The Spectrum Distribution of Transport Operator with Abstract Boundary Conditions in Slab Geometry

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Abstract: In this paper, transport equation with continuous energy, nonhomogeneous medium and abstract boundary conditions is studied in slab geometry. It is to prove that $|Im\lambda| \parallel K(\lambda I - B_H)^{-1}K \parallel (|Im\lambda| \to +\infty)$ is bounded in the trip Γ_ε, and the spectrum of transport operator A_H consists of only finite isolated eigenvalues with a finite algebraic multiplicities in trip Γ_ε. The main methods rely on operators theory, resolvent operators and comparison operators approach.

Key–Words: Transport operator; abstract boundary condition; boundedness; isolated eigenvalues.

1 Related Knowledge

In this paper, we are concerned with the transport equation with continuous energy, nonhomogeneous medium and abstract boundary conditions in slab geometry. The specific model is as follow

$$\begin{aligned} &\frac{\partial\psi(x,v,\mu,t)}{\partial t} = \\ -&\ \mu\frac{\partial\psi(x,v,\mu,t)}{\partial x} - \sigma(x,v)\psi(x,v,\mu,t) \qquad (1) \\ +&\ \int_E dv' \int_{-1}^{1} k(x,v,\mu,v',\mu')\psi(x,v',\mu',t)d\mu', \end{aligned}$$

with the initial condition

$$\psi(x,\mu,v,0) = \psi_0(x,\mu,v), \qquad (2)$$

where the function $\psi(x,v,\mu)$ represents the number density of gas particles having the position x, the particle velocity v and the direction cosine of propagation μ. here $x \in [-a,a]$ for a parameter $0 < a < +\infty$, $v, v' \in E = [v_m, v_M], 0 < v_m < v_M < +\infty$, and the v_m and v_M are called, respectively, minimum velocity and maximum velocity, and $\mu, \mu' \in [-1,1]$, the function $\sigma(.,.)$ is called the collision frequency, and the function $k(.,.,.,.,.)$ is called the scattering kernel. The abstract boundary conditions are modeled by

$$\psi^i = H\psi^0, \qquad (3)$$

here, H is a linear operator in boundary space.

Since Lehner and Wing made some creating work in [1] in 1950's, the research of spectral distribution of the transport equation have been interesting topic in mathematics, physics, biology and sociology. Latrach and Dehici [2] investigated some spectral properties of time-dependent anisotropic transport equation with periodic and perfecting boundary conditions, using the perturbation theory of strongly continuous semigroups. In fact, let X be a Banach space, and the streaming operator B generates a C_o semigroup $(U(t)_{t\geq 0})$. It is well known that if $K \in \mathcal{L}(X)$ is bounded linear operators, then $B + K$ generates a strongly continuous semigroup $(V(t)_{t\geq 0})$, where

$$V(t) = \sum_{j=0}^{n-1} U_j(t) + R_n(t), \qquad (4)$$

where $U_0(t) = U(t)$, and

$$U_i(t) = \int_0^t U(s)KU_{j-1}(t-s)ds,\ j = 1,2,..., \qquad (5)$$

and the remainder term $R_n(t)$ can be expressed by

$$\begin{aligned} R_n(t) &= \sum_{j=n}^{+\infty} U_j(t) \\ &= \int_{t_1+\cdots+t_n\leq t, t_i\geq 0} U(t_1)KU(t_2)K\cdots U(t_n) \\ &\quad \times KV(t-t_1\cdots - t_n)dt_1\cdots dt_n, \qquad (6) \end{aligned}$$

where if $n = 2$, we can get

$$\begin{aligned} R_2(t) &= \int_{t_1+t_2\leq t, t_1\geq 0, t_2\geq 0} U(t_1)KU(t_2) \\ &\quad \times KV(t-t_1-t_2)dt_1dt_2. \qquad (7) \end{aligned}$$

The above method was named by semigroup perturbation approach, and this approach was used by many authors (see, e.g., [3]-[7]). Some authors developed the perturbation technique to the essential spectral radius of transport operators (see, e.g., [8]-[12]) .

Recently, Wang and Ma in [13] discussed the transport operator of anisotropic continuous energy and homogeneous with periodic boundary conditions in slab geometry in L_2 space. They proved that the streaming operator B generates a C_0 semigroup $(U(t)_{t\geq 0})$, the transport operator A generates a C_0 semigroup, and the second-order remained term $R_2(t)$ of the Dyson-Phillips expansion (4) of the C_0 semigroup is compact in L_2 space. Hence the spectra of the transport operator in some vertical strip Γ consists only of finite many isolated eigenvalues that has a finite algebraic multiplicity. Wang and Wu in [14] discussed the transport operator with anisotropic continuous energy and nonhomogeneous with general boundary conditions in slab geometry in $L_p(1 \leq p < \infty)$ space. They proved that the streaming operator B generates a C_0 semigroup $(U(t)_{t\geq 0})$, where $U(t)$ is of the form

$$\begin{aligned}
&U(t)\varphi(x,v,u) = \sum_{n\geq 0}\alpha^{2n}\\
&\times \exp\Big(-\frac{1}{|\mu|}\Big(2n\int_{-a}^{a} + sgn(\mu)\int_{x'}^{x}\Big)\sigma(\xi,v)d\xi\Big)\\
&\times\varphi(sgn(\mu)4na + x - \mu t, v, \mu)\\
&\times\chi_{[(sgn(\mu)x+(4n-1)a)/|\mu|,(sgn(\mu)x+(4n+1)a)/|\mu|]}(t)\\
&+\sum_{n\geq 0}\alpha^{2n+1}\exp\Big(-\frac{2n}{|\mu|}\int_{-a}^{a}\sigma(\xi,v)d\xi\Big) \qquad (8)\\
&\times \exp\Big(-\frac{1}{|\mu|}sgn(\mu)(\int_{-a}^{x} + \int_{-a}^{x'})\sigma(\xi,v)d\xi\Big)\\
&\times\varphi(-sgn(\mu)(4n+2)a - x + \mu t, v, -\mu)\\
&\times\chi_{[(sgn(\mu)x+(4n+1)a)/|\mu|,(sgn(\mu)x+(4n+3)a)/|\mu|]}(t),
\end{aligned}$$

the transport operator A generates a C_0 semigroup, and the second-order remained term $R_2(t)$ of the Dyson-Phillips expansion of the semigroup is compact in $L_p(1 < p < \infty)$ space and weakly compact in L_1 space, It is similar to the result of [13].

It is well-known that if the transport equation with the specific boundary conditions, or abstract boundary conditions, then the bounded perturbation methods will fail. This is because the boundary operator is a unbounded linear operator. So we have to use the resolvent analysis approach to study the transport equation. Latrach and Megdiche in [15] discussed the transport equation with anisotropic and abstract boundary conditions in slab geometry. Under some assumption that, for $r \in [0,1)$

$$\lim_{|\Im\lambda|\to+\infty} \mid \Im\lambda \mid^{r} \| K(\lambda I - B)^{-1}K \| = 0, \qquad (9)$$

uniformly on some vertical strip, they derived various descriptions of the large time behavior of solutions. Latrach et al. in [16] discussed the transport equation with reentry boundary conditions in slab geometry, they derived conditions that ensure the compactness of the remainder term $R_n(t)$ for some integer n, and got the large time asymptotic behavior of the solution to the one-dimensional transport equation. Lately, some authors discussed the transport equation with anisotropic continuous energy and homogeneous in slab geometry, and obtained essential spectrum and isolated spectrum of the transport equation (see, e.g., [17]-[24], [30]-[32]).

In the past years, some authors described the time asymptotic behavior of the solution of a one-velocity transport operator without restriction on the initial data in sphere (see, e.g., [25, 26]). Of course, there are some progresses about the spectral of bizarre transport equation (see, e.g., [27, 28]). The spectral analysis of transport operator in growing cell population (see, e.g., [33-35]). Recent, Abdelmoumen et al. in [29] discussed the transport operator with anisotropic in sphere, and described the large time behavior of solutions to an abstract Cauchy problem under some assumptions. They proved that there exists an integer m_0 and $r_0 \in [0,1)$ such that

$$\mid \Im\lambda \mid^{r_0} \| [(\lambda I - B)^{-1}K]^{m_0} \|, \qquad (10)$$

is bounded uniformly in some vertical strip. A question is what spectral distribution in slab geometry is under the above condition. In this paper, we will discuss, in $L_p(1 \leq p < +\infty)$ space, the transport equation with continuous energy nonhomogeneous medium and abstract boundary conditions in slab geometry. We will prove that operator

$$\mid \Im\lambda \mid \| K(\lambda I - B_H)^{-1}K \|, (|\Im\lambda| \to +\infty), \qquad (11)$$

is bounded on a vertical strip Γ_ε, and the spectrum of transport operator in the strip Γ_ε is composed of finite many isolated eigenvalues of finite algebraic multiplicities.

Let us introduce some notion and notations, and make precise the function setting of the problem. Let space be

$$X = L_p(D, dxdvd\mu), \qquad (12)$$

the norm is defined by

$$\|\psi\|_X = \Big(\int_{-a}^{a}\int_E\int_{-1}^{1}|\psi(x,v,\mu)|^p dxdvd\mu\Big)^{\frac{1}{p}}, \qquad (13)$$

where $D = [-a,a] \times E \times [-1,1], p \in [1,+\infty)$.

We define the following sets representing the incoming and the outgoing boundary of the phase space

$$D^0 = D_1^0 \cup D_2^0 = \{-a\} \times E \times [-1,0] \cup \{a\} \times E \times [0,1], \quad (14)$$

$$D^i = D_1^i \cup D_2^i = \{-a\} \times E \times [0,1] \cup \{a\} \times E \times [-1,0]. \quad (15)$$

Moreover, we introduce the following boundary spaces

$$\begin{aligned} X^0 &= L_p(D^0, |\mu| dv d\mu) \sim L_p(D_1^0, |\mu| dv d\mu) \\ &\quad \oplus L_p(D_2^0, |\mu| dv d\mu) \\ &= X_1^0 \oplus X_2^0, \end{aligned} \quad (16)$$

$$\begin{aligned} X^i &= L_p(D^i, |\mu| dv d\mu) \sim L_p(D_1^i, |\mu| dv d\mu) \\ &\quad \oplus L_p(D_2^i, |\mu| dv d\mu) \\ &= X_1^i \oplus X_2^i, \end{aligned} \quad (17)$$

endowed with the norm

$$\begin{aligned} \|\varphi^0\|_{X^0} &= \Big(\|\varphi_1^0\|_{X_1^0}^p + \|\varphi_2^0\|_{X_2^0}^p\Big)^{\frac{1}{p}} \\ &= \Big(\int_E dv \int_{-1}^0 |\varphi(-a,v,\mu)|^p |\mu| d\mu \\ &\quad + \int_E dv \int_0^1 |\varphi(a,v,\mu)|^p |\mu| d\mu\Big)^{\frac{1}{p}}, \end{aligned} \quad (18)$$

$$\begin{aligned} \|\varphi^i\|_{X^i} &= \Big(\|\varphi_1^i\|_{X_1^i}^p + \|\varphi_2^i\|_{X_2^i}^p\Big)^{\frac{1}{p}} \\ &= \Big(\int_E dv \int_0^1 |\varphi(-a,v,\mu)|^p |\mu| d\mu \\ &\quad + \int_E dv \int_{-1}^0 |\varphi(a,v,\mu)|^p |\mu| d\mu\Big)^{\frac{1}{p}}, \end{aligned} \quad (19)$$

where $\sim$ means the natural identification of the above spaces. We define the streaming operator B_H by

$$\begin{aligned} B_H \psi(x,v,\mu) &= -\mu \frac{\partial \psi(x,v,\mu)}{\partial x} \\ &\quad -\sigma(x,v)\psi(x,v,\mu), \end{aligned} \quad (20)$$

where

$$D(B_H) = \left\{ \psi \in X \,\middle|\, \mu \frac{\partial \psi}{\partial x} \in X, \psi^i = H\psi^0 \right\}, \quad (21)$$

where $\sigma(x,v)$ is a non-negative and measurable function, $\psi^0 = (\psi_1^0, \psi_2^0)^\top$, and $\psi^i = (\psi_1^i, \psi_2^i)^\top$ with ψ_1^0, ψ_2^0, ψ_1^i and ψ_2^i are given by

$$\begin{aligned} \psi_1^i(v,\mu) &= \psi(-a,v,\mu), && (22)\\ \psi_2^i(v,\mu) &= \psi(a,v,\mu), && (23)\\ \psi_1^0(v,\mu) &= \psi(-a,v,\mu), && (24)\\ \psi_2^0(v,\mu) &= \psi(a,v,\mu). && (25) \end{aligned}$$

Moreover, we define the disturbance operators K by

$$\begin{aligned} K\psi(x,v,\mu) &= \int_E \mathrm{d}v' \int_{-1}^1 \\ &\times k(x,v,\mu,v',\mu')\psi(x,v',\mu')\mathrm{d}\mu'. \end{aligned} \quad (26)$$

So, we can define the transport operator A_H by

$$A_H = B_H + K, \quad D(A_H) = D(B_H). \quad (27)$$

Setting

$$\sigma_0 = \operatorname{essinf}\{\sigma(x,v)\}.$$

Let $\varphi \in X$ and consider the resolvent equation for B_H

$$(\lambda I - B_H)\psi = \varphi. \quad (28)$$

Thus, for $\Re\lambda > -\sigma_0$,, the solution of (28) is formally given by

$$\begin{aligned} &\psi(x,v,\mu) \\ =\;& \psi(-a,v,\mu)\exp\Big(\frac{-1}{\mu}\int_{-a}^x (\lambda+\sigma(\xi,v))\mathrm{d}\xi\Big) \\ +\;& \frac{1}{\mu}\int_{-a}^x \exp\Big(\frac{-1}{\mu}\int_{x'}^x (\lambda+\sigma(\xi,v))\mathrm{d}\xi\Big) \\ &\times\varphi(x',v,\mu)dx', \quad \mu\in(0,1), \end{aligned} \quad (29)$$

$$\begin{aligned} &\psi(x,v,\mu) \\ =\;& \psi(a,v,\mu)\exp\Big(\frac{1}{\mu}\int_x^a (\lambda+\sigma(\xi,v))\mathrm{d}\xi\Big) \\ -\;& \frac{1}{\mu}\int_x^a \exp\Big(\frac{1}{\mu}\int_x^{x'} (\lambda+\sigma(\xi,v))\mathrm{d}\xi\Big) \\ &\times\varphi(x',v,\mu)dx', \quad \mu\in(-1,0). \end{aligned} \quad (30)$$

For $x = \pm a$, we can get

$$\begin{aligned} &\psi(a,v,\mu) \\ =\;& \psi(-a,v,\mu)\exp\Big(\frac{-1}{\mu}\int_{-a}^a (\lambda+\sigma(\xi,v))\mathrm{d}\xi\Big) \\ +\;& \frac{1}{\mu}\int_{-a}^a \exp\Big(\frac{-1}{\mu}\int_{x'}^a (\lambda+\sigma(\xi,v))\mathrm{d}\xi\Big) \\ &\times\varphi(x',v,\mu)dx', \end{aligned} \quad (31)$$

$$\begin{aligned}&\psi(-a,v,\mu)\\ =\;&\psi(a,v,\mu)\exp\Big(\frac{1}{\mu}\int_{-a}^{a}(\lambda+\sigma(\xi,v))\mathrm{d}\xi\Big)\\ -\;&\frac{1}{\mu}\int_{-a}^{a}\exp\Big(\frac{1}{\mu}\int_{-a}^{x'}(\lambda+\sigma(\xi,v))\mathrm{d}\xi\Big)\\ &\times\varphi(x',v,\mu)dx'.\end{aligned}\tag{32}$$

Now, we define operators P_λ, Q_λ, D_λ and E_λ as follow

$$P_\lambda : X^i \to X^0;\quad P_\lambda\varphi = (P_\lambda^+\varphi, P_\lambda^-\varphi),\tag{33}$$

where

$$\begin{aligned}&P_\lambda^+\varphi(a,v,\mu)=\varphi(-a,v,\mu)\\ &\times\exp\Big(\frac{-1}{\mu}\int_{-a}^{a}(\lambda+\sigma(\xi,v))\mathrm{d}\xi\Big),\end{aligned}\tag{34}$$

and

$$\begin{aligned}&P_\lambda^-\varphi(-a,v,\mu)=\varphi(a,v,\mu)\\ &\times\exp\Big(\frac{1}{\mu}\int_{-a}^{a}(\lambda+\sigma(\xi,v))\mathrm{d}\xi\Big);\end{aligned}\tag{35}$$

$$Q_\lambda : X^i \to X;\quad Q_\lambda\varphi = (Q_\lambda^+\varphi, Q_\lambda^-\varphi),\tag{36}$$

where

$$\begin{aligned}&Q_\lambda^+\varphi(-a,v,\mu)=\varphi(-a,v,\mu)\\ &\times\exp\Big(\frac{-1}{\mu}\int_{-a}^{x}(\lambda+\sigma(\xi,v))\mathrm{d}\xi\Big),\end{aligned}\tag{37}$$

and

$$\begin{aligned}&Q_\lambda^-\varphi(a,v,\mu)=\varphi(a,v,\mu)\\ &\times\exp\Big(\frac{1}{\mu}\int_{x}^{a}(\lambda+\sigma(\xi,v))\mathrm{d}\xi\Big);\end{aligned}\tag{38}$$

$$D_\lambda : X \to X^0;\quad D_\lambda\varphi = (D_\lambda^+\varphi, D_\lambda^-\varphi),\tag{39}$$

where

$$\begin{aligned}&D_\lambda^+\varphi(x,v,\mu)=\frac{1}{\mu}\int_{-a}^{a}\varphi(x',v,\mu)\\ &\times\exp\Big(\frac{-1}{\mu}\int_{x'}^{a}(\lambda+\sigma(\xi,v))\mathrm{d}\xi\Big)dx',\end{aligned}\tag{40}$$

and

$$\begin{aligned}&D_\lambda^-\varphi(-a,v,\mu)=\frac{1}{\mu}\int_{-a}^{a}\varphi(x',v,\mu)\\ &\times\exp\Big(\frac{1}{\mu}\int_{-a}^{x'}(\lambda+\sigma(\xi,v))d\xi\Big)\mathrm{d}x';\end{aligned}\tag{41}$$

$$E_\lambda : X \to X;\quad E_\lambda\varphi = (E_\lambda^+\varphi, D_\lambda^-\varphi),\tag{42}$$

where

$$\begin{aligned}&E_\lambda^+\varphi(x,v,\mu)=\frac{1}{\mu}\int_{-a}^{x}\varphi(x',v,\mu)\\ &\times\exp\Big(\frac{-1}{\mu}\int_{x'}^{x}(\lambda+\sigma(\xi,v))d\xi\Big)dx,\end{aligned}\tag{43}$$

and

$$\begin{aligned}&E_\lambda^-\varphi(-a,v,\mu)=\frac{1}{\mu}\int_{x}^{a}\varphi(x',v,\mu)\\ &\times\exp\Big(\frac{1}{\mu}\int_{x}^{x'}(\lambda+\sigma(\xi,v))d\xi\Big)dx.\end{aligned}\tag{44}$$

We assume that the boundary operator H satisfies the following condition.

Assumption O_1**:** $H:\ X^0 \to X^i$,

$$H\begin{pmatrix}u_1\\u_2\end{pmatrix}=\begin{pmatrix}0 & H_{12}\\ H_{21} & 0\end{pmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix}.\tag{45}$$

where

$$\begin{cases}H_{12}=\alpha J_1+\beta L_1:\ X_2^0\to X_2^i;\\ H_{12}\in L(X_2^0,X_2^i),\end{cases}\tag{46}$$

$$\begin{cases}H_{21}=\alpha J_2+\beta L_2:\ X_1^0\to X_2^i;\\ H_{21}\in L(X_1^0,X_2^i),\end{cases}\tag{47}$$

$\alpha,\beta\in R^+$, J_1 and J_2 are compact operators. Moreover

$$L_1u(-a,v,\mu)=u(a,v,\mu),\tag{48}$$

$$L_2u(a,v,\mu)=u(-a,v,\mu).\tag{49}$$

So, for $\Re\lambda > -\sigma_0$, we get

$$\begin{aligned}(\lambda I-B_H)^{-1}\;=\;&\chi_{(0,1)}(\mu)R^+(\lambda I,B_H)\\ +\;&\chi_{(-1,0)}(\mu)R^-(\lambda I,B_H),\end{aligned}\tag{50}$$

where,

$$\begin{aligned}&R^+(\lambda I,B_H)\\ =\;&\sum_{n\ge 0}Q_\lambda^+H_{12}(P_\lambda^+H_{12})^nD_\lambda^+ + E_\lambda^+,\end{aligned}\tag{51}$$

$$\begin{aligned}&R^-(\lambda I,B_H)\\ =\;&\sum_{n\ge 0}Q_\lambda^-H_{21}(P_\lambda^-H_{21})^nD_\lambda^+ + E_\lambda^-.\end{aligned}\tag{52}$$

Assumption O_2**:** Operator K is a regular operator in X. So it can be approximated in the uniform operator topology by operators. Thus

$$\begin{aligned}K\varphi(x,v,\mu)\;=\;&\sum_{i\in I}\int_E dv'\int_{-1}^{1}\theta_i(x)f_i(v,\mu)\\ &\times g_i(v',\mu')\varphi(x,v',\mu')d\mu',\end{aligned}\tag{53}$$

where $\theta_i(\cdot) \in L_\infty([-a,a])$, $f_i(\cdot,\cdot) \in L_1(E \times [-1,1])$, $g_i(\cdot,\cdot) \in L_\infty(E \times [-1,1]), I$ is finite set. Setting

$$\lambda_0 = \begin{cases} -\sigma_0, & \| H \| \leq 1, \\ -\sigma_0 + \frac{1}{2a}\log(\| H \|), & \| H \| > 1. \end{cases} \quad (54)$$

Lemma 1. *[15] If the assume O_1 is satisfied, then, for $\Re\lambda > -\sigma_0$, we have $(\lambda I - B_H)^{-1}$ is bounded and*

$$||(\lambda I - B_H)^{-1}|| \leq \frac{1}{Re\lambda + \sigma_0}. \quad (55)$$

Lemma 2. *[5] If for any $\varepsilon > 0$, there exists a $m \in N$, η, such that $[(\lambda I - B_H)^{-1}K]^m$ is compact, and*

$$\lim_{|\Im\lambda| \to +\infty} \| [(\lambda I - B_H)^{-1}K]^m \| = 0. \quad (56)$$

Then, there exists at most finitely many isolated eigenvalues of A_H in the strip $\{\lambda \in \mathbb{C};\ \Re\lambda \geq \eta + \varepsilon\}$ where η is type of C_0 semigroup generated by streaming operator B_H, which are of finite algebraic multiplicity.

2 Main Result

In this section, we will give the main results of this paper. Setting

$$\Gamma_\varepsilon = \{\lambda \in C;\ \Re\lambda \geq -\sigma_0 + \varepsilon\}(\varepsilon > 0). \quad (57)$$

Theorem 3. *If assumptions O_1 and O_2 are satisfied, then*

$$| \Im\lambda ||| K(\lambda I - B_H)^{-1}K \|, \quad (58)$$

is uniformly bounded on Γ_ε.

Proof. We finish the proof by the following serval steps.

Step 1. Because of

$$\begin{aligned} & \| K(\lambda - B_H)^{-1}K \| \\ \leq & \| KE_\lambda^+ K \| + \| KE_\lambda^- K \| \\ + & \sum_{n\geq 0} \| KQ_\lambda^+ H_{12}(P_\lambda^+ H_{12})^n D_\lambda^+ K \| \\ + & \sum_{n\geq 0} \| KQ_\lambda^- H_{21}(P_\lambda^- H_{21})^n D_\lambda^+ K \| . \end{aligned} \quad (59)$$

So, if we prove (57) is bounded uniformly on Γ_ε, we only prove

$$| \Im\lambda ||| KE_\lambda^+ K \|, \quad (60)$$

$$| \Im\lambda ||| KE_\lambda^- K \|, \quad (61)$$

$$| \Im\lambda | \sum_{n\geq 0} \| KQ_\lambda^+ H_{12}(P_\lambda^+ H_{12})^n D_\lambda^+ K \|, \quad (62)$$

$$| \Im\lambda | \sum_{n\geq 0} \| KQ_\lambda^- H_{21}(P_\lambda^- H_{21})^n D_\lambda^- K \| . \quad (63)$$

are all bounded uniformly on Γ_ε.

Step 2. Prove equation (60) is bounded uniformly on Γ_ε. For all $\varphi \in X$,we get

$$\begin{aligned} & E_\lambda^+\varphi(x,v,\mu) = \frac{1}{\mu}\int_{-a}^{x} \varphi(x',v,\mu) \\ \times & \exp\Big(\frac{-1}{\mu}\int_{x'}^{x}(\lambda + \sigma(\xi,v))\mathrm{d}\xi\Big)\mathrm{d}x' \\ = & \frac{1}{\mu}\int_{-a}^{x}\varphi(x',v,\mu) \\ \times & \exp\Big(\frac{-1}{\mu}\Big[(x-x')\lambda + \int_{x'}^{x}\lambda + \sigma(\xi,v)\mathrm{d}\xi\Big]\Big)\mathrm{d}x'. \end{aligned} \quad (64)$$

The change of $s = \frac{x-x'}{\mu}$ gives

$$\begin{aligned} E_\lambda^+\varphi(x,v,\mu) = & \int_0^{+\infty}\varphi(x - s\mu, v, \mu)\chi_{(0,\frac{x+a}{\mu})}(s) \\ & \times \exp\Big(-\lambda s - \int_{x-s\mu}^{x}\sigma(\xi,v)\mathrm{d}\xi\Big)\mathrm{d}s. \end{aligned} \quad (65)$$

Now consider the sequence of operators $E_{\lambda,\varepsilon_n}^+$, where

$$\begin{aligned} & E_{\lambda,\varepsilon_n}\varphi(x,v,\mu) \\ = & \int_{\varepsilon_n}^{+\infty}\varphi(x - s\mu, v, \mu)\chi_{(0,\frac{x+a}{\mu})}(s) \\ & \times \exp\Big(-\lambda s - \int_{x-s\mu}^{x}\sigma(\xi,v)\mathrm{d}\xi\Big)\mathrm{d}s, \end{aligned} \quad (66)$$

where $(\varepsilon_n)_{n\in N}$ is a sequence of non-negative real numbers which converge to zero as $n \to \infty$. Clearly, the sequence $(E_{\lambda,\varepsilon_n})_{n\in N}$ converges to E_λ^+, in the operator topology, uniformly on Γ_ε as $n \to \infty$. So, it suffices to prove that, for $\varepsilon > 0$,

$$| \Im\lambda ||| KE_{\lambda,\varepsilon}^+ K \|,$$

is bounded uniformly on Γ_ε. Because of

$$\begin{aligned} & KE_{\lambda,\varepsilon}^+ K\varphi(x,v,\mu) \\ = & \int_E \mathrm{d}v' \int_0^1 \mathrm{d}\mu' h(v',\mu') f(v,\mu)\chi_{(0,\frac{x+a}{\mu'})}(s) \\ & \times \exp\Big(-\lambda s - \int_{x-s\mu'}^{x}\sigma(\xi,v)\mathrm{d}\xi'\Big) \\ & \times \int_\varepsilon^{+\infty}\int_E\int_{-1}^{1}\theta(x - s\mu')g(v'',\mu'') \\ & \times\theta(x)\varphi(x - s\mu', v'', \mu'')\mathrm{d}s\mathrm{d}v''\mathrm{d}\mu''. \end{aligned} \quad (67)$$

Setting $t = x - \mu' s$, we get

$$\begin{aligned} & KE^+_{\lambda,\varepsilon} K\varphi(x, v, \mu) \\ = \; & \theta(x)\int_E \mathrm{d}v' \int_{-a}^{x} \mathrm{d}\mu' h\Big(v', \frac{x-t}{s}\Big) f(v,\mu) \\ & \times \exp\Big(-\lambda s - \int_t^x \sigma(\xi, v)\mathrm{d}\xi' \Big) \\ & \times \int_\varepsilon^{+\infty} \mathrm{d}s\varphi(t', v'', \mu'').\chi_{(x-t,+\infty)}(s) \\ & \times \int_E \theta(t) g(v'', \mu'')\mathrm{d}v'' \int_{-1}^{1} \mathrm{d}\mu'' \end{aligned} \quad (68)$$

Putting

$$KE^+_{\lambda,\varepsilon} K = A_1 A_\varepsilon A_2, \quad (69)$$

where $A_1 : L_p(-a, a) \to X$,

$$A_1\varphi(x) = \theta(x) f(v, \mu)\varphi(x); \quad (70)$$

$A_2 : L_p(-a, a) \to X$,

$$\begin{aligned} & A_2\varphi(x, v, \mu) \\ = \; & \int_E \mathrm{d}v \int_{-1}^{1} \theta(x) g(v, \mu)\varphi(x, v, \mu)\mathrm{d}\mu; \end{aligned} \quad (71)$$

$A_\varepsilon : L_p(-a, a) \to L_p(-a, a)$,

$$\begin{aligned} A_\varepsilon\varphi(x) = & \int_{-a}^{x} \mathrm{d}t \int_E \mathrm{d}v \int_\varepsilon^{+\infty} \frac{\mathrm{d}s}{s} \\ & \times \exp\Big[-\lambda s - \int_t^x \sigma(x, \xi)\mathrm{d}\xi \Big] \\ & \times h\Big(v, \frac{x-t}{s}\Big)\varphi(t)\chi_{(x-t,+\infty)}(s); \end{aligned} \quad (72)$$

and $A_{\varepsilon,n} : L_p(-a, a) \to L_p(-a, a)$,

$$\begin{aligned} A_{\varepsilon,n}\varphi(x) = & \int_{-a}^{x} \mathrm{d}t \int_E \mathrm{d}v \int_\varepsilon^{+\infty} l_{x-t,v,n}(s) \\ & \times \exp\Big[-(\lambda + \sigma_0 - \frac{\varepsilon}{2})s \Big]\mathrm{d}s \\ & \times h\Big(v, \frac{x-t}{s}\Big)\varphi(t)\chi_{(x-t,+\infty)}(s), \end{aligned} \quad (73)$$

where $l_{x-t,v,n}(\cdot)$ converges to $\varphi_{x-t,v}(\cdot)$.a.e. Because operators A_1 and A_2 are bounded and uniformly on Γ_ε, so, it suffices to prove that, for $\varepsilon > 0$, $\mid \Im\lambda \mid^p \| A_\varepsilon \|^p$ is bounded uniformly on Γ_ε. According to Lemma 1 of [15], it holds that

$$\mid \Im\lambda \mid^p \| A_{\varepsilon,n} \|^p, (n \in N), \quad (74)$$

is bounded uniformly on Γ_ε. For all $\varphi \in L_p(-a.a), n \in N$, we get

$$\begin{aligned} \| A_{\varepsilon,n}\varphi \|^p = & \int_{-a}^{a} \mathrm{d}x \Big| \int_{-a}^{x} \mathrm{d}t \int_E \mathrm{d}v \int_\varepsilon^{+\infty} \\ & \times \exp\Big[-\Big(\lambda + \sigma_0 - \frac{\varepsilon}{2}\Big)s \Big] l_{x-t,v,n}(s) \\ & \times h\Big(v, \frac{x-t}{s}\Big)\varphi(t)\chi_{(x-t,+\infty)}(s)\mathrm{d}s \Big|^p \\ \le \; & \int_{-2a}^{2a} \mathrm{d}x \Big| \int_{-a}^{x} \mathrm{d}t \int_E \mathrm{d}v \int_\varepsilon^{+\infty} \\ & \times \exp\Big[-\Big(\lambda + \sigma_0 - \frac{\varepsilon}{2}\Big)s \Big] l_{x,v,n}(s) \\ & \times h\Big(v, \frac{x}{s}\Big)\varphi(t)\chi_{(x-t,+\infty)}(s)\mathrm{d}s \Big|^p. \end{aligned} \quad (75)$$

The use of the Hölder inequality gives

$$\begin{aligned} & \Big| \int_{-a}^{x} \mathrm{d}t \int_E \mathrm{d}v \int_\varepsilon^{+\infty} \exp\Big[-(\lambda + \sigma_0 - \frac{\varepsilon}{2})s \Big] \cdot \\ & \qquad \times l_{x,v,n}(s) h(v, \frac{x}{s})\varphi(t)\chi_{(x-t,+\infty)}(s)\mathrm{d}s \Big|^p \\ \le & \Big[\int_{-a}^{a} \mathrm{d}t \Big| \int_E \mathrm{d}v \int_\varepsilon^{+\infty} \exp\Big[-(\lambda + \sigma_0 - \frac{\varepsilon}{2})s \Big] \cdot \\ & \qquad \times l_{x,v,n}(s) h(v, \frac{x}{s}) \Big|^q \mathrm{d}s \Big]^{\frac{p}{q}} \int_{-a}^{a} \mid \varphi(t) \mid^p \mathrm{d}t \\ \le & (2a)^{\frac{p}{q}} \Big| \int_E \mathrm{d}v \int_\varepsilon^{+\infty} \exp\Big[-(\lambda_0 + \sigma - \frac{\varepsilon}{2})s \Big] \cdot \\ & \qquad \times l_{x,v,n}(s) h(v, \frac{x}{s})\mathrm{d}s \Big|^p \| \varphi \|^p . \end{aligned} \quad (76)$$

Use of the Hölder inequality again, we get

$$\begin{aligned} \| A_{\varepsilon,n} \|^p \le & (2a)^{\frac{p}{q}} \int_{-2a}^{2a} \mathrm{d}x \Big| \int_E \mathrm{d}v \int_\varepsilon^{+\infty} \\ & \times \exp\Big[-(\lambda + \sigma_0 - \frac{\varepsilon}{2})s \Big] \\ & \times l_{x,v,n}(s) h(v, \frac{x}{s})\varphi(t)\chi_{(x,+\infty)}(s)\mathrm{d}s \Big|^p \\ \le \; & (2a)^{\frac{p}{q}} \int_{-2a}^{2a} \mathrm{d}x \int_E \mathrm{d}v \Big| \int_\varepsilon^{+\infty} \\ & \times \exp\Big[-(\lambda + \sigma_0 - \frac{\varepsilon}{2})s \Big] \\ & \times l_{x,v,n}(s) h(v, \frac{x}{s})\varphi(t)\chi_{(x,+\infty)}(s)\mathrm{d}s \Big|^p. \end{aligned} \quad (77)$$

So, it suffices to prove that, for $\varepsilon > 0$,

$$\begin{aligned} & \mid \Im\lambda \mid^p \int_{-2a}^{2a} dx \int_E dv \mid \int_\varepsilon^{+\infty} ds \\ & \times \exp[-(\lambda + \sigma_0 - \frac{\varepsilon}{2})] s l_{x,v,n}(s) \\ & \times h(v, \frac{x}{s})\varphi(t)\chi_{(x,+\infty)}(s) \mid^p, \end{aligned} \quad (78)$$

is bounded uniformly on Γ_ε. In fact, for $\forall n \in N$, $x \in (-2a, 2a)$ and $v \in E$ be fixed, we define

$$W_{x,v}(\cdot) : (\varepsilon, +\infty) \to R, \tag{79}$$

$$s \mapsto l_{x,v,n}(s)h(v, \frac{x}{s}), \tag{80}$$

where $l_{x,v,n}(s)$ and $h(v, \frac{x}{s})$ are simple functions. Setting $(s_i)_{1\le i\le m}$, for $\forall i \in \{1, 2, \cdots, m-1\}$,$s \in [s_i, s_{i+1})$, we get

$$W_{x,v}(\cdot) = W_{x,v}(s_i), \tag{81}$$

so we can get

$$\begin{aligned}
&\int_\varepsilon^{+\infty} \exp(-(\lambda + \sigma_0 - \frac{\varepsilon}{2})s)W_{x,v}(s)ds \\
= &\sum_{i=1}^{m-1} G_{x,v}(s_i) \int_{s_i}^{s_{i+1}} \exp(-(\lambda + \sigma_0 - \frac{\varepsilon}{2})s)ds \\
= &\sum_{i=1}^{m-1} \Big(\exp(-(\lambda + \sigma_0 - \frac{\varepsilon}{2})s_i) \\
&- \exp(-(\lambda + \sigma_0 - \frac{\varepsilon}{2})s_{i+1})\Big) \\
&\times \frac{1}{\lambda + \sigma - \frac{\varepsilon}{2}} W_{x,v}(s_i).
\end{aligned}$$

So

$$| \int_\varepsilon^{+\infty} W_{x,v}(s)ds | \le \frac{2(m-1)\sup | h(\cdot,\cdot) |}{\varepsilon | \Im\lambda |}, \tag{82}$$

and

$$\begin{aligned}
&| \Im\lambda |^p \int_{-2a}^{2a} dx \int_E dv \Big| \int_\varepsilon^{+\infty} l_{x,v,n}(s)h(v, \frac{x}{s}) \\
&\times \exp(-(\lambda + \sigma_0 - \frac{\varepsilon}{2})s)\varphi(t)\chi_{(x,+\infty)}(s)\Big|^p \\
&\le | \Im\lambda |^p \int_{-2a}^{2a} dx \int_E dv (2(m-1) \\
&\times \sup | h(\cdot,\cdot) |)^p \varepsilon^{-p} | \Im\lambda |^{-p} \\
&\le 4aM(2(m-1)\sup | h(\cdot,\cdot) |)^p \varepsilon^{-p} | \Im\lambda |^{-p},
\end{aligned} \tag{83}$$

where $M = v_M - v_m$. Since

$$4aM(2(m-1)\sup | h(\cdot,\cdot) |)^p \varepsilon^{-p} | \Im\lambda |^{-p},$$

is bounded and uniformly on Γ_ε. This ends the step 2.

Since the (61) and (60) have the same mode, similarly, we can get the equation (61) is bounded uniformly on Γ_ε.

Step 3. (62) is bounded uniformly on Γ_ε. Since $(P_\lambda^+ H_{12})^n$ can be expressed by

$$(P_\lambda^+ H_{12})^n = \sum_{j=1}^{2^n} P_j, \tag{84}$$

where each P_j is the product of n factors involving both $\alpha P_\lambda^+ T_1$ and $\beta P_\lambda^+ L_1$ except the term

$$P_{2^n} = (\beta P_\lambda^+ L_1)^n. \tag{85}$$

So, for $j \in \{1, 2, \cdots, 2^n - 1\}$, the operator T_1 appears at least once in the expression of P_j. While

$$\begin{aligned}
&\| KQ_\lambda^+ H_{12} P_j D_\lambda^+ K \| \\
\le &\| KQ_\lambda^+ H_{12} \| \cdot \| P_j D_\lambda^+ K \|,
\end{aligned} \tag{86}$$

where $j \in \{1, 2, \cdots, 2^n - 1\}$, so if we prove the equation (62) is bounded and uniformly on Γ_ξ. We only need to prove

$$| \Im\lambda ||| P_j D_\lambda^+ K \|, \tag{87}$$

$$| \Im\lambda ||| KQ_\lambda^+ H_{12} P_{2^n} D_\lambda^+ K \|, \tag{88}$$

are all bounded uniformly on Γ_ε, where $j \in \{1, 2, \cdots, 2^n - 1\}$. In fact, according to the hypotheses, there exists $k \in \{1, 2, \cdots, n-1\}$, such that

$$P_j = Q_j P_\lambda^+ T_1 (P_\lambda^+ L_1)^k, \tag{89}$$

where Q_j is bounded and uniformly on Γ_ε. Since

$$\begin{aligned}
&\| Q_j P_\lambda^+ T_1 (P_\lambda^+ L_1)^k D_\lambda^+ K \| \\
\le &\| Q_j P_\lambda^+ \| \cdot \| T_1 (P_\lambda^+ L_1)^k D_\lambda^+ K \|,
\end{aligned} \tag{90}$$

so, it is sufficient to prove

$$| \Im\lambda ||| T_1 (P_\lambda^+ L_1)^k D_\lambda^+ K \|, \tag{91}$$

is bounded and uniformly on Γ_ε. Since J_1 is compact, it is sufficient to establish the result for an operator of rank one, that is $T_1 : \varphi(a, v, \mu) \to T_1\varphi(-a, v, \mu)$,

$$\begin{aligned}
T_1\varphi(-a, v, \mu) = \theta(x) \int_E \mathrm{d}v f(v, \mu) \\
\times \int_0^1 g(v', \mu')\varphi(a, v', \mu') | \mu' | \mathrm{d}\mu',
\end{aligned} \tag{92}$$

where $f(\cdot,\cdot)$, $g(\cdot,\cdot)$ are measurable simple functions. For $\varphi \in X$,

$$\begin{aligned}
&T_1 (P_\lambda^+ L_1)^k D_\lambda^+ K\varphi(x, v, \mu) \\
&= \theta(x) f(v, \mu) \int_E \mathrm{d}v' \int_0^1 \mathrm{d}\mu' g(v', \mu') \\
&\times f(v', \mu') \int_{-a}^{a} \mathrm{d}x\theta(x') \exp\Big[\frac{-1}{\mu'} \int_{x'}^{x} \\
&\times \big(\lambda - \sigma(\xi, v)\big)\mathrm{d}\xi'((2k+1)a - x)\Big] \int_E \mathrm{d}v'' \\
&\times \int_{-1}^{1} g(v'', \mu'')\varphi(x - s\mu', v'', \mu'')\mathrm{d}\mu''.
\end{aligned}$$

(93)

Now, we define the operators by

$$\begin{aligned} B_1 &: \varphi \in X \to \theta(x) \int_E dv \\ & \times \int_{-1}^{1} g(v,\mu)\varphi(x,v,\mu)d\mu, \quad (94) \\ B_2 &: \gamma \in R \to \gamma\eta(v,\mu) \in L^i_{p,1}, \quad (95) \end{aligned}$$

$$\begin{aligned} B_k &: \varphi \in L_p(-a,a) \to \int_E \mathrm{d}v \int_0^1 \mathrm{d}\mu \\ & \times f(v,\mu)g(v,\mu) \int_{-a}^{a} \exp\Big[\frac{(2k+1)a - x}{\mu} \\ & \times \theta(x) \int_{x'}^{x} \big(\lambda - \sigma(\xi,v)\big)\mathrm{d}\xi\Big]\varphi(x)\mathrm{d}x, \quad (96) \end{aligned}$$

so, we can get

$$T_1(P^+_\lambda L_1)^k D^+_\lambda K = B_2 B_k B_1.$$

Clearly,

$$\| T_1(P^+_\lambda L_1)^k D^+_\lambda K \| \le \| B_2 \| \cdot \| B_k \| \cdot \| B_1 \|,$$

because of B_1, B_2, and B_k are all bounded, moreover B_1 and B_2 are independent of λ, we only need to prove that $| \Im\lambda ||| B_k \|$ is bounded uniformly on Γ_ε. Now, we set $\varphi \in L_p((-a,a);dx)$ and $\bar{\varphi}$ denote by its trivial extension to R, so $B_k\varphi$ may be written in the from

$$\begin{aligned} B_k\varphi &= \int_R F_\lambda((2k+1)a - x)\bar{\varphi}dx \\ &= (F_\lambda * \bar{\varphi})((2k+1)a). \end{aligned}$$

the Young inequality gives

$$| B_k\varphi | \le \| F_\lambda \|_{L^q(R)} \cdot \| \bar{\varphi} \|_{L^p(-a,a)},$$

then

$$\begin{aligned} & \| B_k \|^q \le \theta(x) \int_0^{+\infty} \Big| \int_E \mathrm{d}v \\ & \times \int_0^1 \mathrm{d}\mu f(v,\mu)g(v,\mu) \exp\Big[- \frac{-1}{\mu} \\ & \times \int_{x'}^{x} \big(\lambda - \sigma(\xi,v)d\xi\big)\Big]\mathrm{d}\mu\Big|^q \mathrm{d}t. \end{aligned}$$

Since

$$\begin{aligned} & | \Im\lambda |^q \int_\varepsilon^{+\infty} \Big| \int_E \mathrm{d}v \int_0^1 \mathrm{d}\mu f(v,\mu) \\ & \times g(v,\mu)\theta(x) \exp\Big[- \frac{1}{\mu} \int_{x'}^{x} \\ & \times \big(\lambda - \sigma(\xi,v)d\xi\big)\Big]\mathrm{d}\mu\Big|^q \mathrm{d}t, \end{aligned}$$

is bounded uniformly on Γ_ε, we can get the (87) is bounded uniformly on Γ_ε.

Now, we prove that (88) is bounded uniformly on Γ_ε. Since $H_{12} = \alpha T_1 + \beta L_1, T_1$ is compact operator, it suffices to prove that $| \Im\lambda ||| KQ^+_\lambda L_1 P_{2^n} D^+_\lambda K \|$ is bounded uniformly on Γ_ε.

In fact, for all $\varphi \in X$, then

$$\begin{aligned} & KQ^+_\lambda L_1(\beta P^+_\lambda L_1)D^+_\lambda K = \\ & \theta(x)\beta^n f(v,\mu) \int_{-a}^{a} \mathrm{d}x \int_E \mathrm{d}v' \int_0^1 g(v',\mu') \\ & \times f(v',\mu') \exp\Big[\frac{-(2n+1)a + x}{\mu'} \\ & \times \int_{x'}^{x} \big(\lambda - \sigma(\xi',v))\mathrm{d}\xi'\big)\Big]\mathrm{d}\mu' \\ & \times \theta(x') \int_E \mathrm{d}v'' \int_{-1}^{1} g(v'',\mu'') \\ & \times \varphi(x - s\mu', v'', \mu'')\mathrm{d}\mu''. \end{aligned}$$

Setting

$$KQ^+_\lambda L_1(\beta P^+_\lambda L_1)D^+_\lambda K = E_2 E_n E_1,$$

where

$$\begin{aligned} E_1 &: \varphi \in X \to \theta(x) \int_E \mathrm{d}v \int_{-1}^{1} g(v,\mu) \\ & \times \varphi(x,v,\mu)\mathrm{d}\mu \in L_p((-a,a);\mathrm{d}x), \\ E_2 &: \varphi \in L_p((-a,a);\mathrm{d}x) \to \\ & \theta(x)\beta^n f(v,\mu)\varphi(x) \in X_p, \\ E_n &: \varphi \in L_p((-a,a);\mathrm{d}x) \to \\ & \int_{-a}^{a} \mathrm{d}x \int_E \mathrm{d}v \int_0^1 f(v,\mu)g(v,\mu) \\ & \times \exp\Big[\frac{1}{\mu} \int_{x'}^{x} \big(- \lambda + \sigma(\xi,v)\mathrm{d}\xi\big) \\ & \times ((2n+1)a - x)\Big]\mathrm{d}\mu \in L_p((-a,a);\mathrm{d}x). \end{aligned}$$

Since operator E_1, E_2 and E_n are bounded, moreover E_1 and E_2 are independent of λ, so it is easy to prove that, $| \Im\lambda ||| E_n \|$ is bounded uniformly on Γ_ε. This ends the step three.

Finally, since (63) and equation (62) have the same form, so we can get the (63) is bounded uniformly on Γ_ε. This ends the proof. □

Theorem 4. *If assumption O_1 and O_2 are satisfied, then for all $\varepsilon > 0$, and big enough $|\Im\lambda|$, then the spectrum of transport operator A_H consists of, only, finite isolated eigenvalues which have a finite algebraic multiplicities in trip Γ_ε.*

Proof. On one hand, because of hypothesis O_2 and Lemma 1 we can get operators $K(\lambda I - B_H)^{-1}$ and $(\lambda I - B_H)^{-1}K$ are compact operator on X. So, for all $\lambda \in \Gamma_\varepsilon$, the operator $[(\lambda I - B_H)^{-1}K]^2$ is compact operator on X.

On the other hand, since

$$[(\lambda I - B_H)^{-1}K]^2$$
$$= (\lambda I - B_H)^{-1}[K(\lambda I - B_H)^{-1}K],$$

so, according to the Lemma 1 and Theorem 3, we can get

$$\lim_{|\Im\lambda|\to+\infty} \| [(\lambda I - B_H)^{-1}K]^2 \| = 0.$$

Finally, according to the Lemma 2, the desired result follows. □

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