Exact Determinants of the RFPrLrR Circulant Involving Jacobsthal, Jacobsthal-Lucas, Perrin and Padovan Numbers

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Abstract: The row first-plus-$r$last r-right (RFPrLrR) circulant matrix and row last-plus-$r$first r-left (RLPrFrL) circulant matrices are two special pattern matrices. On the basis of the beautiful properties of famous numbers and the inverse factorization of polynomial, we give the exact determinants of the two pattern matrices involving Jacobsthal, Jacobsthal-Lucas, Perrin and Padovan numbers.

Key–Words: Determinant; RFPrLrR circulant matrix; RLPrFrL circulant matrix; Jacobsthal number; Jacobsthal-Lucas number; Perrin number; Padovan number;

1 Introduction

Circulant matrix family occurs in various fields, applied in image processing, communications, signal processing, encoding and preconditioner. Meanwhile, the circulant matrices [1, 2] have been extended in many directions recently. The $f(x)$-circulant matrix is another natural extension of the research category, please refer to [3, 11].

Recently, some authors researched the circulant type matrices with famous numbers. In [3], Shen et al. discussed the explicit determinants of the RFMLR and RLMFL circulant matrices involving certain famous numbers. Jaiswal [4] showed some determinants of circulant matrices whose elements are the generalized Fibonacci numbers. Lind presented the determinants of circulant and skew circulant involving Fibonacci numbers in [5]. Gao et al. [6] considered the determinants and inverses of skew circulant and skew left circulant matrices with Fibonacci and Lucas numbers. Akbulak and Bozkurt [7] proposed some properties of Toeplitz matrices involving Fibonacci and Lucas numbers. In [8], authors considered circulant matrices with Fibonacci and Lucas numbers and proposed their explicit determinants and inverses by constructing the transformation matrices. Jiang and Hong gave exact determinants of some special circulant matrices involving four kinds of famous numbers in [9]. See more of the literatures in [10, 11, 12].

We introduce two new patten matrices, i.e. row first-plus-r last r-right (RFPrLrR) circulant matrix and row last-plus-r first r-left (RLPrFrL) circulant matrix. Based on the characteristic polynomial of the basic RFPrLrR circulant matrix and the Binet formulae of famous numbers, we obtain the exact determinants of two patten matrices with Jacobsthal, Jacobsthal-Lucas, Perrin and Padoven numbers.

A row first-plus-$r$last r-right (RFPrLrR) circulant matrix with the first row $(a_0, a_1, \dots, a_{n-1})$, denoted by $\mathrm{RFP}r\mathrm{LRcirc}_r\mathrm{fr}\big(a_0, a_1, \dots, a_{n-1}\big)$, is meant to be a square matrix of the form

$$A = \begin{pmatrix} a_0 & a_1 & \dots & a_{n-1} \\ ra_{n-1} & a_0 + ra_{n-1} & \dots & a_{n-2} \\ ra_{n-2} & ra_{n-1} + ra_{n-2} & \dots & a_{n-3} \\ \dots & \dots & \dots & \dots \\ ra_1 & ra_2 + ra_1 & \dots & a_0 + ra_{n-1} \end{pmatrix}.$$

Be aware that the RFPrLrR circulant matrix is a $x^n - rx - r$-circulant matrix.

We define $\Theta_{(r,r)}$ as the basic RFPrLrR circulant matrix, that is

$$\Theta_{(r,r)} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ r & r & 0 & \dots & 0 & 0 \end{pmatrix}_{n\times n} = \mathrm{RFP}r\mathrm{LRcirc}_r\mathrm{fr}(0, 1, 0, \dots, 0).$$

Both the minimal polynomial and the characteristic polynomial of $\Theta_{(r,r)}$ are $g(x) = x^n - rx - r$, which has only simple roots, denoted by π_i $(i = 1, 2, \dots, n)$.

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A row last-plus-$r$first r-left (RLPrFrL) circulant matrix with the first row $(a_0, a_1, \ldots, a_{n-1})$, denoted by $\mathrm{RLP}r\mathrm{FLcirc}_r\mathrm{fr}\big(a_0, a_1, \ldots, a_{n-1}\big)$, is meant to be a square matrix of the form

$$B = \begin{pmatrix} a_0 & \ldots & a_{n-2} & a_{n-1} \\ a_1 & \ldots & a_{n-1} + ra_0 & ra_0 \\ a_2 & \ldots & ra_0 + ra_1 & ra_1 \\ \ldots & \ldots & \ldots & \ldots \\ a_{n-1} + ra_0 & \ldots & ra_{n-3} + ra_{n-2} & ra_{n-2} \end{pmatrix}.$$

Let A=$\mathrm{RLP}r\mathrm{FLcirc}_r\mathrm{fr}(a_0, a_1, \ldots, a_{n-1})$ and $B = \mathrm{RFP}r\mathrm{LRcirc}_r\mathrm{fr}(a_{n-1}, a_{n-2}, \ldots, a_0)$. By explicit computation, we find

$$A = B\widehat{I}_n, \tag{1}$$

where $\widehat{I}_n$ is the backward identity matrix of the form

$$\widehat{I}_n = \begin{pmatrix} & \mathbf{0} & & 1 \\ & & 1 & \\ & & \iddots & \\ & 1 & & \mathbf{0} \\ 1 & & & \end{pmatrix}. \tag{2}$$

The Jacobsthal sequences $\{J_n\}$, Jacobsthal-Lucas sequences $\{j_n\}$[13, 14, 15], Perrin sequences $\{R_n\}$ and Padovan sequences $\{\mathbb{P}_n\}$ [16, 17, 18] are defined by the following recurrence relations, respectively:

$$J_n = J_{n-1} + 2J_{n-2}, \quad n \geq 2, \tag{3}$$

$$j_n = j_{n-1} + 2j_{n-2}, \quad n \geq 2, \tag{4}$$

$$R_n = R_{n-2} + R_{n-3}, \quad n \geq 3, \tag{5}$$

$$\mathbb{P}_n = \mathbb{P}_{n-2} + \mathbb{P}_{n-3}, \quad n \geq 3, \tag{6}$$

with the initial condition $J_0 = 0$, $J_1 = 1$, $j_0 = 2$, $j_1 = 1$, $R_0 = 3$, $R_1 = 0$, $R_2 = 2$ and $\mathbb{P}_0 = 1$, $\mathbb{P}_1 = 1, \mathbb{P}_2 = 1$.

The first few members of these sequences are given as follows:

n	0	1	2	3	4	5	6
J_n	0	1	1	3	5	11	21
j_n	2	1	5	7	17	31	65
R_n	3	0	2	3	2	5	5
$\mathbb{P}_n$	1	1	1	2	2	3	4

The sequences $\{J_n\}$ and $\{j_n\}$ are given by the Binet formulae

$$J_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

and

$$j_n = \alpha^n + \beta^n,$$

where α, β are the roots of the equation $x^2 - x - 2 = 0$.

While, recurrences (5) and (6) involve the characteristic equation $x^3 - x - 1 = 0$, its roots are denoted by r_1, r_2, r_3. Then we obtain:

$$\begin{cases} r_1 + r_2 + r_3 = 0, \\ r_1r_2 + r_1r_3 + r_2r_3 = -1, \\ r_1r_2r_3 = 1. \end{cases} \tag{7}$$

In addition, the Binet form for the Perrin sequences is

$$R_n = r_1^n + r_2^n + r_3^n, \tag{8}$$

and the Binet form for Padovan sequences is

$$\mathbb{P}_n = a_1r_1^n + a_2r_2^n + a_3r_3^n, \tag{9}$$

where

$$a_i = \prod_{\substack{j=1 \\ j \neq i}}^{3} \frac{r_j - 1}{r_i - r_j}, \quad i = 1, 2, 3.$$

2 Main Results

By Proposition 5.1 in [19] and properties of RFPrLrR circulant matrices. we deduce the following lemma.

Lemma 1. *Let* $A = \mathrm{RFP}r\mathrm{LRcirc}_r\mathrm{fr}(a_0, a_1, \ldots, a_{n-1})$. *Then the eigenvalues of A are given by*

$$\lambda_i = f(\pi_i) = \sum_{j=0}^{n-1} a_j\pi_i^j, \quad i = 0, 1, 2, \ldots, n-1,$$

where, π_i $\big(i = 1, 2, \ldots, n\big)$ *are the eigenvalues of* $\Theta_{(r,r)}$, *and the determinant of A is given by*

$$\det A = \prod_{i=1}^{n} \lambda_i = \prod_{i=1}^{n}\sum_{j=0}^{n-1} a_j\pi_i^j.$$

Lemma 2. *Assume* π_i $(i = 1, 2, \ldots, n)$ *are the roots of the characteristic polynomial of* $\Theta_{(r,r)}$. *If* $a = 0$, *then*

$$\begin{aligned}&\prod_{i=1}^{n}(a\pi_i^3 + b\pi_i^2 + c\pi_i + d)\\ =&\prod_{i=1}^{n}(b\pi_i^2 + c\pi_i + d)\\ =&d^n - b^{n-1}\big[dr\big(s^{n-1} + t^{n-1}\big) + br\big(s^n + t^n\big)\\ &- r^2\big(d - c + b\big)\big],\end{aligned}$$

where $a, b, c \in \mathbb{R}$ *and*

$$s = \frac{-c+\sqrt{c^2-4bd}}{2b}; \quad t = \frac{-c-\sqrt{c^2-4bd}}{2b}.$$

If $a \neq 0$, *then*

$$\begin{aligned}&\prod_{i=1}^{n}(a\pi_i^3+b\pi_i^2+c\pi_i+d)\\ =&d^n+\frac{(-a)^{n-1}}{2}d\big(rX_{2(n-1)}-rX_{n-1}^2+2r^2X_{n-1}\big)\\ &+\frac{(-a)^n}{2}\big(rX_{2n}+2r^2X_n-rX_n^2-2r^2X_{n+1}\big)\\ &+(-a)^{n-1}\big(r^3\big(a-b+c-d\big)-br^2X_n\big),\end{aligned}$$

where $X_n = x_1^n + x_2^n + x_3^n$, *and* x_1, x_2, x_3 *are the roots of the equation* $a\pi_i^3 + b\pi_i^2 + c\pi_i + d = 0$.

Proof. Since π_i $(i = 1, 2, \ldots, n)$ are the roots of the characteristic polynomial of $\Theta_{(r,r)}$, $g(x) = x^n - rx - r$ can be factored as

$$x^n - rx - r = \prod_{i=1}^{n}(x-\pi_i).$$

Let x_1, x_2, x_3 be the roots of the equation $a\pi_i^3 + b\pi_i^2 + c\pi_i + d = 0$. If $a = 0$, please see [11] for details of the proof. If $a \neq 0$, then

$$\begin{aligned}&\prod_{i=1}^{n}(a\pi_i^3+b\pi_i^2+c\pi_i+d)\\ =&a^n\prod_{i=1}^{n}(\pi_i^3+\frac{b}{a}\pi_i^2+\frac{c}{a}\pi_i+\frac{d}{a})\\ =&a^n\prod_{i=1}^{n}(\pi_i-x_1)(\pi_i-x_2)(\pi_i-x_3)\\ =&(-a)^n\prod_{i=1}^{n}(x_1-\pi_i)\prod_{i=1}^{n}(x_2-\pi_i)\prod_{i=1}^{n}(x_3-\pi_i)\\ =&(-a)^n(x_1^n-rx_1-r)(x_2^n-rx_2-r)(x_3^n-rx_3-r)\\ =&(-a)^n\Big\{(x_1x_2x_3)^n-rx_1x_2x_3\big[(x_1x_2)^{n-1}\\ &+(x_1x_3)^{n-1}+(x_2x_3)^{n-1}\big]-r\big[(x_1x_2)^n+(x_1x_3)^n\\ &+(x_2x_3)^n\big]+r^2x_1x_2x_3(x_1^{n-1}+x_2^{n-1}+x_3^{n-1})\\ &+r^2\big[x_1^n(x_2+x_3)+x_2^n(x_1+x_3)+x_3^n(x_1+x_2)\big]\\ &+r^2(x_1^n+x_2^n+x_3^n)-r^3x_1x_2x_3\\ &-r^3(x_1x_2+x_1x_3+x_2x_3)\\ &-r^3(x_1+x_2+x_3)-r^3\Big\}.\end{aligned}$$

Let $X_n = x_1^n + x_2^n + x_3^n$, we obtain $(x_1x_2)^n + (x_1x_3)^n + (x_2x_3)^n = \frac{X_n^2 - X_{2n}}{2}$ from $(x_1^n + x_2^n + x_3^n)^2 = x_1^{2n} + x_2^{2n} + x_3^{2n} + 2[(x_1x_2)^n + (x_1x_3)^n + (x_2x_3)^n]$. Taking the relation of roots and coefficients

$$\begin{cases} x_1+x_2+x_3 &= -\frac{b}{a}\\ x_1x_2+x_1x_3+x_2x_3 &= \frac{c}{a}\\ x_1x_2x_3 &= -\frac{d}{a}\end{cases}$$

into account, we get that

$$\begin{aligned}&\prod_{i=1}^{n}(a\pi_i^3+b\pi_i^2+c\pi_i+d)\\ &=d^n+\frac{(-a)^{n-1}}{2}d\big(rX_{2(n-1)}-rX_{n-1}^2+2r^2X_{n-1}\big)\\ &+\frac{(-a)^n}{2}\big(rX_{2n}+2r^2X_n-rX_n^2-2r^2X_{n+1}\big)\\ &+(-a)^{n-1}\big(r^3\big(a-b+c-d\big)-br^2X_n\big),\end{aligned}$$

where $X_n = x_1^n + x_2^n + x_3^n$, and x_1, x_2, x_3 are the roots of the equation $a\pi_i^3 + b\pi_i^2 + c\pi_i + d = 0$. □

In the following, we show the exact determinants of the RFPrLrR and RLPrFrL circulant matrices involving some related famous numbers.

3 Determinants of the RFPrLrR and RLPrFrL Circulant Matrices Involving the Jacobsthal Numbers

Theorem 3. *If* $A = \mathrm{RFP}r\mathrm{LRcirc}_r\mathrm{fr}\,(J_0, \ldots, J_{n-1})$, *then*

$$\begin{aligned}&\det A\\ =&\frac{\big(-rJ_n\big)^n+\big(-2rJ_{n-1}\big)^{n-1}\big[r^2J_n\big(s_1^{n-1}+t_1^{n-1}\big)\big]}{1-rj_{n-1}-rj_n}\\ &+\frac{\big(-2rJ_{n-1}\big)^{n-1}\big[2r^2J_{n-1}\big(s_1^n+t_1^n\big)-r^2\big]}{1-rj_{n-1}-rj_n},\end{aligned}$$

where

$$\begin{aligned}s_1=&\frac{rJ_n+2rJ_{n-1}-1}{-4rJ_{n-1}}\\ &+\frac{\sqrt{\big(1-rJ_n-2rJ_{n-1}\big)^2-8r^2J_nJ_{n-1}}}{-4rJ_{n-1}},\end{aligned}$$

$$\begin{aligned}t_1=&\frac{rJ_n+2rJ_{n-1}-1}{-4rJ_{n-1}}\\ &-\frac{\sqrt{\big(1-rJ_n-2rJ_{n-1}\big)^2-8r^2J_nJ_{n-1}}}{-4rJ_{n-1}}.\end{aligned}$$

Proof. The matrix A can be written as

$$A=\begin{pmatrix} J_0 & J_1 & \dots & J_{n-1} \\ rJ_{n-1} & J_0+rJ_{n-1} & \dots & J_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ rJ_2 & rJ_3+rJ_2 & \dots & J_1 \\ rJ_1 & rJ_2+rJ_1 & \dots & J_0+rJ_{n-1} \end{pmatrix}.$$

Using the Lemma 1, the determinant of A is

$$\begin{aligned}&\det A\\ =&\prod_{i=1}^{n}\left(J_0+J_1\pi_i+\cdots+J_{n-1}\pi_i^{n-1}\right)\\ =&\prod_{i=1}^{n}\left(\frac{\alpha-\beta}{\alpha-\beta}\pi_i+\frac{\alpha^2-\beta^2}{\alpha-\beta}\pi_i^2+\cdots\right.\\ &\left.+\frac{\alpha^{n-1}-\beta^{n-1}}{\alpha-\beta}\pi_i^{n-1}\right)\\ =&\prod_{i=1}^{n}\frac{(-2rJ_{n-1})\pi_i^2-(1-rJ_n-2rJ_{n-1})\pi_i-rJ_n}{1-\pi_i-2\pi_i^2}.\end{aligned}$$

According to Lemma 2, we can get

$$\begin{aligned}&\prod_{i=1}^{n}\left[(-2rJ_{n-1})\pi_i^2-(1-rJ_n-2rJ_{n-1})\pi_i-rJ_n\right]\\ =&(-rJ_n)^n-(-2rJ_{n-1})^{n-1}\left[-r^2J_n(s_1^{n-1}+t_1^{n-1})\right.\\ &\left.-2r^2J_{n-1}(s_1^n+t_1^n)+r^2\right].\end{aligned}$$

Then we obtain that

$$\begin{aligned}&\det A\\ =&\frac{(-rJ_n)^n+(-2rJ_{n-1})^{n-1}\left[r^2J_n(s_1^{n-1}+t_1^{n-1})\right]}{1-rj_{n-1}-rj_n}\\ &+\frac{(-2rJ_{n-1})^{n-1}\left[2r^2J_{n-1}(s_1^n+t_1^n)-r^2\right]}{1-rj_{n-1}-rj_n},\end{aligned}$$

where

$$\begin{aligned}s_1=&\frac{rJ_n+2rJ_{n-1}-1}{-4rJ_{n-1}}\\ &+\frac{\sqrt{(1-rJ_n-2rJ_{n-1})^2-8r^2J_nJ_{n-1}}}{-4rJ_{n-1}},\end{aligned}$$

$$\begin{aligned}t_1=&\frac{rJ_n+2rJ_{n-1}-1}{-4rJ_{n-1}}\\ &-\frac{\sqrt{(1-rJ_n-2rJ_{n-1})^2-8r^2J_nJ_{n-1}}}{-4rJ_{n-1}}.\end{aligned}$$

□

Using the method in Theorem 3 similarly, we also have

Theorem 4. *If* $A'=\mathrm{RFP}r\mathrm{LRcirc}_r\mathrm{fr}\left(J_{n-1},\dots,J_0\right)$, *then*

$$\begin{aligned}\det A'=&\frac{(r-2J_{n-1})^n-r(J_n-r)^{n-1}(r-2J_{n-1})}{(-2)^n+2rj_{n-1}-rj_n}\\ &-\frac{r(J_n-r)^n}{(-2)^n+2rj_{n-1}-rj_n}.\end{aligned}$$

Theorem 5. *If* $J=\mathrm{RLP}r\mathrm{FLcirc}_r\mathrm{fr}\left(J_0,\dots,J_{n-1}\right)$, *then we have*

$$\begin{aligned}\det J=&\left[\frac{(r-2J_{n-1})^n-r(J_n-r)^{n-1}(r-2J_{n-1})}{(-2)^n+2rj_{n-1}-rj_n}\right.\\ &\left.-\frac{r(J_n-r)^n}{(-2)^n+2rj_{n-1}-rj_n}\right](-1)^{\frac{n(n-1)}{2}}.\end{aligned}$$

Proof. The matrix J can be written as

$$\begin{aligned}J=&\begin{pmatrix} J_0 & \dots & J_{n-2} & J_{n-1} \\ J_1 & \dots & J_{n-1}+rJ_0 & rJ_0 \\ \vdots & \ddots & \vdots & \vdots \\ J_{n-2} & \dots & rJ_{n-4}+rJ_{n-3} & rJ_{n-3} \\ J_{n-1}+rJ_0 & \dots & rJ_{n-3}+rJ_{n-2} & rJ_{n-2} \end{pmatrix}\\ =&\begin{pmatrix} J_{n-1} & J_{n-2} & \dots & J_0 \\ rJ_0 & J_{n-1}+rJ_0 & \dots & J_1 \\ \vdots & \vdots & \ddots & \vdots \\ rJ_{n-3} & rJ_{n-4}+rJ_{n-3} & \dots & J_{n-2} \\ rJ_{n-2} & rJ_{n-3}+rJ_{n-2} & \dots & J_{n-1}+rJ_0 \end{pmatrix}\\ &\times\begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & \dots & 0 & 1 & 0 \\ \vdots & ⋰ & ⋰ & ⋰ & \vdots \\ 0 & 1 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix},\end{aligned}$$

then we can get

$$\det J=\det A'\det\Gamma,$$

which matrix $A'=\mathrm{RFP}r\mathrm{LRcirc}_r\mathrm{fr}\left(J_{n-1},\dots,J_0\right)$, its determinant could be obtained through Theorem 2.

$$\begin{aligned}\det A'=&\frac{(r-2J_{n-1})^n-r(J_n-r)^{n-1}(r-2J_{n-1})}{(-2)^n+2rj_{n-1}-rj_n}\\ &-\frac{r(J_n-r)^n}{(-2)^n+2rj_{n-1}-rj_n},\end{aligned}$$

and

$$\det\Gamma=(-1)^{\frac{n(n-1)}{2}}.$$

So

$$\begin{aligned}&\mathrm{detJ} = \mathrm{detA'detΓ}\\ =&\Big[\frac{\big(r-2J_{n-1}\big)^n - r\big(J_n - r\big)^{n-1}\big(r-2J_{n-1}\big)}{(-2)^n + 2rj_{n-1} - rj_n}\\ &-\frac{r\big(J_n - r\big)^n}{(-2)^n+2rj_{n-1}-rj_n}\Big](-1)^{\frac{n(n-1)}{2}}.\end{aligned}$$

□

4 Determinants of the RFPrLrR and RLPrFrL Circulant Matrix Involving Jacobsthal-Lucas Numbers

Theorem 6. *If* $B = \mathrm{RFP}r\mathrm{LRcirc}_r\mathrm{fr}\,(j_0,\ldots,j_{n-1})$, *then*

$$\begin{aligned}\det B =& \frac{\big(2-rj_n\big)^n}{1-rj_{n-1}-rj_n}\\ &+\frac{\big(-2rj_{n-1}\big)^{n-1}\big[r\big(rj_n-2\big)\big(s_2^{n-1}+t_2^{n-1}\big)\big]}{1-rj_{n-1}-rj_n}\\ &+\frac{\big(-2rj_{n-1}\big)^{n-1}\big[2r^2\big(s_2^n+t_2^n\big)+3r^2\big]}{1-rj_{n-1}-rj_n},\end{aligned}$$

where

$$\begin{aligned}s_2 =&\frac{1+2rj_{n-1}+rj_n}{-4rj_{n-1}}\\ &+\frac{\sqrt{\big(1+2rj_{n-1}+rj_n\big)^2+16rj_{n-1}-8r^2j_nj_{n-1}}}{-4rj_{n-1}},\\ t_2 =&\frac{1+2rj_{n-1}+rj_n}{-4rj_{n-1}}\\ &-\frac{\sqrt{\big(1+2rj_{n-1}+rj_n\big)^2+16rj_{n-1}-8r^2j_nj_{n-1}}}{-4rj_{n-1}}.\end{aligned}$$

Similarly, we also get the following Theorem 7.

Theorem 7. *If* $B' = \mathrm{RFP}r\mathrm{LRcirc}_r\mathrm{fr}\,(j_{n-1},\ldots,j_0)$, *then*

$$\begin{aligned}\det B' =& \frac{\big(-r-2j_{n-1}\big)^n}{(-2)^n+2rj_{n-1}-rj_n}\\ &+\frac{\big(2r\big)^{n-1}(r^2+2rj_{n-1})\big(s_3^{n-1}+t_3^{n-1}\big)}{(-2)^n+2rj_{n-1}-rj_n}\\ &-\frac{\big(2r\big)^{n-1}[2r^2\big(s_3^n+t_3^n\big)-r^2(j_n-2j_{n-1})]}{(-2)^n+2rj_{n-1}-rj_n},\end{aligned}$$

where

$$\begin{aligned}s_3 =&\frac{j_n-r+\sqrt{\big(r-j_n\big)^2+8r^2+16rj_{n-1}}}{4r},\\ t_3 =&\frac{j_n-r-\sqrt{\big(r-j_n\big)^2+8r^2+16rj_{n-1}}}{4r}.\end{aligned}$$

Theorem 8. *If* $J' = \mathrm{RLP}r\mathrm{FLcirc}_r\mathrm{fr}\,(j_0,\ldots,j_{n-1})$, *then we have*

$$\begin{aligned}\det J' =& \Big[\frac{\big(-r-2j_{n-1}\big)^n}{(-2)^n+2rj_{n-1}-rj_n}\\ &+\frac{\big(2r\big)^{n-1}(r^2+2rj_{n-1})\big(s_3^{n-1}+t_3^{n-1}\big)}{(-2)^n+2rj_{n-1}-rj_n}\\ &-\frac{\big(2r\big)^{n-1}\Big(2r^2\big(s_3^n+t_3^n\big)-r^2(j_n-2j_{n-1})\Big)}{(-2)^n+2rj_{n-1}-rj_n}\Big]\\ &\times(-1)^{\frac{n(n-1)}{2}},\end{aligned}$$

where

$$\begin{aligned}s_3 =&\frac{j_n-r+\sqrt{\big(r-j_n\big)^2+8r^2+16rj_{n-1}}}{4r},\\ t_3 =&\frac{j_n-r-\sqrt{\big(r-j_n\big)^2+8r^2+16rj_{n-1}}}{4r}.\end{aligned}$$

Proof. The matrix J' can be written as

$$\begin{aligned}J' =& \begin{pmatrix} j_0 & \cdots & j_{n-2} & j_{n-1}\\ j_1 & \cdots & j_{n-1}+rj_0 & rj_0\\ \vdots & \ddots & \vdots & \vdots\\ j_{n-2} & \cdots & rj_{n-4}+rJ_{n-3} & rj_{n-3}\\ j_{n-1}+rj_0 & \cdots & rj_{n-3}+rj_{n-2} & rj_{n-2}\end{pmatrix}\\ =&\begin{pmatrix} j_{n-1} & j_{n-2} & \cdots & j_0\\ rj_0 & j_{n-1}+rj_0 & \cdots & j_1\\ \vdots & \vdots & \ddots & \vdots\\ rj_{n-3} & rj_{n-4}+rj_{n-3} & \cdots & j_{n-2}\\ rj_{n-2} & rj_{n-3}+rj_{n-2} & \cdots & j_{n-1}+rj_0\end{pmatrix}\\ &\times\begin{pmatrix} 0 & 0 & \ldots & 0 & 1\\ 0 & \ldots & 0 & 1 & 0\\ \vdots & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \vdots\\ 0 & 1 & 0 & \ldots & 0\\ 1 & 0 & 0 & \ldots & 0\end{pmatrix},\end{aligned}$$

then we can get

$$\det J' = \det B' \det\Gamma,$$

with matrix $B' = \mathrm{RFP}r\mathrm{LRcirc}_r\mathrm{fr}\,(j_{n-1},\ldots,j_0)$, its determinant could be obtained through Theorem 7.

$$\begin{aligned}\det B' = &\frac{\big(-r-2j_{n-1}\big)^n}{(-2)^n+2rj_{n-1}-rj_n}\\ &+\frac{\big(2r\big)^{n-1}(r^2+2rj_{n-1})\big(s_3^{n-1}+t_3^{n-1}\big)}{(-2)^n+2rj_{n-1}-rj_n}\\ &-\frac{\big(2r\big)^{n-1}[2r^2\big(s_3^n+t_3^n\big)-r^2(j_n-2j_{n-1})]}{(-2)^n+2rj_{n-1}-rj_n},\end{aligned}$$

where

$$\begin{aligned}s_3 =&\frac{j_n-r+\sqrt{\big(r-j_n\big)^2+8r^2+16rj_{n-1}}}{4r},\\ t_3 =&\frac{j_n-r-\sqrt{\big(r-j_n\big)^2+8r^2+16rj_{n-1}}}{4r},\end{aligned}$$

and $\det\Gamma = (-1)^{\frac{n(n-1)}{2}}$.

So

$$\begin{aligned}&\det J' = \det B'\det\Gamma\\ =&\Bigg[\frac{\big(-r-2j_{n-1}\big)^n}{(-2)^n+2rj_{n-1}-rj_n}\\ &+\frac{\big(2r\big)^{n-1}(r^2+2rj_{n-1})\big(s_3^{n-1}+t_3^{n-1}\big)}{(-2)^n+2rj_{n-1}-rj_n}\\ &-\frac{\big(2r\big)^{n-1}\Big(2r^2\big(s_3^n+t_3^n\big)-r^2(j_n-2j_{n-1})\Big)}{(-2)^n+2rj_{n-1}-rj_n}\Bigg]\\ &\times(-1)^{\frac{n(n-1)}{2}},\end{aligned}$$

where

$$\begin{aligned}s_3 =&\frac{j_n-r+\sqrt{\big(r-j_n\big)^2+8r^2+16rj_{n-1}}}{4r},\\ t_3 =&\frac{j_n-r-\sqrt{\big(r-j_n\big)^2+8r^2+16rj_{n-1}}}{4r}.\end{aligned}$$

□

5 Determinants of the RFPrLrR and RLPrFrL Circulant Matrix Involving Perrin Numbers

Theorem 9. *Let* $C = \mathrm{RFP}r\mathrm{LRcirc}_r\mathrm{fr}\big(R_1,R_2,\ldots,R_n\big)$. *Then*

$$\begin{aligned}\det C =& \frac{2\big(-rR_{n+1}-rR_{n+2}\big)^n}{\mathbb{Y}'}\\ &+\Bigg[\frac{\big(rR_n\big)^{n-1}\big(rR_{n+1}+rR_{n+2}\big)}{\mathbb{Y}'}\\ &\qquad\times\big(rX_{n-1}^2-rX_{2(n-1)}-2r^2X_{n-1}\big)\Bigg]\\ &-\frac{\big(rR_n\big)^n\big(rX_n^2-rX_{2n}-2r^2X_n+2r^2X_{n+1}\big)}{\mathbb{Y}'}\\ &-\Bigg[\frac{r^3\big(1-2rR_{n+2}\big)+\big(3-rR_n-rR_{n+2}\big)r^2X_n}{\mathbb{Y}'}\\ &\qquad\times 2\big(rR_n\big)^{n-1}\Bigg],\end{aligned}$$

where
$\mathbb{Y}' = -rY_n^2-rY_{n-1}^2+rY_{2n}+rY_{2(n-1)}-2r^2Y_{n+1}+2r^2Y_{n-1}-2r^3+2$.
$X_n = x_1^n+x_2^n+x_3^n$, *and* x_1, x_2, x_3 *are the roots of the equation* $-rR_nx^3+(3-rR_n-rR_{n+2})x^2+\big(2-rR_{n+1}\big)x+\big(-rR_{n+1}-rR_{n+2}\big)=0$.
$Y_n = y_1^n+y_2^n+y_3^n$, y_1, y_2, y_3 *are the roots of the equation* $y^3+y^2-1=0$.

Proof. Obviously, the matrix C has the form

$$C=\begin{pmatrix} R_1 & R_2 & \ldots & R_n\\ rR_n & R_1+rR_n & \ldots & R_{n-1}\\ \vdots & \vdots & \ddots & \vdots\\ rR_3 & rR_4+rR_3 & \ldots & R_2\\ rR_2 & rR_3+rR_2 & \ldots & R_1+rR_n\end{pmatrix}.$$

Owing to Lemma 1, the Binet form (8) and (7),

$$\begin{aligned}\det C =& \prod_{i=1}^n\big(R_1+R_2\pi_i+\cdots+R_n\pi_i^{n-1}\big)\\ =&\prod_{i=1}^n\sum_{k=1}^n\sum_{j=1}^3 r_j^k\pi_i^{k-1}\\ =&\prod_{i=1}^n\sum_{j=1}^3\frac{r_j(1-r_j^n\pi_i^n)}{1-r_j\pi_i}\\ =&\prod_{i=1}^n\Bigg[\frac{-rR_n\pi_i^3+(3-rR_n-rR_{n+2})\pi_i^2}{-\pi_i^3-\pi_i^2+1}\\ &+\frac{\big(2-rR_{n+1}\big)\pi_i-\big(rR_{n+1}+rR_{n+2}\big)}{-\pi_i^3-\pi_i^2+1}\Bigg].\end{aligned}$$

By Lemma 2 and the recurrence (5), we obtain

$$\begin{aligned}
&\prod_{i=1}^{n}\Big[-rR_n\pi_i^3+(3-rR_n-rR_{n+2})\pi_i^2\\
&\qquad+\big(2-rR_{n+1}\big)\pi_i-\big(rR_{n+1}+rR_{n+2}\big)\Big]\\
=&\big(-rR_{n+1}-rR_{n+2}\big)^n\\
&+\Big[\frac{1}{2}\big(rR_n\big)^{n-1}\big(rR_{n+1}+rR_{n+2}\big)\\
&\quad\times\big(rX_{n-1}^2-rX_{2(n-1)}-2r^2X_{n-1}\big)\Big]\\
&-\frac{1}{2}\big(rR_n\big)^n\big(rX_n^2-rX_{2n}-2r^2X_n+2r^2X_{n+1}\big)\\
&-\big(rR_n\big)^{n-1}\Big[r^3\big(1-2rR_{n+2}\big)\\
&\qquad+\big(3-rR_n-rR_{n+2}\big)r^2X_n\Big],
\end{aligned}$$

where $X_n=x_1^n+x_2^n+x_3^n$, and x_1,x_2,x_3 are the roots of the equation $-rR_nx^3+(3-rR_n-rR_{n+2})x^2+\big(2-rR_{n+1}\big)x+\big(-rR_{n+1}-rR_{n+2}\big)=0$. And

$$\begin{aligned}
&\prod_{i=1}^{n}(-\pi_i^3-\pi_i^2+1)\\
=&\frac{1}{2}\big(-rY_n^2-rY_{n-1}^2+rY_{2n}+rY_{2(n-1)}\big)\\
&-r^2Y_{n+1}+r^2Y_{n-1}-r^3+1,
\end{aligned}$$

where $Y_n=y_1^n+y_2^n+y_3^n$, and y_1,y_2,y_3 are the roots of the equation $y^3+y^2-1=0$.

Consequently,

$$\begin{aligned}
\det C=&\frac{2\big(-rR_{n+1}-rR_{n+2}\big)^n}{\mathbb{Y}'}\\
&+\Big[\frac{\big(rR_n\big)^{n-1}\big(rR_{n+1}+rR_{n+2}\big)}{\mathbb{Y}'}\\
&\qquad\times\big(rX_{n-1}^2-rX_{2(n-1)}-2r^2X_{n-1}\big)\Big]\\
&-\frac{\big(rR_n\big)^n\big(rX_n^2-rX_{2n}-2r^2X_n+2r^2X_{n+1}\big)}{\mathbb{Y}'}\\
&-\Big[\frac{r^3\big(1-2rR_{n+2}\big)+\big(3-rR_n-rR_{n+2}\big)r^2X_n}{\mathbb{Y}'}\\
&\qquad\times 2\big(rR_n\big)^{n-1}\Big].
\end{aligned}$$

□

Theorem 10. *Let $C'=\mathrm{RFP}r\mathrm{LRcirc}_r\mathrm{fr}\big(R_n,R_{n-1},\ldots,R_1\big)$. Then*

$$\begin{aligned}
\det C'=&\frac{(R_n-3r)^n}{\mathbb{Y}''}-\Big[\frac{r(R_n-3r)(x_1^{n-1}+x_2^{n-1})}{\mathbb{Y}''}\\
&+\frac{r(R_{n+1}-2r)(x_1^n+x_2^n)}{\mathbb{Y}''}\\
&-\frac{r^2\big(R_n+R_{n+1}-R_{n+2}\big)}{\mathbb{Y}''}\Big]\cdot(R_{n+1}-2r)^{n-1},
\end{aligned}$$

where

$$\begin{aligned}
\mathbb{Y}''=&1-r^3-\frac{1}{2}[rY_{n-1}^2-rY_{2(n-1)}-2r^2Y_{n-1}]\\
&-\frac{1}{2}[rY_n^2-rY_{2n}-2r^2Y_n+2r^2Y_{n+1}]\\
x_1=&\frac{5r-R_{n+2}+\sqrt{\chi}}{2R_{n+1}-4r},\\
x_2=&\frac{5r-R_{n+2}-\sqrt{\chi}}{2R_{n+1}-4r},\\
\chi=&(R_{n+2}-5r)^2-4(R_{n+1}-2r)(R_n-3r),
\end{aligned}$$

$Y_n=y_1^n+y_2^n+y_3^n$, and y_1, y_2, y_3 are the roots of the equation $y^3-y-1=0$.

Proof. The matrix C' has the form

$$\begin{pmatrix}
R_n & R_{n-1} & \ldots & R_1\\
rR_1 & R_n+rR_1 & \ldots & R_2\\
\vdots & \vdots & \ddots & \vdots\\
rR_{n-2} & rR_{n-3}+rR_{n-2} & \ldots & R_{n-1}\\
rR_{n-1} & rR_{n-2}+rR_{n-1} & \ldots & R_n+rR_1
\end{pmatrix}.$$

According to Lemma 1, the Binet form (8) and (7), we have

$$\begin{aligned}
&\det C'=\prod_{i=1}^{n}\big(R_n+R_{n-1}\pi_i+\cdots+R_1\pi_i^{n-1}\big)\\
=&\prod_{i=1}^{n}\sum_{k=0}^{n-1}\sum_{j=1}^{3}r_j^{n-k}\pi_i^k=\prod_{i=1}^{n}\sum_{j=1}^{3}\frac{r_j^{n+1}-r_j\pi_i^n}{r_j-\pi_i}\\
=&\prod_{i=1}^{n}\frac{(R_{n+1}-2r)\pi_i^2+(R_{n+2}-5r)\pi_i+R_n-3r}{-\pi_i^3+\pi_i+1}.
\end{aligned}$$

Using Lemma 2 and the recurrence (5), we obtain

$$\begin{aligned}
&\prod_{i=1}^{n}\left[(R_{n+1}-2r)\pi_i^2+(R_{n+2}-5r)\pi_i+R_n-3r\right]\\
=&(R_n-3r)^n-\Big[r(R_n-3r)(x_1^{n-1}+x_2^{n-1})\\
&+r(R_{n+1}-2r)(x_1^n+x_2^n)\\
&-r^2\big(R_n+R_{n+1}-R_{n+2}\big)\Big](R_{n+1}-2r)^{n-1},
\end{aligned}$$

where

$$x_1 = \frac{5r - R_{n+2} + \sqrt{\chi}}{2R_{n+1} - 4r}, \quad x_2 = \frac{5r - R_{n+2} - \sqrt{\chi}}{2R_{n+1} - 4r},$$
$$\chi = (R_{n+2} - 5r)^2 - 4(R_{n+1} - 2r)(R_n - 3r).$$

And

$$\prod_{i=1}^{n}(-\pi_i^3 + \pi_i + 1)$$
$$= 1 - r^3 - \frac{1}{2}[rY_{n-1}^2 - rY_{2(n-1)} - 2r^2Y_{n-1}] - \frac{1}{2}[rY_n^2 - rY_{2n} - 2r^2Y_n + 2r^2Y_{n+1}],$$

where $Y_n = y_1^n + y_2^n + y_3^n$, and y_1, y_2, y_3 are the roots of the equation $y^3 - y - 1 = 0$. Therefore,

$$\det C' = \frac{(R_n - 3r)^n}{\mathbb{Y}''} - \left[\frac{r(R_n - 3r)(x_1^{n-1} + x_2^{n-1})}{\mathbb{Y}''} + \frac{r(R_{n+1} - 2r)(x_1^n + x_2^n)}{\mathbb{Y}''} - \frac{r^2(R_n + R_{n+1} - R_{n+2})}{\mathbb{Y}''}\right] \cdot (R_{n+1} - 2r)^{n-1}.$$

□

Theorem 11. *Let* $R = \mathrm{RLP}r\mathrm{FLcirc}_r\mathrm{fr}(R_1, R_2, \ldots, R_n)$. *Then*

$$\det R = \left\{\frac{(R_n - 3r)^n}{\mathbb{Y}''} - \left[\frac{r(R_n - 3r)(x_1^{n-1} + x_2^{n-1})}{\mathbb{Y}''} + \frac{r(R_{n+1} - 2r)(x_1^n + x_2^n)}{\mathbb{Y}''} - \frac{r^2(R_n + R_{n+1} - R_{n+2})}{\mathbb{Y}''}\right] \cdot (R_{n+1} - 2r)^{n-1}\right\} \times (-1)^{\frac{n(n-1)}{2}},$$

where

$$\mathbb{Y}'' = 1 - r^3 - \frac{1}{2}[rY_{n-1}^2 - rY_{2(n-1)} - 2r^2Y_{n-1}] - \frac{1}{2}[rY_n^2 - rY_{2n} - 2r^2Y_n + 2r^2Y_{n+1}],$$
$$x_1 = \frac{5r - R_{n+2} + \sqrt{\chi}}{2R_{n+1} - 4r},$$
$$x_2 = \frac{5r - R_{n+2} - \sqrt{\chi}}{2R_{n+1} - 4r},$$
$$\chi = (R_{n+2} - 5r)^2 - 4(R_{n+1} - 2r)(R_n - 3r),$$

$Y_n = y_1^n + y_2^n + y_3^n$, *and* y_1, y_2, y_3 *are the roots of the equation* $y^3 - y - 1 = 0$.

Proof. Since

$$\det \widehat{I}_n = (-1)^{\frac{n(n-1)}{2}},$$

the result can be derived from Theorem 10 and the relation (1). □

6 Determinants of the RFPrLrR and RLPrFrL Circulant Matrix Involving Padovan Numbers

Theorem 12. *Let* $D = \mathrm{RFP}r\mathrm{LRcirc}_r\mathrm{fr}(\mathbb{P}_1, \mathbb{P}_2, \ldots, \mathbb{P}_n)$. *Then*

$$\det D = \frac{(\mathbb{P}_1 - r\mathbb{P}_{n+1})^n}{\mathbb{V}'} - \frac{\frac{1}{2}(r\mathbb{P}_n)^{n-1}}{\mathbb{V}'}(\mathbb{P}_1 - r\mathbb{P}_{n+1})(rU_{n-1}^2 - rU_{2(n-1)} - 2r^2U_{n-1}) - \frac{\frac{1}{2}(r\mathbb{P}_n)^n}{\mathbb{V}'}(rU_n^2 - rU_{2n} - 2r^2U_n + 2r^2U_{n+1}) - \frac{(r\mathbb{P}_n)^{n-1}}{\mathbb{V}'}\Big[r^3(\mathbb{P}_1 - \mathbb{P}_2 + a_1 + a_2 + a_3) + (a_1 + a_2 + a_3 - r\mathbb{P}_n - r\mathbb{P}_{n+2})r^2U_n\Big],$$

where $\mathbb{V}' = 1 - r^3 - \frac{1}{2}(rV_{n-1}^2 - rV_{2(n-1)} - 2r^2V_{n-1}) - \frac{1}{2}(rV_n^2 - rV_{2n} + 2r^2V_{n+1} - 4r^2V_n)$, $U_n = u_1^n + u_2^n + u_3^n$, u_1, u_2, u_3 *are the roots of the equation* $-r\mathbb{P}_nx^3 + (a_1 + a_2 + a_3 - r\mathbb{P}_n - r\mathbb{P}_{n+2})x^2 + (\mathbb{P}_2 - r\mathbb{P}_{n+1} - r\mathbb{P}_{n+2}x + \mathbb{P}_1 - r\mathbb{P}_{n+1} = 0$ *and* $V_n = v_1^n + v_2^n + v_3^n$, v_1, v_2, v_3 *are the roots of the equation* $y^3 + y^2 - 1 = 0$.

Proof. The matrix D has the form

$$D = \begin{pmatrix} \mathbb{P}_1 & \mathbb{P}_2 & \ldots & \mathbb{P}_n \\ r\mathbb{P}_n & \mathbb{P}_1 + r\mathbb{P}_n & \ldots & \mathbb{P}_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ r\mathbb{P}_3 & r\mathbb{P}_4 + r\mathbb{P}_3 & \ldots & \mathbb{P}_2 \\ r\mathbb{P}_2 & r\mathbb{P}_3 + r\mathbb{P}_2 & \ldots & \mathbb{P}_1 + r\mathbb{P}_n \end{pmatrix}.$$

$$\det D = \prod_{i=1}^{n}(\mathbb{P}_1 + \mathbb{P}_2\pi_i + \cdots + \mathbb{P}_n\pi_i^{n-1})$$
$$= \prod_{i=1}^{n}\sum_{k=1}^{n}\sum_{j=1}^{3} a_j r_j^k \pi_i^{k-1}$$
$$= \prod_{i=1}^{n}\sum_{j=1}^{3}\frac{a_j r_j(1 - r_j^n\pi_i^n)}{1 - r_j\pi_i}$$
$$= \prod_{i=1}^{n}\left[\frac{-r\mathbb{P}_n\pi_i^3 + (a_1 + a_2 + a_3 - r\mathbb{P}_n - r\mathbb{P}_{n+2})\pi_i^2}{-\pi_i^3 - \pi_i^2 + 1} + \frac{(\mathbb{P}_2 - r\mathbb{P}_{n+1} - r\mathbb{P}_{n+2})\pi_i + \mathbb{P}_1 - r\mathbb{P}_{n+1}}{-\pi_i^3 - \pi_i^2 + 1}\right],$$

from Lemma 1, the Binet form (9) and (7). Using

Lemma 2 and the recurrence (6), we obtain

$$\prod_{i=1}^{n}\Big[-r\mathbb{P}_n\pi_i^3+\big(a_1+a_2+a_3-r\mathbb{P}_n-r\mathbb{P}_{n+2}\big)\pi_i^2 + \big(\mathbb{P}_2-r\mathbb{P}_{n+1}-r\mathbb{P}_{n+2}\big)\pi_i+\mathbb{P}_1-r\mathbb{P}_{n+1}\Big]$$
$$=\big(\mathbb{P}_1-r\mathbb{P}_{n+1}\big)^n-\frac{1}{2}\big(r\mathbb{P}_n\big)^{n-1}\big(\mathbb{P}_1-r\mathbb{P}_{n+1}\big) \times\big[rU_{n-1}^2-rU_{2(n-1)}-2r^2U_{n-1}\big] -\frac{1}{2}\big(r\mathbb{P}_n\big)^n\big[rU_n^2-rU_{2n}-2r^2U_n+2r^2U_{n+1}\big] -\big(r\mathbb{P}_n\big)^{n-1}\big[r^3\big(\mathbb{P}_1-\mathbb{P}_2+a_1+a_2+a_3\big) +\big(a_1+a_2+a_3-r\mathbb{P}_n-r\mathbb{P}_{n+2}\big)r^2U_n\big],$$

and

$$\prod_{i=1}^{n}\big[-\pi_i^3-\pi_i^2+1\big] =1-r^3-\frac{1}{2}\big[rV_{n-1}^2-rV_{2(n-1)}-2r^2V_{n-1}\big] -\frac{1}{2}\big[rV_n^2-rV_{2n}+2r^2V_{n+1}-4r^2V_n\big],$$

where $U_n=u_1^n+u_2^n+u_3^n$, u_1, u_2, u_3 are the roots of the equation $-r\mathbb{P}_nx^3+\big(a_1+a_2+a_3-r\mathbb{P}_n-r\mathbb{P}_{n+2}\big)x^2+\big(\mathbb{P}_2-r\mathbb{P}_{n+1}-r\mathbb{P}_{n+2}\big)x+\mathbb{P}_1-r\mathbb{P}_{n+1}=0$.

We have the following results:

$$\det D =\frac{\big(\mathbb{P}_1-r\mathbb{P}_{n+1}\big)^n}{\mathbb{V}'}-\frac{\frac{1}{2}\big(r\mathbb{P}_n\big)^{n-1}}{\mathbb{V}'}\big(\mathbb{P}_1 -r\mathbb{P}_{n+1}\big)\big(rU_{n-1}^2-rU_{2(n-1)}-2r^2U_{n-1}\big) -\frac{\frac{1}{2}\big(r\mathbb{P}_n\big)^n}{\mathbb{V}'}\big(rU_n^2-rU_{2n}-2r^2U_n+2r^2U_{n+1}\big) -\frac{\big(r\mathbb{P}_n\big)^{n-1}}{\mathbb{V}'}\Big[r^3\big(\mathbb{P}_1-\mathbb{P}_2+a_1+a_2+a_3\big) +\big(a_1+a_2+a_3-r\mathbb{P}_n-r\mathbb{P}_{n+2}\big)r^2U_n\Big],$$

where $\mathbb{V}'=1-r^3-\frac{1}{2}\big(rV_{n-1}^2-rV_{2(n-1)}-2r^2V_{n-1}\big)-\frac{1}{2}\big(rV_n^2-rV_{2n}+2r^2V_{n+1}-4r^2V_n\big)$, $U_n=u_1^n+u_2^n+u_3^n$, u_1, u_2, u_3 are the roots of the equation $-r\mathbb{P}_nx^3+\big(a_1+a_2+a_3-r\mathbb{P}_n-r\mathbb{P}_{n+2}\big)x^2+\big(\mathbb{P}_2-r\mathbb{P}_{n+1}-r\mathbb{P}_{n+2}x+\mathbb{P}_1-r\mathbb{P}_{n+1}=0$ and $V_n=v_1^n+v_2^n+v_3^n$, v_1, v_2, v_3 are the roots of the equation $y^3+y^2-1=0$. □

Theorem 13. *Let* $D'=\mathrm{RFP}r\mathrm{LRcirc}_r\mathrm{fr}(\mathbb{P}_n,\mathbb{P}_{n-1},\ldots,\mathbb{P}_1)$. *Then*

$$\det D' =\frac{\big[\mathbb{P}_n-r\big(a_1+a_2+a_3\big)\big]^n}{\mathbb{V}''}-\frac{\frac{1}{2}r^{n-1}}{\mathbb{V}''}\Big[\mathbb{P}_n-r\big(a_1 +a_2+a_3\big)\Big]\cdot\big(rU_{n-1}^2-rU_{2(n-1)}-2r^2U_{n-1}\big) -\frac{\frac{1}{2}r^n}{\mathbb{V}''}\cdot\big(rU_n^2-rU_{2n}-2r^2U_n+2r^2U_{n+1}\big) -\frac{r^{n-1}}{\mathbb{V}''}\big[r^3\big(\mathbb{P}_n+\mathbb{P}_{n+1}-\mathbb{P}_{n+2}-r\mathbb{P}_2\big) +\big(\mathbb{P}_{n+1}-r\mathbb{P}_1-r\mathbb{P}_2\big)r^2U_n\big],$$

where $\mathbb{V}''=1-r^3-\frac{1}{2}\big[rV_{n-1}^2-rV_{2(n-1)}-2r^2V_{n-1}\big]-\frac{1}{2}\big[rV_n^2-rV_{2n}-2r^2V_n+2r^2V_{n+1}\big]$, $U_n=u_1^n+u_2^n+u_3^n$, u_1, u_2, u_3 *are the roots of the equation* $-r\mathbb{P}_1x^3+\big(\mathbb{P}_{n+1}-r\mathbb{P}_1-r\mathbb{P}_2\big)x^2+\big[\mathbb{P}_{n+2}-r\big(a_1+a_2+a_3\big)-r\mathbb{P}_2\big]x+\mathbb{P}_n-r\big(a_1+a_2+a_3\big)=0$. *and* $V_n=v_1^n+v_2^n+v_3^n$, v_1, v_2, v_3 *are the roots of the equation* $y^3-y-1=0$.

Proof. The matrix D' has the form

$$\begin{pmatrix} \mathbb{P}_n & \mathbb{P}_{n-1} & \ldots & \mathbb{P}_1 \\ r\mathbb{P}_1 & \mathbb{P}_n+r\mathbb{P}_1 & \ldots & \mathbb{P}_2 \\ \vdots & \vdots & \ddots & \vdots \\ r\mathbb{P}_{n-2} & r\mathbb{P}_{n-3}+r\mathbb{P}_{n-2} & \ldots & \mathbb{P}_{n-1} \\ r\mathbb{P}_{n-1} & r\mathbb{P}_{n-2}+r\mathbb{P}_{n-1} & \ldots & \mathbb{P}_n+r\mathbb{P}_{n-1} \end{pmatrix}.$$

According to Lemma 1, the Binet form (9) and (7), we have

$$\det D'=\prod_{i=1}^{n}\big(\mathbb{P}_n+\mathbb{P}_{n-1}\pi_i+\cdots+\mathbb{P}_1\pi_i^{n-1}\big) =\prod_{i=1}^{n}\sum_{k=0}^{n-1}\sum_{j=1}^{3}a_jr_j^{n-k}\pi_i^k=\prod_{i=1}^{n}\sum_{j=1}^{3}\frac{a_jr_j^{n+1}-a_jr_j\pi_i^n}{r_j-\pi_i} =\prod_{i=1}^{n}\Bigg[\frac{-r\pi_i^3+\big(\mathbb{P}_{n+1}-2r\big)\pi_i^2}{-\pi_i^3+\pi_i+1} +\frac{\big[\mathbb{P}_{n+2}-r\big(a_1+a_2+a_3\big)-r\big]\pi_i}{-\pi_i^3+\pi_i+1} +\frac{\mathbb{P}_n-r\big(a_1+a_2+a_3\big)}{-\pi_i^3+\pi_i+1}\Bigg].$$

Using Lemma 2 and (9), we obtain

$$\prod_{i=1}^{n}\Big[-r\pi_i^3+\big(\mathbb{P}_{n+1}-2r\big)\pi_i^2+\big[\mathbb{P}_{n+2}-r\big(a_1+a_2+a_3\big)-r\big]\pi_i+\mathbb{P}_n-r\big(a_1+a_2+a_3\big)\Big]$$
$$=\big[\mathbb{P}_n-r\big(a_1+a_2+a_3\big)\big]^n-\frac{1}{2}\big(r\mathbb{P}_1\big)^{n-1}\big[\mathbb{P}_n-r\big(a_1+a_2+a_3\big)\big]\big(rU_{n-1}^2-rU_{2(n-1)}-2r^2U_{n-1}\big)-\frac{1}{2}\big(r\mathbb{P}_1\big)^n\big(rU_n^2-rU_{2n}-2r^2U_n+2r^2U_{n+1}\big)-\big(r\mathbb{P}_1\big)^{n-1}\big[r^3\big(\mathbb{P}_n+\mathbb{P}_{n+1}-\mathbb{P}_{n+2}-r\mathbb{P}_2\big)+\big(\mathbb{P}_{n+1}-r\mathbb{P}_1-r\mathbb{P}_2\big)r^2U_n\big],$$

where $U_n=u_1^n+u_2^n+u_3^n$, u_1, u_2, u_3 are the roots of the equation $-r\mathbb{P}_1x^3+\big(\mathbb{P}_{n+1}-r\mathbb{P}_1-r\mathbb{P}_2\big)x^2+\big[\mathbb{P}_{n+2}-r\big(a_1+a_2+a_3\big)-r\mathbb{P}_2\big]x+\mathbb{P}_n-r\big(a_1+a_2+a_3\big)=0$.

$$\prod_{i=1}^{n}\big(-\pi_i^3+\pi_i+1\big)=1-r^3-\frac{1}{2}\big[rV_{n-1}^2-rV_{2(n-1)}-2r^2V_{n-1}\big]-\frac{1}{2}\big[rV_n^2-rV_{2n}-2r^2V_n+2r^2V_{n+1}\big],$$

where $V_n=v_1^n+v_2^n+v_3^n$, v_1, v_2, v_3 are the roots of the equation $y^3-y-1=0$.

Then we have the following results:

$$\det D'=\frac{\big[\mathbb{P}_n-r\big(a_1+a_2+a_3\big)\big]^n}{\mathbb{V}''}-\frac{\frac{1}{2}r^{n-1}}{\mathbb{V}''}\Big[\mathbb{P}_n-r\big(a_1+a_2+a_3\big)\Big]\cdot\big(rU_{n-1}^2-rU_{2(n-1)}-2r^2U_{n-1}\big)-\frac{\frac{1}{2}r^n}{\mathbb{V}''}\cdot\big(rU_n^2-rU_{2n}-2r^2U_n+2r^2U_{n+1}\big)-\frac{r^{n-1}}{\mathbb{V}''}\big[r^3\big(\mathbb{P}_n+\mathbb{P}_{n+1}-\mathbb{P}_{n+2}-r\mathbb{P}_2\big)+\big(\mathbb{P}_{n+1}-r\mathbb{P}_1-r\mathbb{P}_2\big)r^2U_n\big],$$

where $\mathbb{V}''=1-r^3-\frac{1}{2}\big[rV_{n-1}^2-rV_{2(n-1)}-2r^2V_{n-1}\big]-\frac{1}{2}\big[rV_n^2-rV_{2n}-2r^2V_n+2r^2V_{n+1}\big]$, $U_n=u_1^n+u_2^n+u_3^n$, u_1, u_2, u_3 are the roots of the equation $-r\mathbb{P}_1x^3+\big(\mathbb{P}_{n+1}-r\mathbb{P}_1-r\mathbb{P}_2\big)x^2+\big[\mathbb{P}_{n+2}-r\big(a_1+a_2+a_3\big)-r\mathbb{P}_2\big]x+\mathbb{P}_n-r\big(a_1+a_2+a_3\big)=0$. and $V_n=v_1^n+v_2^n+v_3^n$, v_1, v_2, v_3 are the roots of the equation $y^3-y-1=0$. □

Theorem 14. *Let* $\mathbb{P}=\mathrm{RLP}r\mathrm{FLcirc}_r\mathrm{fr}\big(\mathbb{P}_1,\mathbb{P}_2,\ldots,\mathbb{P}_n\big)$. *Then*

$$\det\mathbb{P}=\Big\{\frac{\big[\mathbb{P}_n-r\big(a_1+a_2+a_3\big)\big]^n}{\mathbb{V}''}-\frac{\frac{1}{2}r^{n-1}}{\mathbb{V}''}\Big[\mathbb{P}_n-r\big(a_1+a_2+a_3\big)\Big]\cdot\big(rU_{n-1}^2-rU_{2(n-1)}-2r^2U_{n-1}\big)-\frac{\frac{1}{2}r^n}{\mathbb{V}''}\cdot\big(rU_n^2-rU_{2n}-2r^2U_n+2r^2U_{n+1}\big)-\frac{r^{n-1}}{\mathbb{V}''}\big[r^3\big(\mathbb{P}_n+\mathbb{P}_{n+1}-\mathbb{P}_{n+2}-r\mathbb{P}_2\big)+\big(\mathbb{P}_{n+1}-r\mathbb{P}_1-r\mathbb{P}_2\big)r^2U_n\big]\Big\}\cdot(-1)^{\frac{n(n-1)}{2}},$$

where $\mathbb{V}''=1-r^3-\frac{1}{2}\big[rV_{n-1}^2-rV_{2(n-1)}-2r^2V_{n-1}\big]-\frac{1}{2}\big[rV_n^2-rV_{2n}-2r^2V_n+2r^2V_{n+1}\big]$, $U_n=u_1^n+u_2^n+u_3^n$, u_1, u_2, u_3 *are the roots of the equation* $-r\mathbb{P}_1x^3+\big(\mathbb{P}_{n+1}-r\mathbb{P}_1-r\mathbb{P}_2\big)x^2+\big[\mathbb{P}_{n+2}-r\big(a_1+a_2+a_3\big)-r\mathbb{P}_2\big]x+\mathbb{P}_n-r\big(a_1+a_2+a_3\big)=0$. *and* $V_n=v_1^n+v_2^n+v_3^n$, v_1, v_2, v_3 *are the roots of the equation* $y^3-y-1=0$.

Proof. The theorem can be proved by using Theorem 13 and the relation (1). □

7 Conclusion

In this paper, we introduce RFPrLrR circulant matrix and RLPrFrL circulant matrices, which are two special pattern matrices. Based on the Binet formulaes of famous numbers and the characteristic polynomial of the basic RFPrLrR circulant matrix, we give the exact determinants of the two pattern matrices involving Jacobsthal, Jacobsthal-Lucas, Perrin and Padovan numbers in section 3, 4, 5 and 6.

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