \\ + & ? & 0 & 0 & 0 & 0 \\ - & 0 & ? & 0 & 0 & 0 \\ - & 0 & 0 & ? & + & 0 \\ 0 & 0 & 0 & + & ? & + \\ 0 & 0 & 0 & 0 & + & ? \end{bmatrix},$$

$$\mathcal{A}_{35} = \begin{bmatrix} ? & + & - & + & 0 & 0 \\ + & ? & 0 & 0 & 0 & 0 \\ - & 0 & ? & 0 & 0 & 0 \\ + & 0 & 0 & ? & - & 0 \\ 0 & 0 & 0 & - & ? & + \\ 0 & 0 & 0 & 0 & + & ? \end{bmatrix},$$

$$\mathcal{A}_{36} = \begin{bmatrix} ? & + & - & + & 0 & 0 \\ + & ? & 0 & 0 & 0 & 0 \\ - & 0 & ? & 0 & 0 & 0 \\ + & 0 & 0 & ? & + & 0 \\ 0 & 0 & 0 & + & ? & - \\ 0 & 0 & 0 & 0 & - & ? \end{bmatrix},$$

$$\mathcal{A}_{45} = \begin{bmatrix} ? & + & + & - & 0 & 0 \\ + & ? & 0 & 0 & 0 & 0 \\ + & 0 & ? & 0 & 0 & 0 \\ - & 0 & 0 & ? & - & 0 \\ 0 & 0 & 0 & - & ? & + \\ 0 & 0 & 0 & 0 & + & ? \end{bmatrix},$$

$$\mathcal{A}_{46} = \begin{bmatrix} ? & + & + & - & 0 & 0 \\ + & ? & 0 & 0 & 0 & 0 \\ + & 0 & ? & 0 & 0 & 0 \\ - & 0 & 0 & ? & + & 0 \\ 0 & 0 & 0 & + & ? & - \\ 0 & 0 & 0 & 0 & - & ? \end{bmatrix},$$

and

$$\mathcal{A}_{56} = \begin{bmatrix} ? & + & + & + & 0 & 0 \\ + & ? & 0 & 0 & 0 & 0 \\ + & 0 & ? & 0 & 0 & 0 \\ + & 0 & 0 & ? & - & 0 \\ 0 & 0 & 0 & - & ? & - \\ 0 & 0 & 0 & 0 & - & ? \end{bmatrix}.$$

Proof: By a way of contradiction, assume that sign patterns $\mathcal{A}_{23}$, $\mathcal{A}_{24}$, $\mathcal{A}_{25}$, $\mathcal{A}_{26}$, $\mathcal{A}_{34}$, $\mathcal{A}_{35}$, $\mathcal{A}_{36}$, $\mathcal{A}_{45}$, $\mathcal{A}_{46}$ and $\mathcal{A}_{56}$ are potentially eventually positive. Let $\hat{\mathcal{A}_{23}}$, $\hat{\mathcal{A}_{24}}$, $\hat{\mathcal{A}_{25}}$, $\hat{\mathcal{A}_{26}}$, $\hat{\mathcal{A}_{34}}$, $\hat{\mathcal{A}_{35}}$, $\hat{\mathcal{A}_{36}}$, $\hat{\mathcal{A}_{45}}$, $\hat{\mathcal{A}_{46}}$ and $\hat{\mathcal{A}_{56}}$ be the sign patterns obtained from sign patterns $\mathcal{A}_{23}$, $\mathcal{A}_{24}$, $\mathcal{A}_{25}$, $\mathcal{A}_{26}$, $\mathcal{A}_{34}$, $\mathcal{A}_{35}$, $\mathcal{A}_{36}$, $\mathcal{A}_{45}$, $\mathcal{A}_{46}$ and $\mathcal{A}_{56}$ by changing all diagonal entries to be $+$, respectively. By Lemma 2, sign patterns $\hat{\mathcal{A}_{23}}$, $\hat{\mathcal{A}_{24}}$, $\hat{\mathcal{A}_{25}}$, $\hat{\mathcal{A}_{26}}$, $\hat{\mathcal{A}_{34}}$, $\hat{\mathcal{A}_{35}}$, $\hat{\mathcal{A}_{36}}$, $\hat{\mathcal{A}_{45}}$, $\hat{\mathcal{A}_{46}}$ and $\hat{\mathcal{A}_{56}}$ are potentially eventually positive. But the previous sign patterns $\hat{\mathcal{A}_{23}}$, $\hat{\mathcal{A}_{24}}$, $\hat{\mathcal{A}_{25}}$, $\hat{\mathcal{A}_{26}}$, $\hat{\mathcal{A}_{34}}$, $\hat{\mathcal{A}_{35}}$, $\hat{\mathcal{A}_{36}}$, $\hat{\mathcal{A}_{45}}$, $\hat{\mathcal{A}_{46}}$ and $\hat{\mathcal{A}_{56}}$ are a proper subpattern of a checkerboard block sign patterns

$$\begin{bmatrix} [+]_{1\times1} & [-] & [+] \\ [-] & [+]_{2\times2} & [-] \\ [+] & [-] & [+]_{3\times3} \end{bmatrix},$$

$$\begin{bmatrix} [+]_{1\times1} & [-] & [+] & [-] & [+] \\ [-] & [+]_{1\times1} & [-] & [+] & [-] \\ [+] & [-] & [+]_{1\times1} & [-] & [+] \\ [-] & [+] & [-] & [+]_{1\times1} & [-] \\ [+] & [-] & [+] & [-] & [+]_{2\times2} \end{bmatrix},$$

$$\begin{bmatrix} [+]_{1\times1} & [-] & [+] & [-] \\ [-] & [+]_{1\times1} & [-] & [+] \\ [+] & [-] & [+]_{2\times2} & [-] \\ [-] & [+] & [-] & [+]_{2\times2} \end{bmatrix},$$

$$\begin{bmatrix} [+]_{1\times1} & [-] & [+] & [-] \\ [-] & [+]_{1\times1} & [-] & [+] \\ [+] & [-] & [+]_{3\times3} & [-] \\ [-] & [+] & [-] & [+]_{1\times1} \end{bmatrix},$$

$$\begin{bmatrix} [+]_{2\times2} & [-] \\ [-] & [+]_{4\times4} \end{bmatrix},$$

$$\begin{bmatrix} [+]_{2\times2} & [-] & [+] & [-] \\ [-] & [+]_{1\times1} & [-] & [+] \\ [+] & [-] & [+]_{1\times1} & [-] \\ [-] & [+] & [-] & [+]_{2\times2} \end{bmatrix},$$

$$\begin{bmatrix} [+]_{2\times2} & [-] & [+] & [-] \\ [-] & [+]_{1\times1} & [-] & [+] \\ [+] & [-] & [+]_{2\times2} & [-] \\ [-] & [+] & [-] & [+]_{1\times1} \end{bmatrix},$$

$$\begin{bmatrix} [+]_{3\times3} & [-] & [+] \\ [-] & [+]_{1\times1} & [-] \\ [+] & [-] & [+]_{2\times2} \end{bmatrix},$$

$$\begin{bmatrix} [+]_{3\times3} & [-] & [+] \\ [-] & [+]_{2\times2} & [-] \\ [+] & [-] & [+]_{1\times1} \end{bmatrix},$$

and

$$\begin{bmatrix} [+]_{4\times4} & [-] & [+] \\ [-] & [+]_{1\times1} & [-] \\ [+] & [-] & [+]_{1\times1} \end{bmatrix},$$

respectively. It follows from Lemma 4 that sign patterns $\hat{\mathcal{A}_{23}}$, $\hat{\mathcal{A}_{24}}$, $\hat{\mathcal{A}_{25}}$, $\hat{\mathcal{A}_{26}}$, $\hat{\mathcal{A}_{34}}$, $\hat{\mathcal{A}_{35}}$, $\hat{\mathcal{A}_{36}}$, $\hat{\mathcal{A}_{45}}$, $\hat{\mathcal{A}_{46}}$ and $\hat{\mathcal{A}_{56}}$ are not potentially eventually positive. Consequently, sign patterns $\mathcal{A}_{23}$, $\mathcal{A}_{24}$, $\mathcal{A}_{25}$, $\mathcal{A}_{26}$, $\mathcal{A}_{34}$, $\mathcal{A}_{35}$, $\mathcal{A}_{36}$, $\mathcal{A}_{45}$, $\mathcal{A}_{46}$ are not potentially eventually positive; a contradiction. □

Lemma 10. *The following ten tree sign patterns are not potentially eventually positive:*

$$\mathcal{A}_{234} = \begin{bmatrix} ? & - & - & - & 0 & 0 \\ - & ? & 0 & 0 & 0 & 0 \\ - & 0 & ? & 0 & 0 & 0 \\ - & 0 & 0 & ? & + & 0 \\ 0 & 0 & 0 & + & ? & + \\ 0 & 0 & 0 & 0 & + & ? \end{bmatrix},$$

$$\mathcal{A}_{235} = \begin{bmatrix} ? & - & - & + & 0 & 0 \\ - & ? & 0 & 0 & 0 & 0 \\ - & 0 & ? & 0 & 0 & 0 \\ + & 0 & 0 & ? & - & 0 \\ 0 & 0 & 0 & - & ? & + \\ 0 & 0 & 0 & 0 & + & ? \end{bmatrix},$$

$$\mathcal{A}_{236} = \begin{bmatrix} ? & - & - & + & 0 & 0 \\ - & ? & 0 & 0 & 0 & 0 \\ - & 0 & ? & 0 & 0 & 0 \\ + & 0 & 0 & ? & + & 0 \\ 0 & 0 & 0 & + & ? & - \\ 0 & 0 & 0 & 0 & - & ? \end{bmatrix},$$

$$\mathcal{A}_{245} = \begin{bmatrix} ? & - & + & - & 0 & 0 \\ - & ? & 0 & 0 & 0 & 0 \\ + & 0 & ? & 0 & 0 & 0 \\ - & 0 & 0 & ? & - & 0 \\ 0 & 0 & 0 & - & ? & + \\ 0 & 0 & 0 & 0 & + & ? \end{bmatrix},$$

$$\mathcal{A}_{246} = \begin{bmatrix} ? & - & + & - & 0 & 0 \\ - & ? & 0 & 0 & 0 & 0 \\ + & 0 & ? & 0 & 0 & 0 \\ - & 0 & 0 & ? & + & 0 \\ 0 & 0 & 0 & + & ? & - \\ 0 & 0 & 0 & 0 & - & ? \end{bmatrix},$$

$$\mathcal{A}_{256} = \begin{bmatrix} ? & - & + & + & 0 & 0 \\ - & ? & 0 & 0 & 0 & 0 \\ + & 0 & ? & 0 & 0 & 0 \\ + & 0 & 0 & ? & - & 0 \\ 0 & 0 & 0 & - & ? & - \\ 0 & 0 & 0 & 0 & - & ? \end{bmatrix},$$

$$\mathcal{A}_{345} = \begin{bmatrix} ? & + & - & - & 0 & 0 \\ + & ? & 0 & 0 & 0 & 0 \\ - & 0 & ? & 0 & 0 & 0 \\ - & 0 & 0 & ? & - & 0 \\ 0 & 0 & 0 & - & ? & + \\ 0 & 0 & 0 & 0 & + & ? \end{bmatrix},$$

$$\mathcal{A}_{346} = \begin{bmatrix} ? & + & - & - & 0 & 0 \\ + & ? & 0 & 0 & 0 & 0 \\ - & 0 & ? & 0 & 0 & 0 \\ - & 0 & 0 & ? & + & 0 \\ 0 & 0 & 0 & + & ? & - \\ 0 & 0 & 0 & 0 & - & ? \end{bmatrix},$$

$$\mathcal{A}_{356} = \begin{bmatrix} ? & + & - & + & 0 & 0 \\ + & ? & 0 & 0 & 0 & 0 \\ - & 0 & ? & 0 & 0 & 0 \\ + & 0 & 0 & ? & - & 0 \\ 0 & 0 & 0 & - & ? & - \\ 0 & 0 & 0 & 0 & - & ? \end{bmatrix},$$

and

$$\mathcal{A}_{456} = \begin{bmatrix} ? & + & + & - & 0 & 0 \\ + & ? & 0 & 0 & 0 & 0 \\ + & 0 & ? & 0 & 0 & 0 \\ - & 0 & 0 & ? & - & 0 \\ 0 & 0 & 0 & - & ? & - \\ 0 & 0 & 0 & 0 & - & ? \end{bmatrix}.$$

Proof: By a way of contradiction, assume that sign patterns $\mathcal{A}_{234}$, $\mathcal{A}_{235}$, $\mathcal{A}_{236}$, $\mathcal{A}_{245}$, $\mathcal{A}_{246}$, $\mathcal{A}_{256}$, $\mathcal{A}_{345}$, $\mathcal{A}_{346}$, $\mathcal{A}_{356}$ and $\mathcal{A}_{456}$ are potentially eventually positive. Let $\hat{\mathcal{A}}_{234}$, $\hat{\mathcal{A}}_{235}$, $\hat{\mathcal{A}}_{236}$, $\hat{\mathcal{A}}_{245}$, $\hat{\mathcal{A}}_{246}$, $\hat{\mathcal{A}}_{256}$, $\hat{\mathcal{A}}_{345}$, $\hat{\mathcal{A}}_{346}$, $\hat{\mathcal{A}}_{356}$ and $\hat{\mathcal{A}}_{456}$ be the sign patterns obtained from sign patterns $\mathcal{A}_{234}$, $\mathcal{A}_{235}$, $\mathcal{A}_{236}$, $\mathcal{A}_{245}$, $\mathcal{A}_{246}$, $\mathcal{A}_{256}$, $\mathcal{A}_{345}$, $\mathcal{A}_{346}$, $\mathcal{A}_{356}$ and $\mathcal{A}_{456}$ by changing all diagonal entries to be $+$, respectively. By Lemma 2, sign patterns $\hat{\mathcal{A}}_{234}$, $\hat{\mathcal{A}}_{235}$, $\hat{\mathcal{A}}_{236}$, $\hat{\mathcal{A}}_{245}$, $\hat{\mathcal{A}}_{246}$, $\hat{\mathcal{A}}_{256}$, $\hat{\mathcal{A}}_{345}$, $\hat{\mathcal{A}}_{346}$, $\hat{\mathcal{A}}_{356}$ and $\hat{\mathcal{A}}_{456}$ are potentially eventually positive. But sign patterns $\hat{\mathcal{A}}_{234}$, $\hat{\mathcal{A}}_{235}$, $\hat{\mathcal{A}}_{236}$, $\hat{\mathcal{A}}_{245}$, $\hat{\mathcal{A}}_{246}$, $\hat{\mathcal{A}}_{256}$, $\hat{\mathcal{A}}_{345}$, $\hat{\mathcal{A}}_{346}$, $\hat{\mathcal{A}}_{356}$ and $\hat{\mathcal{A}}_{456}$ are a proper subpattern of a checkerboard block sign patterns

$$\begin{bmatrix} [+]_{1\times1} & [-] \\ [-] & [+]_{5\times5} \end{bmatrix},$$

$$\begin{bmatrix} [+]_{1\times1} & [-] & [+] & [-] \\ [-] & [+]_{2\times2} & [-] & [+] \\ [+] & [-] & [+]_{1\times1} & [-] \\ [-] & [+] & [-] & [+]_{2\times2} \end{bmatrix},$$

$$\begin{bmatrix} [+]_{1\times1} & [-] & [+] & [-] \\ [-] & [+]_{2\times2} & [-] & [+] \\ [+] & [-] & [+]_{2\times2} & [-] \\ [-] & [+] & [-] & [+]_{1\times1} \end{bmatrix},$$

$$\begin{bmatrix} [+]_{1\times1} & [-] & [+] & [-] & [+] \\ [-] & [+]_{1\times1} & [-] & [+] & [-] \\ [+] & [-] & [+]_{1\times1} & [-] & [+] \\ [-] & [+] & [-] & [+]_{1\times1} & [-] \\ [+] & [-] & [+]_{1\times1} & [-] & [+]_{2\times2} \end{bmatrix},$$

$$\begin{bmatrix} + & - & + & - & + & - \\ - & + & - & + & - & + \\ + & - & + & - & + & - \\ - & + & - & + & - & + \\ + & - & + & - & + & - \\ - & + & - & + & - & + \end{bmatrix},$$

$$\begin{bmatrix} [+]_{1\times1} & [-] & [+] & [-] & [+] \\ [-] & [+]_{1\times1} & [-] & [+] & [-] \\ [+] & [-] & [+]_{2\times2} & [-] & [+] \\ [-] & [+] & [-] & [+]_{1\times1} & [-] \\ [+] & [-] & [+]_{1\times1} & [-] & [+]_{1\times1} \end{bmatrix},$$

$$\begin{bmatrix} [+]_{2\times 2} & [-] & [+] \\ [-] & [+]_{2\times 2} & [-] \\ [+] & [-] & [+]_{2\times 2} \end{bmatrix},$$

$$\begin{bmatrix} [+]_{2\times 2} & [-] & [+] \\ [-] & [+]_{3\times 3} & [-] \\ [+] & [-] & [+]_{1\times 1} \end{bmatrix},$$

$$\begin{bmatrix} [+]_{2\times 2} & [-] & [+] & [-] & [+] \\ [-] & [+]_{1\times 1} & [-] & [+] & [-] \\ [+] & [-] & [+]_{1\times 1} & [-] & [+] \\ [-] & [+] & [-] & [+]_{1\times 1} & [-] \\ [+] & [-] & [+]_{1\times 1} & [-] & [+]_{1\times 1} \end{bmatrix},$$

and

$$\begin{bmatrix} [+]_{3\times 3} & [-] & [+] & [-] \\ [-] & [+]_{1\times 1} & [-] & [+] \\ [+] & [-] & [+]_{1\times 1} & [-] \\ [-] & [+] & [-] & [+]_{1\times 1} \end{bmatrix},$$

respectively. It follows from Lemma 4 that sign patterns $\hat{\mathcal{A}}_{234}$, $\hat{\mathcal{A}}_{235}$, $\hat{\mathcal{A}}_{236}$, $\hat{\mathcal{A}}_{245}$, $\hat{\mathcal{A}}_{246}$, $\hat{\mathcal{A}}_{256}$, $\hat{\mathcal{A}}_{345}$, $\hat{\mathcal{A}}_{346}$, $\hat{\mathcal{A}}_{356}$ and $\hat{\mathcal{A}}_{456}$ are not potentially eventually positive. Consequently, sign patterns $\mathcal{A}_{234}$, $\mathcal{A}_{235}$, $\mathcal{A}_{236}$, $\mathcal{A}_{245}$, $\mathcal{A}_{246}$, $\mathcal{A}_{256}$, $\mathcal{A}_{345}$, $\mathcal{A}_{346}$, $\mathcal{A}_{356}$ and $\mathcal{A}_{456}$ are not potentially eventually positive; a contradiction. □

Lemma 11. *The following five sign patterns are not potentially eventually positive:*

$$\mathcal{A}_{2345} = \begin{bmatrix} ? & - & - & - & 0 & 0 \\ - & ? & 0 & 0 & 0 & 0 \\ - & 0 & ? & 0 & 0 & 0 \\ - & 0 & 0 & ? & - & 0 \\ 0 & 0 & 0 & - & ? & + \\ 0 & 0 & 0 & 0 & + & ? \end{bmatrix},$$

$$\mathcal{A}_{2346} = \begin{bmatrix} ? & - & - & - & 0 & 0 \\ - & ? & 0 & 0 & 0 & 0 \\ - & 0 & ? & 0 & 0 & 0 \\ - & 0 & 0 & ? & + & 0 \\ 0 & 0 & 0 & + & ? & - \\ 0 & 0 & 0 & 0 & - & ? \end{bmatrix},$$

$$\mathcal{A}_{2356} = \begin{bmatrix} ? & - & - & + & 0 & 0 \\ - & ? & 0 & 0 & 0 & 0 \\ - & 0 & ? & 0 & 0 & 0 \\ + & 0 & 0 & ? & - & 0 \\ 0 & 0 & 0 & - & ? & - \\ 0 & 0 & 0 & 0 & - & ? \end{bmatrix},$$

$$\mathcal{A}_{2456} = \begin{bmatrix} ? & - & + & - & 0 & 0 \\ - & ? & 0 & 0 & 0 & 0 \\ + & 0 & ? & 0 & 0 & 0 \\ - & 0 & 0 & ? & - & 0 \\ 0 & 0 & 0 & - & ? & - \\ 0 & 0 & 0 & 0 & - & ? \end{bmatrix},$$

and

$$\mathcal{A}_{3456} = \begin{bmatrix} ? & + & - & - & 0 & 0 \\ + & ? & 0 & 0 & 0 & 0 \\ - & 0 & ? & 0 & 0 & 0 \\ - & 0 & 0 & ? & - & 0 \\ 0 & 0 & 0 & - & ? & - \\ 0 & 0 & 0 & 0 & - & ? \end{bmatrix}.$$

Proof: Assume that sign patterns $\mathcal{A}_{2345}$, $\mathcal{A}_{2346}$, $\mathcal{A}_{2356}$, $\mathcal{A}_{2456}$ and $\mathcal{A}_{3456}$ are potentially eventually positive. Let $\hat{\mathcal{A}}_{2345}$, $\hat{\mathcal{A}}_{2346}$, $\hat{\mathcal{A}}_{2356}$, $\hat{\mathcal{A}}_{2456}$ and $\hat{\mathcal{A}}_{3456}$ are the sign patterns obtained from $\mathcal{A}_{2345}$, $\mathcal{A}_{2346}$, $\mathcal{A}_{2356}$, $\mathcal{A}_{2456}$ and $\mathcal{A}_{3456}$ by changing all 0 and $-$ diagonal entries to $+$, respectively. For sign patterns $\hat{\mathcal{A}}_{2345}$, $\hat{\mathcal{A}}_{2346}$, $\hat{\mathcal{A}}_{2356}$, $\hat{\mathcal{A}}_{2456}$ and $\hat{\mathcal{A}}_{3456}$ are a proper subpattern of the following checkerboard block sign patterns

$$\begin{bmatrix} [+]_{1\times 1} & [-] & [+] \\ [-] & [+]_{3\times 3} & [-] \\ [+] & [-] & [+]_{2\times 2} \end{bmatrix},$$

$$\begin{bmatrix} [+]_{1\times 1} & [-] & [+] \\ [-] & [+]_{4\times 4} & [-] \\ [+] & [-] & [+]_{1\times 1} \end{bmatrix},$$

$$\begin{bmatrix} + & - & + & - & + & - \\ - & + & - & + & - & + \\ + & - & + & - & + & - \\ - & + & - & + & - & + \\ + & - & + & - & + & - \\ - & + & - & + & - & + \end{bmatrix},$$

$$\begin{bmatrix} [+]_{1\times 1} & [-] & [+] & [-] & [+] \\ [-] & [+]_{2\times 2} & [-] & [+] & [-] \\ [+] & [-] & [+]_{1\times 1} & [-] & [+] \\ [-] & [+] & [-] & [+]_{1\times 1} & [-] \\ [+] & [-] & [+] & [-] & [+]_{1\times 1} \end{bmatrix},$$

and

$$\begin{bmatrix} [+]_{2\times 2} & [-] & [+] & [-] \\ [-] & [+]_{2\times 2} & [-] & [+] \\ [+] & [-] & [+]_{1\times 1} & [-] \\ [-] & [+] & [-] & [+]_{1\times 1} \end{bmatrix},$$

respectively. By Lemmas 4 and 1, sign patterns $\hat{\mathcal{A}}_{2345}$, $\hat{\mathcal{A}}_{2346}$, $\hat{\mathcal{A}}_{2356}$, $\hat{\mathcal{A}}_{2456}$ and $\hat{\mathcal{A}}_{3456}$ are not potentially eventually positive. By Lemma 2, sign patterns $\mathcal{A}_{2345}$, $\mathcal{A}_{2346}$, $\mathcal{A}_{2356}$, $\mathcal{A}_{2456}$ and $\mathcal{A}_{3456}$ are not potentially eventually positive; a contradiction. □

Our main results depend readily on the following proposition.

Proposition 12. *If tree sign pattern $\mathcal{A}$ is potentially eventually positive, then the nonzero off-diagonal entries of $\mathcal{A}$ are all positive, that is to say,* $\alpha_{12} = \alpha_{21} = +$, $\alpha_{13} = \alpha_{31} = +$, $\alpha_{14} = \alpha_{41}+$, $\alpha_{45} = \alpha_{54} = +$, $\alpha_{56} = \alpha_{65} = +$.

Proof: Assume that tree sign pattern $\mathcal{A}$ is potentially eventually positive. Then sign pattern $\mathcal{A}$ is symmetric by Proposition 7. To complete the proof, it suffices to show that nonzero off-diagonal entries $\alpha_{12} = \alpha_{13} = \alpha_{14} = \alpha_{45} = \alpha_{56} = +$. It is clear that if all nonzero off-diagonal entries are $-$, then $\mathcal{A}$ has at most six positive entries and hence $\mathcal{A}$ is not potentially eventually positive. Thus, $\mathcal{A}$ has at least two positive off-diagonal entries. The following cases are considered.

Case 1. *Sign pattern $\mathcal{A}$ has exactly two negative off-diagonal entry.* Then sign pattern $\mathcal{A}$ is one of sign patterns $\mathcal{A}_{12}$, $\mathcal{A}_{13}$, $\mathcal{A}_{14}$, $\mathcal{A}_{15}$ and $\mathcal{A}_{16}$. By Lemma 8, $\mathcal{A}$ is not potentially eventually positive; a contradiction.

Case 2. *Sign pattern $\mathcal{A}$ has exactly four negative off-diagonal entries.* Then sign pattern $\mathcal{A}$ is one of sign patterns $\mathcal{A}_{23}$, $\mathcal{A}_{24}$, $\mathcal{A}_{25}$, $\mathcal{A}_{26}$, $\mathcal{A}_{34}$, $\mathcal{A}_{35}$, $\mathcal{A}_{36}$, $\mathcal{A}_{45}$, $\mathcal{A}_{46}$ and $\mathcal{A}_{56}$. By Lemma 9, $\mathcal{A}$ is not potentially eventually positive; a contradiction.

Case 3. *Sign pattern $\mathcal{A}$ has exactly six negative off-diagonal entries.* Then sign pattern $\mathcal{A}$ is one of sign patterns $\mathcal{A}_{234}$, $\mathcal{A}_{235}$, $\mathcal{A}_{236}$, $\mathcal{A}_{245}$, $\mathcal{A}_{246}$, $\mathcal{A}_{256}$, $\mathcal{A}_{345}$, $\mathcal{A}_{346}$, $\mathcal{A}_{356}$ and $\mathcal{A}_{456}$. By Lemma 10, $\mathcal{A}$ is not potentially eventually positive; a contradiction.

Case 4. *Sign pattern $\mathcal{A}$ has exactly eight negative off-diagonal entries.* Then sign pattern $\mathcal{A}$ is one of sign patterns $\mathcal{A}_{2345}$, $\mathcal{A}_{2346}$, $\mathcal{A}_{2356}$, $\mathcal{A}_{2456}$ and $\mathcal{A}_{3456}$. By Lemma 11, $\mathcal{A}$ is not potentially eventually positive; a contradiction.

As discussed above, if sign pattern $\mathcal{A}$ has at least one negative off-diagonal, then $\mathcal{A}$ is not potentially eventually positive. It follows that if $\mathcal{A}$ is potentially eventually positive, then all the nonzero off-diagonal entries of $\mathcal{A}$ are positive. □

3 The Minimality of Potentially Eventually Positive Sign Pattern $\mathcal{A}$

Recall that an n-by-n sign pattern $\mathcal{A}$ is said to be a minimal potentially eventually positive sign pattern (MPEP sign pattern) if $\mathcal{A}$ is potentially eventually positive and no proper subpattern of $\mathcal{A}$ is potentially eventually positive. To consider the minimality of potentially eventually positive sign pattern $\mathcal{A}$, the follow proposition is necessary. To state clearly, let

$$\mathcal{A}_1 = \begin{bmatrix} + & + & + & + & 0 & 0 \\ + & 0 & 0 & 0 & 0 & 0 \\ + & 0 & 0 & 0 & 0 & 0 \\ + & 0 & 0 & 0 & + & 0 \\ 0 & 0 & 0 & + & 0 & + \\ 0 & 0 & 0 & 0 & + & 0 \end{bmatrix},$$

$$\mathcal{A}_2 = \begin{bmatrix} 0 & + & + & + & 0 & 0 \\ + & + & 0 & 0 & 0 & 0 \\ + & 0 & 0 & 0 & 0 & 0 \\ + & 0 & 0 & 0 & + & 0 \\ 0 & 0 & 0 & + & 0 & + \\ 0 & 0 & 0 & 0 & + & 0 \end{bmatrix},$$

$$\mathcal{A}_3 = \begin{bmatrix} 0 & + & + & + & 0 & 0 \\ + & 0 & 0 & 0 & 0 & 0 \\ + & 0 & 0 & 0 & 0 & 0 \\ + & 0 & 0 & + & + & 0 \\ 0 & 0 & 0 & + & 0 & + \\ 0 & 0 & 0 & 0 & + & 0 \end{bmatrix},$$

$$\mathcal{A}_4 = \begin{bmatrix} 0 & + & + & + & 0 & 0 \\ + & 0 & 0 & 0 & 0 & 0 \\ + & 0 & 0 & 0 & 0 & 0 \\ + & 0 & 0 & 0 & + & 0 \\ 0 & 0 & 0 & + & + & + \\ 0 & 0 & 0 & 0 & + & 0 \end{bmatrix},$$

and

$$\mathcal{A}_5 = \begin{bmatrix} 0 & + & + & + & 0 & 0 \\ + & 0 & 0 & 0 & 0 & 0 \\ + & 0 & 0 & 0 & 0 & 0 \\ + & 0 & 0 & 0 & + & 0 \\ 0 & 0 & 0 & + & 0 & + \\ 0 & 0 & 0 & 0 & + & + \end{bmatrix}.$$

Proposition 13. *Tree sign patterns $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$, $\mathcal{A}_4$ and $\mathcal{A}_5$ are minimal potentially eventually positive sign patterns.*

Proof: Tree sign patterns $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$, $\mathcal{A}_4$ and $\mathcal{A}_5$ are potentially eventually positive for their positive parts are primitive, respectively. If some nonzero off-diagonal entries are changed to be 0, then the corresponding subpatterns are not potentially eventually positive by Proposition 12. By Proposition 6, the diagonal entries of $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$, $\mathcal{A}_4$ and $\mathcal{A}_5$ can not be changed to be 0. Therefore, no proper subpatterns of $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$, $\mathcal{A}_4$ and $\mathcal{A}_5$ are potentially eventually positive. It follows that sign patterns $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$, $\mathcal{A}_4$ and $\mathcal{A}_5$ are minimal potentially eventually positive sign patterns. □

Proposition 14. *If $\mathcal{A}$ is a minimal potentially eventually positive sign pattern, then $\mathcal{A}$ has exactly one positive diagonal entry.*

Proof: By a way of contradiction, assume that $\mathcal{A}$ has at least two positive diagonal entries. Then by Proposition 12, all nonzero off-diagonal entries of $\mathcal{A}$ are positive. Consequently, $\mathcal{A}$ must be a superpattern of one of $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$, $\mathcal{A}_4$ and $\mathcal{A}_5$. By Proposition 13, $\mathcal{A}$ is not a minimal potentially eventually positive sign pattern; a contradiction. Hence, $\mathcal{A}$ has exactly one positive diagonal entry. □

In the following Theorem 15, all minimal potentially eventually positive sign patterns are identified. In Theorem 16, all potentially eventually positive sign patterns are classified.

Theorem 15. *Tree sign pattern $\mathcal{A}$ is a minimal potentially eventually positive sign pattern if and only if $\mathcal{A}$ is equivalent to one of sign patterns $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$, $\mathcal{A}_4$ and $\mathcal{A}_5$.*

Proof: The sufficiency follows from Proposition 13. For the necessity, if tree sign pattern $\mathcal{A}$ is a minimal potentially eventually positive sign pattern, then all nonzero off-diagonal entries are positive by Proposition 12 and $\mathcal{A}$ has exactly one positive diagonal entry by Proposition 14. Thus, up to equivalence, $\mathcal{A}$ is one of sign patterns $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$, $\mathcal{A}_4$ and $\mathcal{A}_5$. □

Theorem 16. *Tree sign pattern $\mathcal{A}$ is potentially eventually positive if and only if $\mathcal{A}$ is equivalent to one of superpatterns of sign patterns $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$, $\mathcal{A}_4$ or $\mathcal{A}_5$.*

Proof: Theorem 16 follows readily from Theorem 15 and Lemma 1. □

Recall that an n-by-n tree sign pattern $\mathcal{A}$ is said to require an eventually positive matrix (REP, for short), if every matrix $A \in Q(\mathcal{A})$ is eventually positive; see e.g., [11]. It is obvious that sign pattern $\mathcal{A}$ is REP, then $\mathcal{A}$ is potentially eventually positive. But the converse is not true. We end this section with an interesting corollary.

Corollary 17. *For the tree sign pattern $\mathcal{A}$, the following statements are equivalent:*

(1) $\mathcal{A}$ is a minimal potentially eventually positive sign pattern;

(2) $\mathcal{A}$ requires an eventually positive matrix;

(3) $\mathcal{A}$ is nonnegative and primitive, and has exactly one positive diagonal entry.

Proof: Corollary 17 follows readily from Theorem 15 and Theorem 2.3 in [11]. □

4 Concluding Remarks

In this article, we have identified all the minimal potentially eventually positive sign patterns as five specific tree sign patterns. Consequently, we have classified all the potentially eventually positive sign patterns as the superpatterns of the previous specific tree sign patterns. However, it seems that the difficulty in identifying and classifying the (minimal) potentially eventually positive sign patterns is great increasing, when the order of sign patterns to be discussed is increasing. In a following paper, we will consider the potential eventual positivity of tree sign patterns with bigger orders.

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