Pricing forward starting options under regime switching jump diffusion models

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Abstract: This paper studies the pricing of forward starting options under regime switching jump diffusion models. We suppose that a market economy has only two states, one is a stable state, the other is a high volatility state. The dynamics of a risky asset is modeled by a geometry Brownian motion when the market state is stable, otherwise, it follows a jump diffusion model. We propose two types of regime switching jump diffusion models: one is a two-state regime switching Merton jump diffusion model, and the other is a two-state regime switching double exponential jump diffusion model. Finally, some analytic formulas for pricing forward starting options are derived under these regime switching jump diffusion models.

Key–Words: Jump diffusion; Option pricing; Regime switching

1 Introduction

Option pricing is an important research field in mathematical finance. The traditional Black-Scholes option pricing formula has been widely used for pricing options in finance industry. However, there are a large number of empirical results indicate that the asset price process follows a geometric Brownian motion is not realistic, since it cannot explain the two empirical phenomena: the asymmetric leptokurtic features and the "volatility smile" phenomenon. In order to overcome these disadvantages, many different models had been proposed, for example, stochastic volatility models, stochastic interest rate models, and regime-switching models et al. For example, Li et al.[1], Wang et al.[2], Chang et al.[3] and Duan et al.[4]. Moreover, many empirical results also show that the risky asset price processes have jumps since some rare events can lead to brusque variations in prices. Hence, many jump diffusion models had been presented to describe the dynamics of the risky assets. Studies include those of Merton [5], Kou [6, 7], Ahn et al. [8] and others.

Recently, regime switching models have received much attention by researchers(see, e.g. [9, 10, 11, 12, 13]). It is especially important to incorporate the regime switching effect since there could be substantial changes in the economic condition over a long period of time. Edwards [14] incorporated regime switching into the risky asset price process and presented a new financial model to describe the dynamics of the risky assets. He supposed that an economy had two states, a state was high volatility, and the other was stable. When the state of the market economy was high volatility, the risky assets price followed a Lévy process, otherwise, the dynamics of the risky assets satisfied a geometry Brownian motion.

Inspired by Edwards [14], we combine the jump diffusion model with Black-Scholes model and provide a two-state regime switching jump diffusion model to describe the dynamics of the risky assets. We suppose that a market economy has two states, a stable state and a high volatility state. The dynamics of a risky asset follows different stochastic processes in different states of the market economy: the risky asset price is driven by a geometric Brownian motion when the market is stable, while the risky asset price follows a jump diffusion process if the market is high volatility. The regime switching jump diffusion model describes two different types of jumps. The regime switching incorporate structural changes in asset price dynamics which are attributed to structural changes in economic conditions, and the jump diffusion describes jumps which are attributed to some financial news and lead to spikes in prices.

As an application, we investigate the pricing of forward starting options when the dynamics of the risky assets are described by a two-state regime switching jump diffusion process. Although the forward starting options are a class of quite simple exotic derivatives, their pricing can be demanding. Until recently, there is only a few researches on this subject. Kruse and Nögel [15] gave a pricing formula of forward starting call option in Heston's stochastic volatility model. Ahlip and Rutkowski [16] extended the model in [15] to a stochastic interest rate framework, and obtained the price of forward starting call option. Ramponi [17] employed the Fourier transform methods to pricing forward starting options under a regime switching jump diffusion model. The major differences between their paper and this one are that: first, the model considered here is different from that in [17]. Second, we employ a method which is different from that in [17], and provides the analytic formulas for pricing forward starting options. This paper consists of two parts. The first part supposes that the dynamics of the risky asset is described by a two-state regime switching Merton jump diffusion process, and provides the analytic pricing formula of forward starting options. The other part supposes that the risky asset is modeled by a two-state regime switching double exponential jump diffusion process, and the analytic pricing formulas of forward starting options are also obtained.

This paper is organized as follows: a two-state regime switching jump diffusion model is introduced in Section 2. Section 3 presents the pricing formulas of forward starting options under a two-state regime switching Merton jump diffusion model. Section 4 provides the pricing formulas of forward starting options under a two-state regime switching double exponential jump diffusion model. The final section gives a conclusion.

2 A two-state regime switching jump diffusion model

We consider a finite time horizon $[0,T]$ and a probability space $(\Omega, \mathcal{F}, P)$, where P is a real world measure. Uncertainty is represented by a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, P)$. The states of the market economy are modeled by a stationary continuous time Markov chain process $\varepsilon = \{\varepsilon_t\}_{t\in[0,T]}$. To simplify our financial model, we suppose that ε has only two states $\{e_1, e_2\}$, where state $e_1 = (1,0)' \in \mathcal{R}^2$ and $e_2 = (0,1)' \in \mathcal{R}^2$ of the chain can be explained to represent a "high volatility " state and a "stable" state, respectively. Assume that the Markov chain process ε has a generator $A = (a_{ij})_{i,j=1,2}$ with stationary transition probability given by the following

$$\begin{aligned}\overline{P}_{ij}(\Delta t) &= a_{ij}\Delta t + o(\Delta t), \;\; i \neq j, \\ \overline{P}_{ii}(\Delta t) &= 1 - a_{ii}\Delta t + o(\Delta t),\end{aligned}$$

where

$$\overline{P}_{ij}(\Delta t) = P\left(\varepsilon_{t+\Delta t} = e_j | \varepsilon_t = e_i\right). \tag{1}$$

We consider a financial market with two traded assets, a riskless bond and a stock. The riskless bond price process $B = (B_t)$ satisfy the following

$$dB_t = rB_t dt, \tag{2}$$

where r is the instantaneous interest rate. For sake of convenience, we assume $B_0 = 1$.

In this paper, we adopt a two-state regime switching jump diffusion model to describe the dynamics of a stock. We assume that the stock price follows a jump diffusion process when the state of the market economy is e_1, whereas the stock price follows the Black-Scholes model when the state of market economy is e_2. Let J_t denote the occupation time of ε in state e_1 over the time period $[0,t]$, then

$$J_t = \int_0^t I_{\{\varepsilon_s = e_1\}} ds, \tag{3}$$

where $I_{\{\cdot\}}$ is a indictor function. The stock price process $S = (S_t)$ according to

$$S_t = S_0 \exp\left\{\mu t + \sigma \widetilde{W}_t - \frac{1}{2}\sigma^2 t + \sum_{j=1}^{N_{J_t}} Z_j - \lambda\beta J_t\right\}, \tag{4}$$

where N_t is a Poisson process with intensity λ, $\widetilde{W}$ is a standard P Brownian motion, $\mu > 0$ and $\sigma > 0$ are the appreciation rate and the volatility of the stock S, respectively, and $\{Z_j\}_{j=1,2,\ldots}$ are a sequence of independent and identically distribution random variables with probability density function $f(z)$. The term $\lambda\beta J_t$ is included explicitly in (4) to compensate for the presence of the jumps in share price, hence it is chosen such that $\beta = \mathrm{E}_P\left[e^{Z_j} - 1\right]$. Moreover, we assume that $\widetilde{W}, \varepsilon, N$, and $Z = (Z_j)_{j=1,2,\ldots}$ are mutually independent.

3 Risk neutral martingale measures

In this paper, we consider a financial market in which only a stock and a riskless bond can be traded. The share price of stock is described by a regime switching jump diffusion model, with more stochastic risk factors than tradable assets. Hence, this financial market is incomplete, and the risk neutral martingale measure is not unique. For pricing forward starting options, we need choose a risk neutral martingale measure from the set of risk neutral martingale measures. Let $(\mathcal{F}_t^\varepsilon)_{0\le t\le T}$ denote the P augmentations of the natural filtration generated by Markov-chain ε and $\mathcal{H}_t$ denote the σ-algebra $\mathcal{F}_T^\varepsilon \vee \mathcal{F}_t$. Now, we first define a new probability measure by the following

$$\frac{dQ}{dP}\Big|_{\mathcal{H}_t} = \Lambda_t,$$

and Λ_t is given by

$$\begin{aligned}\Lambda_t &= \exp\Big\{\theta \widetilde{W}_t - \frac{1}{2}\theta^2 t + \vartheta \sum_{j=1}^{N_{J_t}} Z_j \\ &- \lambda J_t \mathrm{E}_P\left[e^{\vartheta Z_j} - 1\right]\Big\}, \end{aligned} \tag{5}$$

where θ, ϑ are explained as the market prices of Brownian motion risk and jump risk.

Under an equivalent martingale measure, the discounted stock price process $\widetilde{S}_t = e^{-rt}S_t$ is a martingale, this is called the martingale condition. As in [11], due to the presence of the uncertainty generated by the Markov-chain process ε, the martingale condition is defined with respect to the enlarged filtration $\mathcal{H}$.

Proposition 1 *The martingale condition holds if θ and ϑ satisfy,*

$$\theta = \frac{r-\mu}{\sigma}, \tag{6}$$

$$\mathrm{E}_P\left[e^{(\vartheta+1)Z_j}\right] = \beta + \mathrm{E}_P\left[e^{\vartheta Z_j}\right]. \tag{7}$$

Proof. For $s < t$, using the Bayes formula, we have

$$\begin{aligned}\mathrm{E}_Q\left[\widetilde{S}_t\Big|\mathcal{H}_s\right] &= \frac{\mathrm{E}_P\left[\widetilde{S}_t\Lambda_t\Big|\mathcal{H}_s\right]}{\mathrm{E}_P\left[\Lambda_t\Big|\mathcal{H}_s\right]} \\ &= \mathrm{E}_P\left[\frac{\widetilde{S}_t\Lambda_t}{\Lambda_s}\Big|\mathcal{H}_s\right] \\ &= \widetilde{S}_s\mathrm{E}_P\left[\frac{\widetilde{S}_t\Lambda_t}{\widetilde{S}_s\Lambda_s}\Big|\mathcal{H}_s\right].\end{aligned}$$

By (4) and (5)

$$\begin{aligned}\frac{\widetilde{S}_t\Lambda_t}{\widetilde{S}_s\Lambda_s} &= \exp\Big\{\left[\mu - r - \frac{1}{2}(\sigma^2+\theta^2)\right](t-s) \\ &+ (\theta+\sigma)(\widetilde{W}_t - \widetilde{W}_s) + \sum_{j=N_{J_s}+1}^{N_{J_t}} (\vartheta+1)Z_j \\ &- \lambda(J_t - J_s)\left(\beta + \mathrm{E}_P\left[e^{\vartheta Z_j} - 1\right]\right)\Big\},\end{aligned}$$

then

$$\begin{aligned}\mathrm{E}_P\left[\frac{\widetilde{S}_t\Lambda_t}{\widetilde{S}_s\Lambda_s}\Big|\mathcal{H}_s\right] &= \exp\Big\{(\mu - r + \theta\sigma)(t-s) \\ &+ \lambda(J_t - J_s)\Big(\mathrm{E}_P\left[e^{(\vartheta+1)Z_j} - 1\right] \\ &- \left(\beta + \mathrm{E}_P\left[e^{\vartheta Z_j} - 1\right]\right)\Big)\Big\}.\end{aligned}$$

Thus, if θ and ϑ satisfy (6) and (7), the martingale condition holds. □

For simplicity, we adopt the assumption of Merton [5] that the jumps are diversifiable in this paper, that is $\vartheta = 0$. Hence, under the risk neutral martingale measure Q, the dynamics of stock S is given by

$$S_t = S_0 \exp\left\{rt + \sigma W_t - \frac{1}{2}\sigma^2 t + \sum_{j=1}^{N_{J_t}} Z_j - \lambda\beta J_t\right\}, \tag{8}$$

where $W_t = \widetilde{W}_t - \frac{r-\mu}{\sigma}t$ is a Q Brownian motion. In addition, we also note that W, ε, N, and $Z = (Z_j)_{j=1,2,\ldots}$ are mutually independent from measure P to measure Q.

4 Pricing the forward starting options

4.1 Forward starting options

A forward starting call option is a simple exotic option. The strike of a forward starting option is relation to the underlying asset, not as the same as the European call option. The payoff of a forward starting call option is given by

$$\Psi^{call}(S_T) = (S_T - KS_{t^*})^+, \tag{9}$$

where $K \in (0,1)$ and $t^* < T$ are constants. In this paper, we only consider the case of $t < t^*$. If $t^* \le t$, a forward starting call option become a European call option with strike KS_{t^*}.

Let $\Psi^{put}(S_T)$ denote the payoff of a forward starting put option, it is given by

$$\Psi^{put}(S_T) = (KS_{t^*} - S_T)^+. \tag{10}$$

4.2 A two-state regime switching Merton jump diffusion model

In this subsection, we suppose that stock price S satisfies a two-state regime switching Merton jump diffusion process. The amplitude Z_j of log jump of S satisfies the normal distribution, and we assume that mean and variance are 0 and γ^2, respectively, that is $f(z) = \frac{1}{\sqrt{2\pi}\gamma}e^{-\frac{z^2}{2\gamma^2}}$, $-\infty < z < \infty$. Moreover, $\beta = e^{\frac{\gamma^2}{2}} - 1$ in (8) since $\beta = \mathrm{E}_P[e^{Z_i} - 1]$.

We start by defining a new probability measure, which will be used latter to pricing forward starting options.

$$\begin{aligned}\eta_t &= \frac{dQ_S}{dQ}\Big|_{\mathcal{H}_t} = \frac{\widetilde{S}_t}{\mathrm{E}_Q[\widetilde{S}_t|\mathcal{F}_T^\varepsilon]} \\ &= \exp\left\{\sigma W_t - \frac{1}{2}\sigma^2 t + \sum_{j=1}^{N_{J_t}} Z_j - \lambda J_t \beta\right\}.\end{aligned}$$

Proposition 2 *Under the probability measure Q_S and conditional on $\mathcal{F}_T^\varepsilon$,*

$$W_t^S = W_t - \sigma t$$

is a Brownian motion, the intensity λ^* of Poisson process N and the probability density function $f^*(z)$ of Z_j are given by

$$\lambda^* = \lambda(\beta + 1)$$

and

$$f^*(z) = \frac{e^z f(z)}{\beta + 1}.$$

Proof: Using the Bayes formula, we have

$$\begin{aligned}&\mathrm{E}_{Q_S}\left[e^{iu\sum_{j=1}^{N_{J_T}} Z_j}\Big|\mathcal{F}_T^\varepsilon\right] \\ =\ &\mathrm{E}_Q\left[e^{(iu+1)\sum_{j=1}^{N_{J_T}} Z_j - \lambda J_T\beta + \sigma W_T - \frac{1}{2}\sigma^2 T}\Big|\mathcal{F}_T^\varepsilon\right] \\ =\ &e^{\lambda J_T\left\{\mathrm{E}_Q[e^{(iu+1)Z_j}] - 1 - \beta\right\}} \\ =\ &e^{\lambda^* J_T\left\{\int_{-\infty}^{\infty}(e^{iuz}-1)f^*(z)dz\right\}}. \qquad (11)\end{aligned}$$

The second equality holds since W, ε, N, and $Z = (Z_j)_{j=1,2,\ldots}$ are mutually independent. By (11), we find that, under the probability measure Q_S and conditional on $\mathcal{F}_T^\varepsilon$, the intensity of N is $\lambda^* = \lambda(\beta + 1)$, and the probability density function of Z_j is $f^*(z) = \frac{e^z f(z)}{\beta+1}$. In addition, by the Girsanov theorem, we have that $W_t^S = W_t - \sigma t$ is a Q_S Brownian motion. □

From Proposition 2, we can obtain the following result.

Remark 3 *If the stock price S satisfies a two-state regime switching Merton jump diffusion process, under the probability measure Q_S, the Poisson process N has intensity $\tilde{\lambda} = \lambda e^{\frac{\gamma^2}{2}}$, and $Z = (Z_j)_{j=1,2,\ldots}$ have probability density $\tilde{f}(z) = \frac{1}{\sqrt{2\pi}}e^{-\frac{(z-\gamma^2)^2}{2\gamma^2}}$, $z \in (-\infty, \infty)$.*

Let $\phi_{J_T - J_t|\varepsilon_t = e_i}(y)$ denote the condition probability density of $J_T - J_t$ under the measure P and given $\varepsilon_t = e_i$, Yoon et al.[18] provided the analytic formula of $\phi_{J_T - J_t|\varepsilon = e_i}(y)$, that is, for $\forall y(0 \le y \le T - t)$

$$\begin{cases}\phi_{J_T-J_t|\varepsilon_t=e_1}(y) = \exp\left[-a_{22}(T-t-y) - a_{11}y\right] \\ \times\left\{\left(\frac{a_{11}a_{22}y}{T-t-y}\right)^{\frac{1}{2}} I_1\left(2\left(a_{11}a_{22}y(T-t-y)\right)^{\frac{1}{2}}\right)\right. \\ \left. + a_{11}I_0\left(2\left(a_{11}a_{22}y(T-t-y)\right)^{\frac{1}{2}}\right)\right\}, \\ \phi_{J_T-J_t|\varepsilon_t=e_2}(y) = \exp\left[-a_{22}(T-t-y) - a_{11}y\right] \\ \times\left\{\left(\frac{a_{11}a_{22}(T-t-y)}{y}\right)^{\frac{1}{2}} I_1\left(2\left(a_{11}a_{22}y(T-t-y)\right)^{\frac{1}{2}}\right)\right. \\ \left. + a_{22}I_0\left(2\left(a_{11}a_{22}y(T-t-y)\right)^{\frac{1}{2}}\right)\right\},\end{cases}$$

and $\phi_{J_T-J_t|\varepsilon_t=e_1}(0) = 0$, $\phi_{J_T-J_t|\varepsilon_t=e_1}(T-t) = e^{-a_{11}(T-t)}$, $\phi_{J_T-J_t|X=e_2}(T-t) = e^{-a_{22}(T-t)}$, $\phi_{J_T-J_t|\varepsilon_t=e_2}(T-t) = 0$, where $I_b(x) := (\frac{x}{2})^b\sum_{n=0}^{\infty}\frac{(\frac{x}{2})^{2n}}{n!\Gamma(b+n+1)}$.

Given $\varepsilon_t = e_i$, let $C(t, e_i)$ denote the conditional price of forward starting call option at time t, it is given as the following theorem.

Theorem 4 *For $0 \le t < t^* < T$ and $\forall i = 1, 2$, the value of forward starting call option at time t and given $\varepsilon_t = e_i$ is*

$$\begin{aligned}C(t, e_i) = \sum_{j=1}^{2}\int_0^{T-t^*}&\mathcal{J}(t, y) \\ \times\ &\overline{P}_{ij}(t - t^*)\phi_{J_T - J_{t^*}|\varepsilon_{t^*} = e_j}(y)dy,\end{aligned}$$

where

$$\begin{aligned}\mathcal{J}(t, y) &= S_t\sum_{n=0}^{\infty}\Big(\pi_n(\tilde{\lambda}, y)\mathcal{N}(d_1(n, y)) \\ &- Ke^{-r(T-t^*)}\pi_n(\lambda, y)\mathcal{N}(d_2(n, y))\Big)\end{aligned}$$

$$\begin{aligned}
\pi_n(\lambda, y) &= e^{-\lambda y}\frac{(\lambda y)^n}{n!},\\
d_1(n,y) &= \frac{(r+\frac{1}{2}\sigma^2)(T-t^*)-\lambda\beta y}{\sqrt{\sigma^2(T-t^*)+n\gamma^2}}\\
&+ \frac{n\gamma^2-\ln K}{\sqrt{\sigma^2(T-t^*)+n\gamma^2}},\\
d_2(n,y) &= d_1(n,y)-\frac{\sigma^2(T-t^*)}{\sqrt{\sigma^2(T-t^*)+n\gamma^2}},\\
\overline{P}_{11}(t) &= \frac{a_{21}}{a_{21}-a_{11}}+\frac{a_{11}}{a_{11}-a_{21}}e^{-(a_{21}-a_{11})t},\\
\overline{P}_{22}(t) &= \frac{a_{11}}{a_{11}-a_{21}}+\frac{a_{21}}{a_{21}-a_{11}}e^{-(a_{21}-a_{11})t},\\
\overline{P}_{12}(t) &= 1-\overline{P}_{11}(t),\\
\overline{P}_{21}(t) &= 1-\overline{P}_{22}(t),
\end{aligned}$$

$\mathcal{N}(\cdot)$ denotes the cumulative normal distribution function.

Proof: For simplicity of notation, let $\mathrm{E}_Q^{t,S,i}[\cdot]$ denote the conditional expectation under the risk neutral martingale measure Q, given $S_t = S$ and $\varepsilon_t = e_i$.

Based on the risk neutral pricing theorem, we have

$$C(t,e_i) = \mathrm{E}_Q^{t,S,i}\left[e^{-r(T-t)}\left(S_T - KS_{t^*}\right)^+\right]. \quad (12)$$

Define

$$\mathcal{J}(t, J_T - J_{t^*}) = \mathrm{E}_Q\left[e^{-r(T-t)}\left(S_T - KS_{t^*}\right)^+ |\mathcal{H}_t\right].$$

Then, using the law of iterated expectations, we get

$$C(t,e_i) = \mathrm{E}_Q^{t,S,i}\left[\mathcal{J}(t, J_T - J_{t^*})\right]. \quad (13)$$

We first derive $\mathcal{J}(t, J_T - J_{t^*})$, by Bayes formula,

$$\begin{aligned}
&\mathcal{J}(t, J_T - J_{t^*})\\
=\;& S_t\mathrm{E}_{Q_S}\left[I_{\left\{\frac{S_T}{S_{t^*}}\ge K\right\}}\Big|\mathcal{H}_t\right]\\
-\;& Ke^{-r(T-t)}\mathrm{E}_Q\left[S_{t^*}I_{\left\{\frac{S_T}{S_{t^*}}\ge K\right\}}\Big|\mathcal{H}_t\right]\\
=\;& S_t\mathrm{E}_{Q_S}\left[I_{\left\{\frac{S_T}{S_{t^*}}\ge K\right\}}\Big|\mathcal{H}_t\right]\\
-\;& Ke^{-r(T-t)}S_t\mathrm{E}_Q\left[\frac{S_{t^*}}{S_t}I_{\left\{\frac{S_T}{S_{t^*}}\ge K\right\}}\Big|\mathcal{H}_t\right]\\
=\;& S_t\mathrm{E}_{Q_S}\left[I_{\left\{\frac{S_T}{S_{t^*}}\ge K\right\}}\Big|\mathcal{H}_t\right]\\
-\;& Ke^{-r(T-t^*)}S_t\mathrm{E}_Q\left[I_{\left\{\frac{S_T}{S_{t^*}}\ge K\right\}}\Big|\mathcal{H}_t\right].
\end{aligned}$$

Combing the Proposition 2 with (8), the above equation can rewrite as

$$\begin{aligned}
&\mathcal{J}(t, J_T - J_{t^*})\\
=\;& S_tQ_S\Bigg((r+\frac{1}{2}\sigma^2)(T-t^*)-\lambda\beta(J_T-J_{t^*})\\
+\;& \sigma(W_T^S - W_{t^*}^S)+\sum_{j=N_{J_{t^*}}+1}^{N_{J_T}} Z_j \ge \ln K\Big|\mathcal{H}_t\Bigg)\\
-\;& KS_te^{-r(T-t^*)}Q\Bigg((r-\frac{1}{2}\sigma^2)(T-t^*)\\
-\;& \lambda\beta(J_T-J_{t^*})+\sigma(W_T-W_{t^*})\\
+\;& \sum_{j=N_{J_{t^*}}+1}^{N_{J_T}} Z_j \ge \ln K\Big|\mathcal{H}_t\Bigg)\\
=\;& \Pi_1 - \Pi_2. \qquad (14)
\end{aligned}$$

We obtain from Remark 3 that

$$\begin{aligned}
\Pi_1 \;=\;& S_t\sum_{n=0}^{\infty}\pi_n(\tilde{\lambda}, J_T - J_{t^*})\\
\times\;& Q_S\Bigg((r+\frac{1}{2}\sigma^2)(T-t^*)-\lambda\beta(J_T\\
-\;& J_{t^*})+\sigma(W_T^S-W_{t^*}^S)+\sum_{j=1}^{n}Z_j \ge \ln K\Big|\mathcal{H}_t\Bigg).
\end{aligned}$$

Noting that for $j = 1, 2, ...,$ Z_j satisfies the normal distribution and independent of W^S, we have

$$\Pi_1 = S_t\sum_{n=0}^{\infty}\pi_n(\tilde{\lambda}, J_T - J_{t^*})\mathcal{N}(d_1(n, J_T - J_{t^*})). \quad (15)$$

Furthermore, employing the same method, we can obtain

$$\begin{aligned}
\Pi_2 \;=\;& Ke^{-r(T-t^*)}S_t\sum_{n=0}^{\infty}\pi_n(\lambda, J_T - J_{t^*})\\
\times\;& \mathcal{N}(d_2(n, J_T - J_{t^*})). \qquad (16)
\end{aligned}$$

$\phi_{J_T - J_{t^*}|\varepsilon_t = e_i}(y)$ denotes the condition probability density of $J_T - J_{t^*}$ under the measure P and conditional on $\varepsilon_t = e_i$. However, the transition probability of the Markov chain ε is not altered from the real word probability measure P to the risk neutral martingale measure Q since Brownian motion $\widetilde{W}$ and Markov chain ε are assumed to be independent. Then by (13), we have

$$C(t,e_i) = \int_0^{T-t^*}\mathcal{J}(t,y)\phi_{J_T - J_{t^*}|\varepsilon_t = e_i}(y)dy.$$

In addition, since

$$\begin{aligned}\phi_{J_T-J_{t^*}|\varepsilon_t=e_i}(y) &= \sum_{j=1}^{2}\phi_{J_T-J_{t^*}|\varepsilon_{t^*}=e_j}(y)\\ &\times P(\varepsilon_{t^*}=e_j|\varepsilon_t=e_i),\end{aligned}$$

and by the homogeneous property of Markov chain ε, we have

$$\begin{aligned}C(t,e_i) &= \sum_{j=1}^{2}\int_0^{T-t^*}\mathcal{J}(t,y)\overline{P}_{ij}(t-t^*)\\ &\times \phi_{J_T-J_{t^*}|\varepsilon_{t^*}=e_j}(y)dy. \qquad (17)\end{aligned}$$

In what follows, we first calculate $\overline{P}_{ij}(t-t^*)$. By the Kolomogrov forward equation, we get

$$\begin{aligned}\frac{d\overline{P}_{11}(t)}{dt} &= \overline{P}_{11}(t)a_{11}+\overline{P}_{12}(t)a_{21}\\ &= \overline{P}_{11}(t)a_{11}+a_{21}-\overline{P}_{11}(t)a_{21}.\end{aligned}$$

Hence

$$\begin{aligned}&de^{(a_{21}-a_{11})t}\overline{P}_{11}(t)=(a_{21}-a_{11})e^{(a_{21}-a_{11})t}\\ \times\ &\overline{P}_{11}(t)dt+e^{(a_{21}-a_{11})t}d\overline{P}_{11}(t)\\ =\ &a_{21}e^{(a_{21}-a_{11})t}dt,\end{aligned}$$

then

$$\overline{P}_{11}(t)=\frac{a_{21}}{a_{21}-a_{11}}+\frac{a_{11}}{a_{11}-a_{21}}e^{-(a_{21}-a_{11})t}.$$

By the same way, we have

$$\overline{P}_{22}(t)=\frac{a_{11}}{a_{11}-a_{21}}+\frac{a_{21}}{a_{21}-a_{11}}e^{-(a_{21}-a_{11})t}.$$

Specifically, $\overline{P}_{12}(t)=1-\overline{P}_{11}(t)$, $\overline{P}_{21}(t)=1-\overline{P}_{22}(t)$. Therefore, we complete the proof of Theorem 4. □

Let $P(t,e_i)$ denote the value of forward starting put option at time t and given $\varepsilon_t=e_i$. The following Proposition 5 provides the value of forward starting put option.

Proposition 5 *The value of forward starting put option at time t under a two-state regime switching Merton jump diffusion model is given by*

$$P(t,e_i)=C(t,e_i)+S_t\left(Ke^{-r(T-t^*)}-1\right).$$

Proof: From (9) and (10), we can obtain

$$\Psi^{put}(S_T)-\Psi^{call}(S_T)=KS_{t^*}-S_T,$$

then

$$\begin{aligned}&P(t,e_i)-C(t,e_i)\\ =\ &\mathrm{E}_Q\left[e^{-r(T-t)}\left(\Psi^{put}(S_T)-\Psi^{call}(S_T)\right)\Big|\mathcal{F}_t\right]\\ =\ &\mathrm{E}_Q\left[e^{-r(T-t)}\left(KS_{t^*}-S_T\right)\Big|\mathcal{F}_t\right]. \qquad (18)\end{aligned}$$

Furthermore, since Q is a risk neutral martingale measure, the discount price process $e^{-rt}S_t$ is a Q martingale with respect to $\mathcal{F}_t$. Thus

$$\mathrm{E}_Q\left[e^{-r(T-t)}KS_{t^*}\Big|\mathcal{F}_t\right]=Ke^{-r(T-t^*)}S_t, \qquad (19)$$

and

$$\mathrm{E}_Q\left[e^{-r(T-t)}S_T\Big|\mathcal{F}_t\right]=S_t. \qquad (20)$$

By (18), (19) and (20), we obtain the result. □

Remark 6 *If the stock price S follows the Black-Scholes model, the value of forward starting call option is*

$$S_t\mathcal{N}(\bar{d}_1)-Ke^{-r(T-t^*)}\mathcal{N}(\bar{d}_2),$$

where

$$\begin{aligned}\bar{d}_1 &= \frac{(r+\frac{1}{2}\sigma^2)(T-t^*)-\ln K}{\sigma\sqrt{T-t^*}},\\ \bar{d}_2 &= \bar{d}_1-\sigma\sqrt{T-t^*}.\end{aligned}$$

Remark 7 *If the stock price S follows the Merton jump diffusion model without regime switching, the value of forward starting call option is*

$$\begin{aligned}&S_t\sum_{n=0}^{\infty}\Big(\pi_n(\tilde{\lambda},T-t^*)\mathcal{N}(d_1(n,T-t^*))\\ -\ &Ke^{-r(T-t^*)}\pi_n(\lambda,T-t^*)\mathcal{N}(d_2(n,T-t^*))\Big).\end{aligned}$$

4.3 A two-state regime switching double exponential jump diffusion model

In this subsection, we suppose that the amplitude Z_j of log jump of the stock price S satisfies a double exponential distribution with the probability density function $f(z)$, which is given by following

$$f(z)=p\eta_1e^{-\eta_1z}I_{\{z\geq 0\}}+q\eta_2e^{\eta_2z}I_{\{z<0\}}, \qquad (21)$$

where $p,q\geq 0, p+q=1, \eta_1>1, \eta_2>0$, p,q denote the probability of upward and downward, respectively. That is

$$Z_j\overset{d}{=}\left\{\begin{array}{ll}\xi^+, & \text{with probability} \quad p\\ \xi^-, & \text{with probability} \quad q\end{array}\right\},$$

where ξ^+ and ξ^- are random variables which follow exponential distribution with mean $\frac{1}{\eta_1}$ and $\frac{1}{\eta_2}$, respectively. The term $\lambda\beta J_t$ is included explicitly in (4) to compensate for the presence of the jumps in share price, hence it is chosen such that $\beta = \mathrm{E}_P\left[e^{Z_j}-1\right]$, then $\beta = \frac{p\eta_1}{\eta_1-1}+\frac{q\eta_2}{\eta_2+1}-1$.

In what follows, we provide two lemmas which are Proposition B.1 and B.3 in Kou [6] for proving Theorem 12.

Lemma 8 *For $\forall\ n \geq 1$, we can obtain the following decomposition*

$$\sum_{i=1}^{n} Z_i \stackrel{d}{=} \left\{ \begin{array}{c} \sum\limits_{i=1}^{m}\xi_i^+, \quad \text{with probability} P_{n,m}, \\ m=1,2,...n \\ -\sum\limits_{i=1}^{m}\xi_i^-, \quad \text{with probability} Q_{n,m}, \\ m=1,2,...n \end{array} \right\},$$

where $P_{n,m}$ and $Q_{n,m}$ are given as following

$$\begin{aligned} P_{n,m} &= \sum_{i=m}^{n-1}\begin{pmatrix} n-m-1 \\ i-m \end{pmatrix}\begin{pmatrix} n \\ i \end{pmatrix}\left(\frac{\eta_1}{\eta_1+\eta_2}\right)^{i-m} \\ &\times \left(\frac{\eta_2}{\eta_1+\eta_2}\right)^{n-i} p^i q^{n-i}, 1 \leq m \leq n-1, \\ Q_{n,m} &= \sum_{i=m}^{n-1}\begin{pmatrix} n-m-1 \\ i-m \end{pmatrix}\begin{pmatrix} n \\ i \end{pmatrix}\left(\frac{\eta_1}{\eta_1+\eta_2}\right)^{n-i} \\ &\times \left(\frac{\eta_2}{\eta_1+\eta_2}\right)^{i-m} p^{n-i} q^i, 1 \leq m \leq n-1, \end{aligned}$$

$P_{n,n} = p^n$, $Q_{n,n} = q^n$ *and* $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ *is defined as* 1, $\left\{\xi_i^+, i=1,2,...\right\}$ *and* $\left\{\xi_i^-, i=1,2,...\right\}$ *are two independent and identically distribution random variables sequence, and satisfy the exponential distributions with parameters η_1 and η_2, respectively.*

Definition 9 ***Hh*** *function is defined by*

$$\begin{aligned} Hh_m(x) &= \int_x^{\infty} Hh_{m-1}(y)dy \\ &= \frac{1}{m!}\int_x^{\infty}(t-x)^m e^{-\frac{t^2}{2}}dt, \end{aligned}$$

where $m = 0,1,2,...$, $Hh_{-1}(x) = e^{-\frac{x^2}{2}} = \sqrt{2\pi}\varphi(x)$ and $Hh_0(x) = \sqrt{2\pi}\int_{-\infty}^{-x}\varphi(y)dy$.

Lemma 10 *Assume that $\{\xi_1, \xi_2, ...\}$ is a sequence of independent and identically distribution random variables, and satisfy the exponential distributions with parameter $\eta > 0$, random variable Z satisfies the normal distribution with mean 0 and variance σ^2, then for $\forall\ n \geq 1$, we have*

1. The probability density functions of $Z + \sum\limits_{i=1}^{n}\xi_i$ and $Z - \sum\limits_{i=1}^{n}\xi_i$ are given by

$$\begin{aligned} f_{Z+\sum\limits_{i=1}^{n}\xi_i}(t) &= (\sigma\eta)^n \frac{e^{\frac{(\sigma\eta)^2}{2}}}{\sigma\sqrt{2\pi}} \\ &\times e^{-t\eta} Hh_{n-1}\left(-\frac{t}{\sigma}+\sigma\eta\right), \\ f_{Z-\sum\limits_{i=1}^{n}\xi_i}(t) &= (\sigma\eta)^n \frac{e^{\frac{(\sigma\eta)^2}{2}}}{\sigma\sqrt{2\pi}} \\ &\times e^{t\eta} Hh_{n-1}\left(\frac{t}{\sigma}+\sigma\eta\right). \end{aligned}$$

2. The tail probabilities of $Z + \sum\limits_{i=1}^{n}\xi_i$ and $Z - \sum\limits_{i=1}^{n}\xi_i$ are given by

$$\begin{aligned} P(Z+\sum_{i=1}^{n}\xi_i \geq x) &= (\sigma\eta)^n \frac{e^{\frac{(\sigma\eta)^2}{2}}}{\sigma\sqrt{2\pi}} \\ &\times I_{n-1}\left(x; -\eta, -\frac{1}{\sigma}, -\sigma\eta\right), \\ P(Z-\sum_{i=1}^{n}\xi_i \geq x) &= (\sigma\eta)^n \frac{e^{\frac{(\sigma\eta)^2}{2}}}{\sigma\sqrt{2\pi}} \\ &\times I_{n-1}\left(x; \eta, \frac{1}{\sigma}, -\sigma\eta\right), \end{aligned}$$

where

$$I_m(c;\alpha,\beta,\delta) := \int_c^{\infty} e^{\alpha x} Hh_m(\beta x - \delta)dx, \quad m \geq 0,$$

and c, α, β, δ are constants.

From Proposition 2, it is easily obtain the following proposition.

Proposition 11 *Under the probability measure Q_S and conditional on $\mathcal{F}_T^{\varepsilon}$, the intensity of Poisson process N is $\hat{\lambda} = \lambda\left(\frac{p\eta_1}{\eta_1-1}+\frac{q\eta_2}{\eta_2+1}\right)$, and the probability density function of Z_j is*

$$\hat{f}(z) = \hat{p}\hat{\eta}_1 e^{-\hat{\eta}_1 z} I_{\{z\geq 0\}} + \hat{q}\hat{\eta}_2 e^{\hat{\eta}_2 z} I_{\{z<0\}},$$

where $\hat{\eta}_1 = \eta_1 - 1$, $\hat{\eta}_2 = \eta_2 + 1$, $\hat{p} = \frac{p\eta_1}{(1+\beta)(\eta_1-1)}$, $\hat{q} = \frac{q\eta_2}{(1+\beta)(\eta_2+1)} = 1-\hat{p}$.

Proof: In light of Proposition 2, we obtain

$$\begin{aligned}\hat{f}(z) &= \frac{e^z f(z)}{\beta+1} = \frac{p\eta_1}{(1+\beta)} e^{-(\eta_1-1)z} I_{\{z\geq 0\}} \\ &+ \frac{q\eta_2}{(1+\beta)} e^{(1+\eta_2)z} I_{\{z<0\}} \\ &= \frac{p\eta_1}{(1+\beta)(\eta_1-1)}(\eta_1-1)e^{-(\eta_1-1)z} I_{\{z\geq 0\}} \\ &+ \frac{q\eta_2}{(1+\beta)(\eta_2+1)}(\eta_2+1)e^{(1+\eta_2)z} I_{\{z<0\}} \\ &= \hat{p}\hat{\eta}_1 e^{-\hat{\eta}_1 z} I_{\{z\geq 0\}} + \hat{q}\hat{\eta}_2 e^{\hat{\eta}_2 z} I_{\{z<0\}}.\end{aligned}$$

In addition, since $\beta = \frac{p\eta_1}{\eta_1-1} + \frac{q\eta_2}{\eta_2+1} - 1$, we have $\hat{q} = \frac{q\eta_2}{(1+\beta)(\eta_2+1)} = 1 - \hat{p}$, and $\hat{\lambda} = \lambda(\beta+1) = \lambda\left(\frac{p\eta_1}{\eta_1-1} + \frac{q\eta_2}{\eta_2+1}\right)$. Hence, we complete the proof of Proposition 11. □

Let $\widehat{C}(t, e_i)$ denote the value of forward starting call option at time t under a two-state regime switching double exponential jump diffusion model and given $\varepsilon_t = e_i$. Now we will prepare to calculate $\widehat{C}(t, e_i)$.

Theorem 12 *For $0 \leq t < t^* < T$ and $\forall i = 1, 2$, the value of forward starting call option at time t and conditional on $\varepsilon_t = e_i$ is given by*

$$\begin{aligned}\widehat{C}(t, e_i) &= \sum_{j=1}^{2}\int_0^{T-t^*} \widehat{\mathcal{J}}(t, y)\overline{P}_{ij}(t-t^*) \\ &\times \phi_{J_T - J_{t^*}|\varepsilon_{t^*}=e_j}(y)dy,\end{aligned}$$

where

$$\begin{aligned}&\delta_1(y) = \ln K - (r + \frac{1}{2}\sigma^2)(T-t^*) + \lambda\beta y, \\ &\delta_2(y) = \delta_1(y) + \sigma^2(T-t^*), \\ &d_3(y) = \frac{-\delta_1(y)}{\sigma\sqrt{T-t^*}}, \\ &d_4(y) = d_3(y) - \sigma\sqrt{T-t^*}, \\ &\widehat{\mathcal{J}}(t, y) = e^{-\hat{\lambda}y} S_t \mathcal{N}(d_3(y)) \\ + \ &S_t \Upsilon(\delta_1(y), \sigma, \hat{\lambda}, \hat{p}, \hat{\eta}_1, \hat{\eta}_2, T-t^*) \\ - \ &K e^{-r(T-t^*)-\lambda y} S_t \mathcal{N}(d_4(y)) - K S_t \\ \times \ &e^{-r(T-t^*)}\Upsilon(\delta_2(y), \sigma, \lambda, p, \eta_1, \eta_2, T-t^*),\end{aligned}$$

and for $i = 1, 2$

$$\begin{aligned}&\Upsilon(\delta_i(y), \sigma, \lambda, p, \eta_1, \eta_2, T-t^*) \\ = \ &\sum_{n=1}^{\infty}\pi_n(\lambda, y)\sum_{m=1}^{n} P_{n,m} \\ \times \ &\frac{(\sigma\sqrt{T-t^*}\eta_1)^m}{\sigma\sqrt{2\pi(T-t^*)}} e^{\frac{(\sigma\eta_1)^2(T-t^*)}{2}} \\ \times \ &I_{m-1}\left(\delta_i(y); -\eta_1, -\frac{1}{\sigma\sqrt{T-t^*}}, -\sigma\sqrt{T-t^*}\eta_1\right) \\ + \ &\sum_{n=1}^{\infty}\pi_n(\lambda, y)\sum_{m=1}^{n} Q_{n,m} \\ \times \ &\frac{(\sigma\sqrt{T-t^*}\eta_2)^m}{\sigma\sqrt{2\pi(T-t^*)}} e^{\frac{(\sigma\eta_2)^2(T-t^*)}{2}} \\ \times \ &I_{m-1}\Big(\delta_i(y); \eta_2, \frac{1}{\sigma\sqrt{T-t^*}}, -\sigma\sqrt{T-t^*}\eta_2\Big).\end{aligned}$$

Proof: For the forward starting call option with the expiration time T, we have the following price formula:

$$\widehat{C}(t, e_i) = \mathrm{E}_Q^{t,S,i}\left[e^{-r(T-t)}\left(S_T - KS_{t^*}\right)^+\right]. \quad (22)$$

Define

$$\widehat{\mathcal{J}}(t, J_T - J_{t^*}) = \mathrm{E}_Q\left[e^{-r(T-t)}\left(S_T - KS_{t^*}\right)^+ \Big| \mathcal{H}_t\right].$$

Then by the above equation and (22) we get

$$\widehat{C}(t, e_i) = \mathrm{E}_Q^{t,S,i}\left[\widehat{\mathcal{J}}(t, J_T - J_{t^*})\right]. \quad (23)$$

Using the similar method in the proof of Theorem 4, we can obtain

$$\begin{aligned}&\widehat{\mathcal{J}}(t, J_T - J_{t^*}) \\ = \ &S_t Q_S\Bigg((r + \frac{1}{2}\sigma^2)(T-t^*) - \lambda\beta(J_T - J_{t^*}) \\ + \ &\sigma(W_T^S - W_{t^*}^S) + \sum_{j=N_{J_{t^*}}+1}^{N_{J_T}} Z_j \geq \ln K\Big|\mathcal{H}_t\Bigg) \\ - \ &K S_t e^{-r(T-t^*)} Q\Bigg((r - \frac{1}{2}\sigma^2)(T-t^*) \\ - \ &\lambda\beta(J_T - J_{t^*}) + \sigma(W_T - W_{t^*}) \\ + \ &\sum_{j=N_{J_{t^*}}+1}^{N_{J_T}} Z_j \geq \ln K\Big|\mathcal{H}_t\Bigg) \\ = \ &\Pi_3 - \Pi_4. \quad (24)\end{aligned}$$

In the following, we calculate Π_3 and Π_4, respectively.

$$\begin{aligned}\Pi_3 &= S_t\sum_{n=0}^{\infty}\pi_n(\hat{\lambda}, J_T - J_{t^*}) \\ &\times Q_S\Bigg((r + \frac{1}{2}\sigma^2)(T-t^*) - \lambda\beta(J_T - J_{t^*}) \\ &+ \sigma(W_T^S - W_{t^*}^S) + \sum_{j=1}^{n} Z_j \geq \ln K\Big|\mathcal{H}_t\Bigg). \quad (25)\end{aligned}$$

It follows from Proposition 11 and Lemma 8 that Π_3 is equivalent to

$$\begin{aligned}
& S_t\pi_0(\hat{\lambda}, J_T - J_{t^*})Q_S\Big(\sigma(W_T^S - W_{t^*}^S) \geq \ln K \\
- & (r+\frac{1}{2}\sigma^2)(T-t^*) + \lambda\beta(J_T - J_{t^*})\Big|\mathcal{H}_t\Big) \\
+ & S_t\sum_{n=1}^{\infty}\pi_n(\hat{\lambda}, J_T - J_{t^*})\sum_{m=1}^{n}P_{n,m} \\
\times & Q_S\Bigg(\sigma(W_T^S - W_{t^*}^S) + \sum_{j=1}^{m}\xi_j^+ \geq \ln K \\
- & (r+\frac{1}{2}\sigma^2)(T-t^*) + \lambda\beta(J_T - J_{t^*})\Big|\mathcal{H}_t\Bigg) \\
+ & S_t\sum_{n=1}^{\infty}\pi_n(\hat{\lambda}, J_T - J_{t^*})\sum_{m=1}^{n}Q_{n,m} \\
\times & Q_S\Bigg(\sigma(W_T^S - W_{t^*}^S) - \sum_{j=1}^{m}\xi_j^- \geq \ln K \\
- & (\frac{1}{2}\sigma^2 + r)(T-t^*) + \lambda\beta(J_T - J_{t^*})\Big|\mathcal{H}_t\Bigg). \quad (26)
\end{aligned}$$

In addition, since ξ_j^+ and ξ_j^- follow the exponential distribution with parameters $\hat{\eta}_1$ and $\hat{\eta}_2$, respectively, by the Lemma 10, we have

$$\begin{aligned}
& Q_S\Bigg(\sigma(W_T^S - W_{t^*}^S) + \sum_{j=1}^{m}\xi_j^+ \geq \ln K \\
- & (\frac{1}{2}\sigma^2 + r)(T-t^*) + \lambda\beta(J_T - J_{t^*})\Big|\mathcal{H}_t\Bigg) \\
= & \frac{(\sigma\sqrt{T-t^*}\hat{\eta}_1)^m e^{\frac{(\sigma\hat{\eta}_1)^2(T-t^*)}{2}}}{\sigma\sqrt{2\pi(T-t^*)}} \\
\times & I_{m-1}\Big(\delta_1(J_T - J_{t^*}); -\hat{\eta}_1, \frac{-1}{\sigma\sqrt{T-t^*}}, \\
- & \sigma\sqrt{T-t^*}\hat{\eta}_1\Big), \quad (27)
\end{aligned}$$

and

$$\begin{aligned}
& Q_S\Bigg(\sigma(W_T^S - W_{t^*}^S) - \sum_{j=1}^{m}\xi_j^- \geq \ln K \\
- & (\frac{1}{2}\sigma^2 + r)(T-t^*) + \lambda\beta(J_T - J_{t^*})\Big|\mathcal{H}_t\Bigg) \\
= & \frac{(\sigma\sqrt{T-t^*}\hat{\eta}_2)^m e^{\frac{(\sigma\hat{\eta}_2)^2(T-t^*)}{2}}}{\sigma\sqrt{2\pi(T-t^*)}} \\
\times & I_{m-1}\Big(\delta_1(J_T - J_{t^*}); \hat{\eta}_2, \frac{1}{\sigma\sqrt{T-t^*}}, \\
- & \sigma\sqrt{T-t^*}\hat{\eta}_2\Big), \quad (28)
\end{aligned}$$

where

$$\begin{aligned}
\delta_1(J_T - J_{t^*}) &= \ln K - (r+\frac{1}{2}\sigma^2)(T-t^*) \\
&+ \lambda\beta(J_T - J_{t^*}).
\end{aligned}$$

Moreover, it is easily to get

$$\begin{aligned}
& S_t\pi_0(\hat{\lambda}, J_T - J_{t^*})Q_S\Big(\sigma(W_T^S - W_{t^*}^S) \geq \ln K \\
- & (r+\frac{1}{2}\sigma^2)(T-t^*) + \lambda\beta(J_T - J_{t^*})\Big|\mathcal{H}_t\Big) \\
= & S_t e^{-\hat{\lambda}(J_T - J_{t^*})}\mathcal{N}(d_3(J_T - J_{t^*})). \quad (29)
\end{aligned}$$

Let

$$\begin{aligned}
& \Upsilon(\delta_1(J_T - J_{t^*}), \sigma, \hat{\lambda}, \hat{p}, \hat{\eta}_1, \hat{\eta}_2, T-t^*) \\
= & \sum_{n=1}^{\infty}\pi_n(\hat{\lambda}, J_T - J_{t^*})\sum_{m=1}^{n}P_{n,m} \\
\times & \frac{(\sigma\sqrt{T-t^*}\hat{\eta}_1)^m}{\sigma\sqrt{2\pi(T-t^*)}}e^{\frac{(\sigma\hat{\eta}_1)^2(T-t^*)}{2}} \\
\times & I_{m-1}\Big(\delta_1(J_T - J_{t^*}); -\hat{\eta}_1, \\
- & \frac{1}{\sigma\sqrt{T-t^*}}, -\sigma\sqrt{T-t^*}\hat{\eta}_1\Big) \\
+ & \sum_{n=1}^{\infty}\pi_n(\hat{\lambda}, J_T - J_{t^*})\sum_{m=1}^{n}Q_{n,m}e^{\frac{(\sigma\hat{\eta}_2)^2(T-t^*)}{2}} \\
\times & \frac{(\sigma\sqrt{T-t^*}\hat{\eta}_2)^m}{\sigma\sqrt{2\pi(T-t^*)}}I_{m-1}\Big(\delta_1(J_T - J_{t^*}); \hat{\eta}_2, \\
& \frac{1}{\sigma\sqrt{T-t^*}}, -\sigma\sqrt{T-t^*}\hat{\eta}_2\Big), \quad (30)
\end{aligned}$$

using (25), (26), (27), (28), (29) and (30), we can derive the following result

$$\begin{aligned}
\Pi_3 = S_t\Big[& e^{-\hat{\lambda}(J_T - J_{t^*})}\mathcal{N}(d_3(J_T - J_{t^*})) + \\
& \Upsilon(\delta_1(J_T - J_{t^*}), \sigma, \hat{\lambda}, \hat{p}, \hat{\eta}_1, \hat{\eta}_2, T-t^*)\Big]. \quad (31)
\end{aligned}$$

Furthermore, we use the same method to obtain

$$\begin{aligned}
\Pi_4 &= Ke^{-r(T-t^*)}S_t\Big[e^{-\lambda(J_T - J_{t^*})} \\
&\times \mathcal{N}(d_4(J_T - J_{t^*})) \quad (32) \\
&+ \Upsilon(\delta_2(J_T - J_{t^*}), \sigma, \lambda, p, \eta_1, \eta_2, T-t^*)\Big].
\end{aligned}$$

Finally, it follows from (23), (24), (31) and (32) that

$$\begin{aligned}
& \widehat{C}(t, e_i) \\
= & S_t\mathrm{E}_Q^{t,S,i}\Big[e^{-\hat{\lambda}(J_T - J_{t^*})}\mathcal{N}(d_3(J_T - J_{t^*})) \\
+ & \Upsilon(\delta_1(J_T - J_{t^*}), \sigma, \hat{\lambda}, \hat{p}, \hat{\eta}_1, \hat{\eta}_2, T-t^*)\Big] \\
- & Ke^{-r(T-t^*)}S_t\mathrm{E}_Q^{t,S,i}\Big[e^{-\lambda(J_T - J_{t^*})} \\
\times & \mathcal{N}(d_4(J_T - J_{t^*})) + \Upsilon(\delta_2(J_T - J_{t^*}), \\
& \sigma, \lambda, p, \eta_1, \eta_2, T-t^*)\Big].
\end{aligned}$$

Therefore, we complete the proof of Theorem 12. □

Let $\widehat{P}(t, e_i)$ denote the value of forward starting put option at time t under a two-state regime switching double exponential jump diffusion model and given $\varepsilon_t = e_i$.

Proposition 13 *The value of forward starting put option at time t under a two-state regime switching double exponential jump diffusion model is*

$$\widehat{P}(t, e_i) = \widehat{C}(t, e_i) + S_t\left(Ke^{-r(T-t^*)} - 1\right).$$

The proof of Proposition 13 is similar to that of Proposition 5. Here, we will no longer provide the proof.

Remark 14 *If the stock price S follows a double exponential jump diffusion model without regime switching, the value of forward starting call option is*

$$\begin{aligned} & e^{-\hat{\lambda}(T-t^*)}S_t\mathcal{N}(d_3(T-t^*)) + S_t \\ \times \quad & \Upsilon(\delta_1(T-t^*), \sigma, \hat{\lambda}, \hat{p}, \hat{\eta}_1, \hat{\eta}_2, T-t^*) \\ - \quad & Ke^{-r(T-t^*)}S_t\Big[e^{-\lambda(T-t^*)}\mathcal{N}(d_4(T-t^*)) \\ + \quad & \Upsilon(\delta_2(T-t^*), \sigma, \lambda, p, \eta_1, \eta_2, T-t^*)\Big]. \end{aligned}$$

5 Conclusion

In this paper, the two-state regime switching jump diffusion models are presented to model the dynamics of the risky asset. Since the regime switching and jumps render the financial market incomplete, we discuss the problem of choosing risk neutral martingale measures. Using the risk neutral pricing technique, we obtain the closed form formulas of pricing forward starting options under the two cases of regime switching jump diffusion models, namely, a two-state regime switching Merton jump diffusion model and a two-state regime switching double exponential jump diffusion model. In further studies, the parameters of the model in our paper can be considered as not constants. Moreover, the valuation of other derivatives under this model can be studied.

Acknowledgements: This work was supported by Zhejiang Provincial Natural Science Foundation of China under Grant No.(LQ12A01006), Humanity and Social Science Youth Foundation of Ministry of Education of China (12YJC910009).

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