# Time-fractional Klein-Gordon equation: formulation and solution using variational methods

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*Abstract:* This paper presents the formulation of time-fractional Klein-Gordon equation using the Euler-Lagrange variational technique in the Riesz derivative sense and derives an approximate solitary wave solution. Our results witness that He's variational iteration method was very efficient and powerful technique in finding the solution of the proposed equation. The basic idea described in this paper is efficient and powerful in solving wide classes of nonlinear fractional high order evolution equations.

*Key–Words:* Riesz fractional derivative; Euler-Lagrange equation; Klein-Gordon equation; He's variational iteration method; Solitary wave

## 1 Introduction

Our aim is to apply the reliable treatment of He's variational iteration method [1–4] to obtain the solution of the initial value problem of the time-fractional Klein-Gordon equation of the form with the initial conditions

$$\begin{cases} {}_0^RD_t^\alpha u - a(u^2)_{xx} + b(u^2)_{xxxx} = 0, \\ u(x,0) = \varphi_1(x), \\ u_t(x,0) = \varphi_2(x), \end{cases} \tag{1}$$

where $1 < \alpha < 2$, $a, b > 0$, $u = u(x,t)$ is a field variable, $x \in \Omega$ ($\Omega \subseteq \mathbb{R}$) is a space coordinate in the propagation direction of the field and $t \in T(= [0, t_0]$ $(t_0 > 0))$ is the time, ${}_0^RD_t^\alpha$ is the Riesz fractional derivative, $\varphi_1(x)$ and $\varphi_2(x)$ are given functions. The subscripts denote the partial differentiation of the function $u$ with respect to the parameter $x$ and $t$.

Fractional calculus have attracted much attention during recent years due to their numerous applications in physics and engineering [5–8], such as, the nonlinear oscillation of earthquake can be modeled with fractional derivatives [9], and the fluid dynamic traffic model with fractional derivatives can eliminate the deficiency arising from the assumption of continuum traffic flow [10]. Finding of a new mathematical algorithm to construct exact solution of nonlinear fractional order evolution equations is important and might have significant impact on future research, we notice that the Lagrangian of conservative system is constructed using fractional derivatives, the resulting equations of motion can be nonconservative. Therefore, in many cases, the real physical processes could be modeled in a reliable manner using fractional-order differential equations [11]. Based on the stochastic embedding theory, Cresson [12] defined the fractional embedding of differential operators and provided a fractional Euler-Lagrange equation for Lagrangian systems, then investigated a fractional Noether theorem and a fractional Hamiltonian formulation of fractional Lagrangian systems. Herzallah and Baleanu [13] presented the necessary and sufficient optimality conditions for the Euler-Lagrange fractional equations of fractional variational problems with determining in which spaces the functional must exist. Malinowska [14] proposed the Euler-Lagrange equations for fractional variational problems with multiple integrals and proved the fractional Noether-type theorem for conservative and nonconservative generalized physical systems. Riewe [15,16] formulated a version of the Euler-Lagrange equation for problems of calculus of variation with fractional derivatives. Wu and Baleanu [17] developed some new variational iteration formulae to find approximate solutions of fractional differential equations and determined the Lagrange multiplier in a more accurate way. For generalized fractional Euler-Lagrange equations we can refer to the works by Odzijewicz [18]. Other the known results we can see Baleanu et al [19] and Inokuti et al [20]. Thanks to the most of physical phenomena may be considered as nonconservative, then they can be described using fractional-order differential equations.

Klein-Gordon equation describes relativistic electrons, and correctly describes the spinless pion, the equation was first considered as a quantum wave equation by Schrödinger in his search for an equation describing de Broglie waves, and plays a significant role in mathematical physics and many scientific applications such as solid-state physics, nonlinear optics, and quantum field theory [21, 22]. The equation has attracted much attention in studying solitons and condensed matter physics [23–25], in investigating the interaction of solitons in a collisionless plasma, the recurrence of initial states, and in examining the nonlinear wave equations [26]. Recently, several methods have been used to solve fractional Klein-Gordon equation using techniques of nonlinear analysis, such as homotopy perturbation method [27], homotopy analysis method [28], Riccati expansion method [29], new iterative method [30,31], the first integral method [32] and series expansion method [33] et al. It was mentioned that the variational iteration method has been used successfully to solve fractional KdV equation [34]. The objective of this paper is to extend the application of the fractional variational iteration method to formulate a high-order time-fractional Klein-Gordon equation and derive an approximate solitary wave solution.

This paper is organized as follows: Section 2 states some background material from fractional calculus. Section 3 presents the principle of He's variational iteration method. Section 4 is devoted to describe the formulation of the time-fractional Klein-Gordon equation using the Euler-Lagrange variational technique and to derive an approximate solitary wave solution. Section 5 makes some analysis for the obtained graphs and figures and discusses the present work.

## 2 Preliminaries

We recall the necessary definitions for the fractional calculus (see [35–37]) which is used throughout the remaining sections of this paper.

**Definition 1** *A real multivariable function $f(x,t)$, $t > 0$ is said to be in the space $C_\gamma$, $\gamma \in \mathbb{R}$ with respect to t if there exists a real number $p(> \gamma)$, such that $f(x,t) = t^p f_1(x,t)$, where $f_1(x,t) \in C(\Omega \times T)$. Obviously, $C_\gamma \subset C_\delta$ if $\delta \le \gamma$.*

**Definition 2** *The left-hand side Riemann-Liouville fractional integral of a function $f \in C_\gamma$, $(\gamma \ge -1)$ is defined by*

$$
{}_0I_t^\alpha f(x,t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(x,\tau)d\tau,\ \alpha > 0,
$$
$$
{}_0I_t^0 f(x,t) = f(x,t).
$$

**Definition 3** *The Riemann-Liouville fractional derivatives of the order $n-1 \le \alpha < n$ of a function $f \in C_\gamma$, $(\gamma \ge -1)$ are defined as*

$$
\begin{aligned}
&{}_0D_t^\alpha f(x,t) \\
&\quad = \frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial t^n} \int_0^t (t-\tau)^{n-\alpha-1} f(x,\tau)d\tau, \\
&{}_tD_{t_0}^\alpha f(x,t) \\
&\quad = \frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial t^n} \int_t^{t_0} (\tau-t)^{n-\alpha-1} f(x,\tau)d\tau.
\end{aligned}
$$

**Lemma 4** *The integration of Riemann-Liouville fractional derivative of the order $0 < \alpha < 1$ of the functions $f$, $g$, ${}_tD_{t_0}^\alpha f(x,t)$, ${}_0D_t^\alpha g(x,t) \in C(\Omega \times T)$ by parts is given by the rule*

$$
\int_T f(x,t)_0D_t^\alpha g(x,t)dt = \int_T g(x,t)_tD_{t_0}^\alpha f(x,t)dt.
$$

**Definition 5** *The Riesz fractional integral of the order $n-1 \le \alpha < n$ of a function $f \in C_\gamma$, $(\gamma \ge -1)$ is defined as*

$$
\begin{aligned}
{}_0^RI_t^\alpha f(x,t) &= \frac{1}{2}\big({}_0I_t^\alpha f(x,t) + {}_tI_{t_0}^\alpha f(x,t)\big) \\
&= \frac{1}{2\Gamma(\alpha)} \int_0^{t_0} |t-\tau|^{\alpha-1} f(x,\tau)d\tau,
\end{aligned}
$$

*where ${}_0I_t^\alpha$ and ${}_tI_{t_0}^\alpha$ are respectively the left- and right-hand side Riemann-Liouville fractional integral operators.*

**Definition 6** *The Riesz fractional derivative of the order $n-1 \le \alpha < n$ of a function $f \in C_\gamma$, $(\gamma \ge -1)$ is defined by*

$$
\begin{aligned}
{}_0^RD_t^\alpha f(x,t) &= \frac{1}{2}\big({}_0D_t^\alpha f(x,t) + (-1)^n {}_tD_{t_0}^\alpha f(x,t)\big) \\
&= \frac{1}{2\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^{t_0} |t-\tau|^{n-\alpha-1} \\
&\quad \times f(x,\tau)d\tau,
\end{aligned}
$$

where ${}_0D_t^\alpha$ and ${}_tD_{t_0}^\alpha$ are respectively the left- and right-hand side Riemann-Liouville fractional differential operators.

**Lemma 7** *Let $\alpha > 0$ and $\beta > 0$ be such that $n-1 < \alpha < n$, $m-1 < \beta < m$ and $\alpha+\beta < n$, and let $f \in L_1(\Omega \times T)$ and ${}_0I_t^{m-\alpha} f \in AC^m(\Omega \times T)$. Then we have the following index rule:*

$$
{}_0^R D_t^{\alpha}\big({}_0^R D_t^{\beta} f(x,t)\big) = {}_0^R D_t^{\alpha+\beta} f(x,t) - \sum_{i=1}^{m} {}_0^R D_t^{\beta-i} f(x,t)|_{t=0} \frac{t^{-\alpha-i}}{\Gamma(1-\alpha-i)}.
$$

**Remark 8** *One can express the Riesz fractional differential operator ${}_0^R D_t^{\alpha-1}$ of the order $0 < \alpha < 1$ as the Riesz fractional integral operator ${}_0^R I_\tau^{1-\alpha}$, i.e.*

$$
{}_0^R D_t^{\alpha-1} f(x,t) = {}_0^R I_t^{1-\alpha} f(x,t), \quad t \in T.
$$

# 3 Variational-iteration method

The variational iteration method provides an effective procedure for explicit and solitary wave solutions of a wide and general class of differential systems representing real physical problems. Moreover, the variational iteration method can overcome the foregoing restrictions and limitations of approximate techniques so that it provides us with a possibility to analyze strongly nonlinear evolution equations. Therefore, we extend this method to solve the time-fractional Klein-Gordon-type equation. The basic features of the variational iteration method outlined as follows.

Considering a nonlinear evolution equation consists of a linear part $\mathcal{L}u$, nonlinear part $\mathcal{N}u$, and a free term $g(= g(x,t))$ represented as

$$
\mathcal{L}u + \mathcal{N}u = g. \tag{2}
$$

According to the variational iteration method, the $n+1$-th approximate solution of (2) can be read using iteration correction functional as

$$
u_{n+1} = u_n + \int_0^t \lambda(\tau)\big(\mathcal{L}\tilde{u} + \mathcal{N}\tilde{u} - g\big)d\tau, \tag{3}
$$

where $\lambda(\tau)$ is a Lagrangian multiplier and $\tilde{u} = \tilde{u}(x,t)$ is considered as a restricted variation function, i.e., $\delta\tilde{u} = 0$. Extreming the variation of the correction functional (3) leads to the Lagrangian multiplier $\lambda(\tau)$. The initial iteration $u_0$ can be used as the initial value $u(x,0)$. As $n$ tends to infinity, the iteration leads to the solitary wave solution of (2), i.e.

$$
u = \lim_{n\to\infty} u_n.
$$

# 4 Time-fractional Klein-Gordon equation

The Klein-Gordon equation is given as

$$
u_{tt} - a(u^2)_{xx} + b(u^2)_{xxxx} = 0. \tag{4}
$$

Employing a potential function $v$ on the field variable $u$, set $u = v_x$ yields the potential equation of the Klein-Gordon equation (4) in the form,

$$
v_{xtt} - a(v_x^2)_{xx} + b(v_x^2)_{xxxx} = 0. \tag{5}
$$

The Lagrangian of this Klein-Gordon equation (4) can be defined using the semi-inverse method [38, 39] as follows. The functional of the potential equation (5) can be represented as

$$
J(v) = \int_\Omega dx \int_T dt\big(c_1 v v_{xtt} - ac_2 v(v_x^2)_{xx} + bc_3 v(v_x^2)_{xxxx}\big), \tag{6}
$$

with $c_i$ $(i = 1,2,3)$ is unknown constant to be determined later. Integrating (6) by parts and taking

$$
v_x|_{\partial T} = v_{xt}|_{\partial T} = v_x|_{\partial\Omega} = (v_x^2)_x|_{\partial\Omega} = (v_x^2)_{xxx}|_{\partial\Omega} = 0 \tag{7}
$$

yield

$$
J(v) = \int_\Omega dx \int_T dt\big(c_1 v_x v_{tt} - ac_2(v_x^2)v_{xx} + bc_3(v_x^2)_{xx}v_{xx}\big). \tag{8}
$$

The constants $c_i$ $(i = 1,2,3)$ can be determined taking the variation of the functional (8) to make it optimal. By applying the variation of the functional, integrating each term by parts, and making use of the variation optimum condition of the functional $J(v)$, it yields the following representation

$$
2c_1 v_{xtt} - ac_2(v_x^2)_{xx} + bc_3(v_x^2)_{xxxx} = 0. \tag{9}
$$

We notice that the obtained result (9) is equivalent to (5), so one has that the constants $c_i$ $(i = 1,2,\ldots,6)$ are respectively

$$
c_1 = \frac{1}{2}, \quad c_2 = c_3 = 1.
$$

Then the functional becomes

$$
J(v) = \int_\Omega dx \int_T dt\big(\frac{1}{2}v_x v_{tt} - a(v_x^2)v_{xx} + b(v_x^2)_{xx}v_{xx}\big).
$$

In addition, the functional expression given by (8) obtains directly the Lagrangian form of the Klein-Gordon equation,

$$N(v_{tt}, v_x, v_{xx}, v_{xxx}) = \frac{1}{2} v_{tt} v_x - a v_x^2 v_{xx} + 2b v_{xx}^3 + 2b v_x v_{xx} v_{xxx}.$$

The Lagrangian of the time-fractional version of the Klein-Gordon equation (1) could be read as

$$F({}_0D_t^{\alpha} v, v_x, v_{xx}, v_{xxx}) = \frac{1}{2} {}_0D_t^{\alpha} v v_x - a v_x^2 v_{xx} + 2b v_{xx}^3 + 2b v_x v_{xx} v_{xxx}, \quad \alpha \in ]1, 2]. \tag{10}$$

Then the functional of the time-fractional Klein-Gordon equation will take the representation

$$J(v) = \int_{\Omega} dx \int_T F({}_0D_t^{\alpha} v, v_x, v_{xx}, v_{xxx}) dt, \tag{11}$$

where the time-fractional Lagrangian $F({}_0D_t^{\alpha} v, v_x, v_{xx}, v_{xxx})$ is given by (10). Following Agrawal's method [40–42], the variation of functional (11) with respect to $v$ leads to

$$\delta J(v) = \int_{\Omega} dx \int_T dt \Big( \frac{\partial F}{\partial {}_0D_t^{\alpha} v} \delta({}_0D_t^{\alpha} v) + \frac{\partial F}{\partial v_x} \delta v_x + \frac{\partial F}{\partial v_{xx}} \delta v_{xx} + \frac{\partial F}{\partial v_{xxx}} \delta(v_{xxx}) \Big). \tag{12}$$

By Lemma 4, upon integrating the right-hand side of (12), one has

$$\begin{aligned}\delta J(v) = \int_{\Omega} dx \int_T dt \Big( & \frac{1}{2} {}_t^C D_T^{\alpha} v_x \delta v + \frac{1}{2} \big( v_{x0} D_t^{\alpha-1} \delta v - v_{xt0} D_t^{\alpha-2} \delta v \big)|_{\partial T} - \frac{1}{2} {}_0D_t^{\alpha} v_x \delta v \\ & - \frac{\partial}{\partial x} \big( -2a v_x v_{xx} + 2b v_{xx} v_{xxx} \big) \delta v \\ & + \frac{\partial^2}{\partial x^2} \big( -a v_x^2 + 2b v_x v_{xxx} + 6b v_{xx}^2 \big) \delta v \\ & - \frac{\partial^3}{\partial x^3} \big( 2b v_x v_{xx} \big) \delta v \Big),\end{aligned}$$

noting that $\delta v|_{\partial T} = \delta v|_{\partial \Omega} = \delta v_x|_{\partial \Omega} = \delta v_{xx}|_{\partial \Omega} = \delta v_x^2|_{\partial \Omega} = \delta v_{xx}|_{\partial T} = 0$. $v_x|_{\partial T} = 0$ in (7) implies

$${}_t^C D_T^{\alpha} v_x = {}_t D_T^{\alpha} v_x.$$

Obviously, optimizing the variation of the functional $J(v)$, i.e., $\delta J(v) = 0$, yields the Euler-Lagrange equation for time-fractional Klein-Gordon equation in the following representation

$$\begin{aligned}&\frac{1}{2} {}_t D_T^{\alpha} v_x - \frac{1}{2} {}_0D_t^{\alpha} v_x + \frac{\partial}{\partial x} \big( 2a v_x v_{xx} - 2b v_{xx} v_{xxx} \big) \\ & - \frac{\partial^2}{\partial x^2} \big( a v_x^2 - 2b v_x v_{xxx} - 6b v_{xx}^2 \big) \\ & - \frac{\partial^3}{\partial x^3} \big( 2b v_x v_{xx} \big) = 0.\end{aligned} \tag{13}$$

Substituting the Lagrangian of the time-fractional Klein-Gordon equation (10) into Euler-Lagrange formula (13) obtains

$$\frac{1}{2} {}_t D_T^{\alpha} v_x - \frac{1}{2} {}_0D_t^{\alpha} v_x + 6b v_{xxx}^2 + 6b v_{xx} v_{xxxx} = 0.$$

Once again, substituting the potential function $v_x$ for $u$, yields the time-fractional Klein-Gordon equation for the state function $u$ as

$$\frac{1}{2} {}_t D_T^{\alpha} u - \frac{1}{2} {}_0D_t^{\alpha} u + 6b u_{xx}^2 + 6b u_x u_{xxx} = 0. \tag{14}$$

According to the Riesz fractional derivative ${}_0^R D_t^{\alpha} u$, the time-fractional Klein-Gordon equation represented in (14) can write as

$${}_0^R D_t^{\alpha} u - 6b u_{xx}^2 - 6b u_x u_{xxx} = 0. \tag{15}$$

Acting from left-hand side by the Riesz fractional operator ${}_0^R D_t^{1-\alpha}$ on (15) leads to

$$\frac{\partial^2 u}{\partial t^2} - \sum_{j=1}^{2} {}_0^R D_t^{2-j} u|_{t=0} \frac{t^{\alpha-2-j}}{\Gamma(\alpha-1-j)} + {}_0^R D_t^{2-\alpha} \Big( -6b \big( \frac{\partial^2 u}{\partial x^2} \big)^2 - 6b \frac{\partial u}{\partial x} \frac{\partial^3 u}{\partial x^3} \Big) = 0, \tag{16}$$

from Lemma 7. In view of the variational iteration method, combining with (16), the $n+1$-th approximate solution of (15) can be read using iteration correction functional as

$$\begin{aligned}u_{n+1}(x,t) = u_n(x,t) - \int_0^t \lambda(\tau) \Big( & \frac{\partial^2}{\partial \tau^2} u_n(x,\tau) \\ & - \frac{\partial}{\partial \tau} u_n(x,\tau)|_{\tau=0} \frac{\tau^{\alpha-3}}{\Gamma(\alpha-2)} \\ & + u_n(x,0) \frac{\tau^{\alpha-4}}{\Gamma(\alpha-3)} \\ & - 6b {}_0^R D_t^{2-\alpha} \Big( \big( \frac{\partial^2}{\partial x^2} \tilde{u}_n(x,\tau) \big)^2 \\ & + \frac{\partial}{\partial x} \tilde{u}_n(x,\tau) \frac{\partial^3}{\partial x^3} \tilde{u}_n(x,\tau) \Big) \Big) d\tau,\end{aligned} \tag{17}$$

where the function $\tilde{u}_n$ is considered as a restricted variation function, i.e., $\delta\tilde{u}_n = 0$. The extreme of the variation of (17) subject to the restricted variation function straightforwardly yields

$$\begin{aligned}\delta u_{n+1} &= \delta u_n + \int_0^t \lambda(\tau)\delta\big(\frac{\partial^2}{\partial\tau^2}u_n(x,\tau)\big)d\tau \\ &= \delta u_n + \lambda(\tau)\delta\big(\frac{\partial}{\partial\tau}u_n(x,\tau)\big)|_{\tau=t} \\ &\quad - \frac{\partial}{\partial\tau}\lambda(\tau)\delta u_n(x,\tau)|_{\tau=t} \\ &\quad + \int_0^t \delta u_n(x,\tau)\frac{\partial^2}{\partial\tau^2}\lambda(\tau)d\tau \\ &= \big(1 - \frac{\partial}{\partial\tau}\lambda(\tau)\big)\delta u_n(x,\tau)|_{\tau=t} \\ &\quad + \lambda(\tau)\delta\big(\frac{\partial}{\partial\tau}u_n(x,\tau)\big)|_{\tau=t} \\ &\quad + \int_0^t \delta u_n(x,\tau)\frac{\partial^2}{\partial\tau^2}\lambda(\tau)d\tau = 0.\end{aligned}$$

This expression reduces the following stationary conditions

$$1 - \frac{\partial}{\partial\tau}\lambda(\tau)|_{\tau=t} = 0, \quad \lambda(\tau)|_{\tau=t} = 0,$$
$$\frac{\partial^2}{\partial\tau^2}\lambda(\tau) = 0,$$

which converted to the Lagrangian multiplier at

$$\lambda(\tau) = \tau - t.$$

Therefore, the correction functional (17) takes the following form

$$\begin{aligned}u_{n+1}(x,t) &= u_n(x,t) - \int_0^t (\tau - t)\Big(\frac{\partial^2}{\partial\tau^2}u_n(x,\tau) \\ &\quad - \frac{\partial}{\partial\tau}u_n(x,\tau)|_{\tau=0}\frac{\tau^{\alpha-3}}{\Gamma(\alpha-2)} \\ &\quad + u_n(x,0)\frac{\tau^{\alpha-4}}{\Gamma(\alpha-3)} \\ &\quad - 6b\,{}_0^R D_\tau^{2-\alpha}\Big(\big(\frac{\partial^2}{\partial x^2}u_n(x,s)\big)^2 \\ &\quad + \frac{\partial}{\partial x}u_n(x,s)\frac{\partial^3}{\partial x^3}u_n(x,s)\Big)\Big)d\tau,\end{aligned} \tag{18}$$

since $\alpha - 2 < 0$, the fractional derivative operator ${}_0^R D_t^{\alpha-2}$ reduces to fractional integral operator ${}_0^R I_t^{2-\alpha}$ by Remark 8.

In view of the right-hand side Riemann-Liouville fractional derivative is interpreted as a future state of the process in physics. For this reason, the right-derivative is usually neglected in applications, when the present state of the process does not depend on the results of the future development, and so the right-derivative is used equal to zero in the following calculations. The zero order solitary wave solution can be taken as the initial value of the state variable from (1), which is taken in this case as

$$\begin{aligned}u_0(x,t) &= \frac{4c^2}{3a}\cosh^2\big(\frac{1}{4}\sqrt{\frac{a}{b}}x + \xi\big) \\ &\quad - \frac{c^3 t}{3\sqrt{ab}}\sinh\big(\frac{1}{2}\sqrt{\frac{a}{b}}x + \xi\big),\end{aligned}$$

where $c$, $\xi$ are the given constants.

Substituting this zero order approximate solitary wave solution into (18) and using the Definition 6 leads to the first order approximate solitary wave solution of the problem (1)

$$\begin{aligned}u_1(x,t) &= \frac{4c^2}{3a}\Big(1 + \frac{t^{\alpha-2}}{\Gamma(\alpha-1)}\Big)\cosh^2\big(\frac{1}{4}\sqrt{\frac{a}{b}}x + \xi\big) \\ &\quad - \frac{c^3}{3\sqrt{ab}}\Big(t - \frac{t^{\alpha-1}}{\Gamma(\alpha)}\Big)\sinh\big(\frac{1}{2}\sqrt{\frac{a}{b}}x + \xi\big) \\ &\quad - \frac{ac^6 t^{\alpha+2}}{12b^2(\alpha+2)\Gamma(\alpha)}\cosh\big(\sqrt{\frac{a}{b}}x + \xi\big) \\ &\quad + \frac{c^5\sqrt{\frac{a}{b}}t^{\alpha+1}}{6b\Gamma(\alpha+2)}\sinh\big(\sqrt{\frac{a}{b}}x + \xi\big) \\ &\quad - \frac{c^4 t^\alpha}{6b\Gamma(\alpha+1)}\cosh\big(\sqrt{\frac{a}{b}}x + \xi\big).\end{aligned}$$

Substituting first order approximate solitary wave solution into (18), using the Definition 6 then leads to the second order approximate solitary wave solution of the problem (1) as follows

$$\begin{aligned}u_2(x,t) &= \frac{4c^2}{3a}\Big(1 + \frac{3t^{\alpha-2}}{\Gamma(\alpha-1)}\Big)\cosh^2\big(\frac{1}{4}\sqrt{\frac{a}{b}}x + \xi\big) \\ &\quad - \frac{c^3}{3\sqrt{ab}}\Big(t - \frac{2t^{\alpha-1}}{\Gamma(\alpha)}\Big)\sinh\big(\frac{1}{2}\sqrt{\frac{a}{b}}x + \xi\big) \\ &\quad - \frac{ac^6 t^{\alpha+2}}{6b^2(\alpha+2)\Gamma(\alpha)}\cosh\big(\sqrt{\frac{a}{b}}x + \xi\big) \\ &\quad + \frac{c^5\sqrt{\frac{a}{b}}t^{\alpha+1}}{3b\Gamma(\alpha+2)}\sinh\big(\sqrt{\frac{a}{b}}x + \xi\big) \\ &\quad - \frac{c^4 t^\alpha}{3b\Gamma(\alpha+1)}\cosh\big(\sqrt{\frac{a}{b}}x + \xi\big) \\ &\quad - \frac{c^4}{6b}\cosh\big(\sqrt{\frac{a}{b}}x + \xi\big)\Big(\frac{2t^{2\alpha-2}}{\Gamma(2\alpha-1)}\end{aligned}$$

$$
\begin{aligned}
&+ \frac{\Gamma(2\alpha-3)t^{3\alpha-4}}{\Gamma^2(\alpha-1)\Gamma(3\alpha-3)} + \frac{t^{\alpha}}{\Gamma(\alpha+1)}\Big) \\
&- \frac{ac^6}{24b^2}\cosh\big(\sqrt{\frac{a}{b}}x+\xi\big)\Big(\frac{2t^{\alpha+2}}{(\alpha+2)\Gamma(\alpha)} \\
&+ \frac{\Gamma(2\alpha-1)t^{3\alpha-2}}{\Gamma^2(\alpha)\Gamma(3\alpha-1)} - \frac{t^{2\alpha}}{\Gamma(2\alpha)}\Big) - \frac{a^4c^{12}}{24b^5} \\
&\times \cosh\big(2\sqrt{\frac{a}{b}}x+\xi\big)\frac{\Gamma(2\alpha+5)t^{3\alpha+4}}{(\alpha+2)^2\Gamma^2(\alpha)\Gamma(3\alpha+5)} \\
&- \frac{a^3c^{10}}{6b^4}\cosh\big(2\sqrt{\frac{a}{b}}x+\xi\big)\frac{\Gamma(2\alpha+1)t^{3\alpha+2}}{\Gamma^2(\alpha+1)\Gamma(3\alpha+3)} \\
&- \frac{a^2c^8}{6b^3}\cosh\big(2\sqrt{\frac{a}{b}}x+\xi\big)\frac{\Gamma(2\alpha+1)t^{3\alpha}}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)} \\
&+ \cdots .
\end{aligned}
$$

Making use of the Definition 6 and some calculation, we obtain $u_3$, $u_4$ and so on, substituting $n-1$ order approximate solitary wave solution into (18), there leads to the $n$ order approximate solitary wave solution. As $n$ tends to infinity, the iteration leads to the solitary wave solution of the time-fractional Klein-Gordon equation (1)

$$u(x,t) = \frac{4c^2}{3a}\cosh^2\big(\frac{1}{4}\sqrt{\frac{a}{b}}(x-ct)+\xi\big).$$

## 5 Discussion

The target of present work is to explore the effect of the fractional order derivative on the structure and propagation of the resulting solitary waves obtained from time-fractional Klein-Gordon equation. We derive the Lagrangian of the Klein-Gordon equation by the semi-inverse method, then take a similar form of Lagrangian to the time-fractional Klein-Gordon equation. Using the Euler-Lagrange variational technique, we continue our calculations until the three-order iteration. During this period, our approximate calculations are carried out concerning the solution of the time-fractional Klein-Gordon equation taking into account the values of the coefficients and some meaningful values namely, $\alpha = 1.98, 1.95, 1.90$ and $1.85$. The solitary wave solution of time-fractional Klein-Gordon equation is obtained. In addition, 3-dimensional representation of the solution $u$ for the time-fractional Klein-Gordon equation with space $x$ and time $t$ for different values of the order $\alpha$ is presented respectively in Figure 1, the solution $u$ is still a single soliton wave solution for all values of the order $\alpha$. It shows that the balancing scenario between nonlinearity and dispersion is still valid. Figure 2 presents the change of amplitude and width of the soliton due

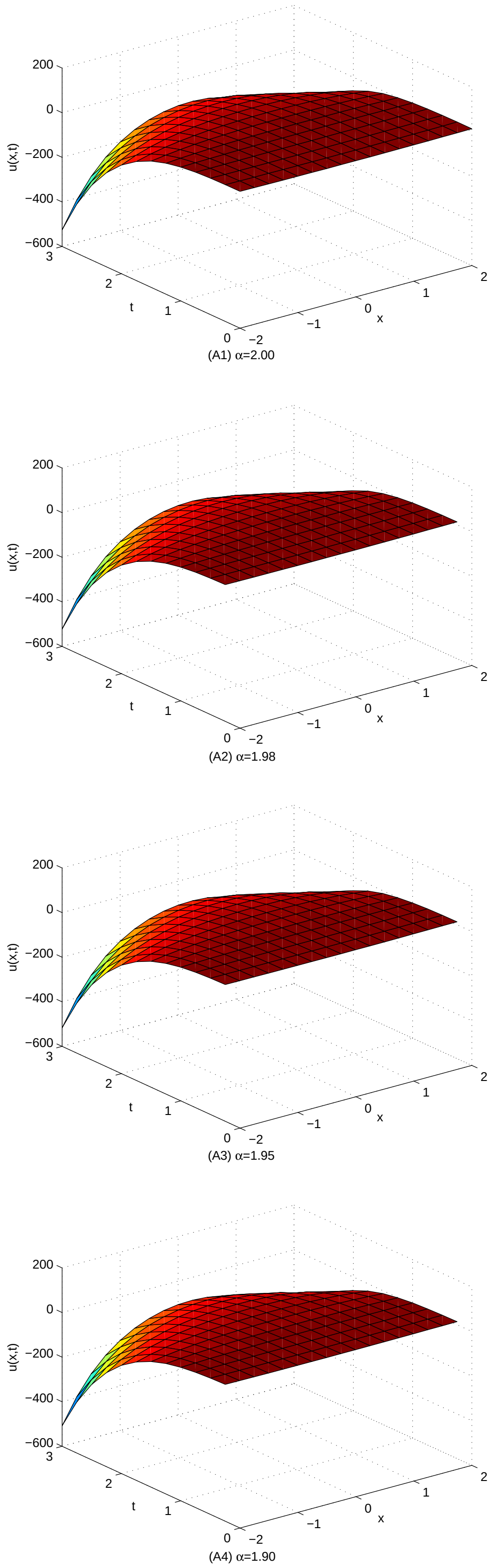


Figure 1 The surfaces of the approximate solutions $u(x,t)$

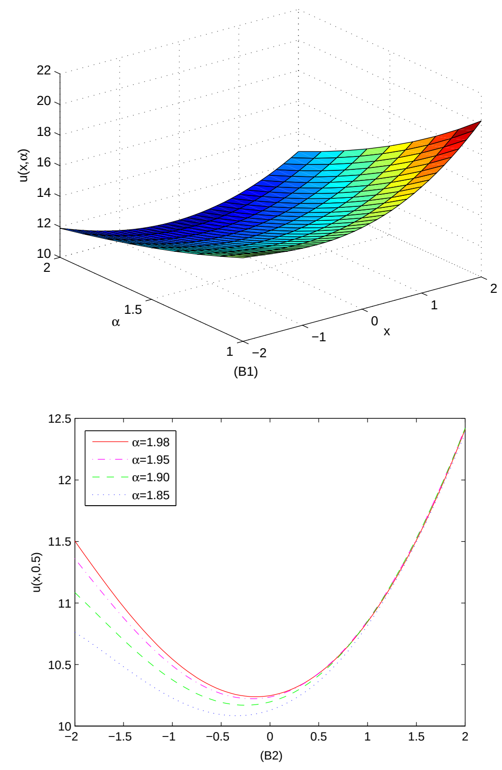


Figure 2 The function $u$ as a function of space $x$ at time $t = 0.5$ for order $\alpha$: (B1) 3-dimensions graph, (B2) 2-dimensions graph

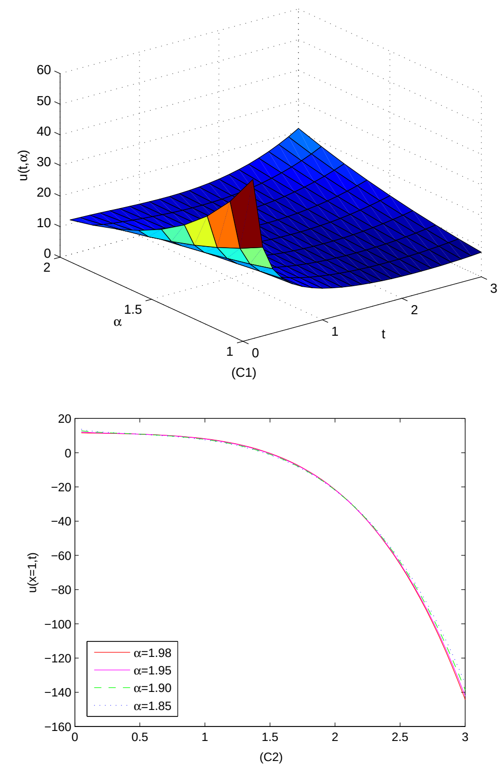


Figure 3 The amplitude of the function $u$ as a function of time $t$ at space $x = 1$ for order $\alpha$: (C1) 3-dimensions graph, (C2) 2-dimensions graph

to the variation of the order $\alpha$, 2- and 3-dimensional graphs depicted the behavior of the solution $u$ at time $t = 0.5$ corresponding to different values of the order $\alpha$. This behavior indicates that the order $\alpha$ can be used to modify the shape of the solitary wave without change of the nonlinearity and the dispersion effects in the medium. Figure 3 devoted to study the expression between the amplitude of the soliton and the fractional order at different time values. These figures show that at the same time, the change of the fractional $\alpha$ decreases the amplitude of the solitary wave.

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