On some properties of certain subclasses of analytic functions defined by using the subordination principle

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Abstract: - In this paper, we introduce some new subclasses of analytic functions related to starlike, convex, close-to-convex and quasi-convex functions defined by using a generalized operator and the differential subordination principle. Inclusion relationships for these subclasses are established. Moreover, we introduce some integral-preserving properties.

Key-Words: - Starlike function; Convex function; Close-to-convex function; Quasi-convex function; Subordination principle.

1 Introduction

Let $\mathbb{A}$ denotes the class of functions $f(z)$ which are analytic in $U=\{z\in\mathbb{C}:|z|<1\}$ and be given by

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n. \qquad (1)$$

Also, for $0\leq\alpha,\beta<1$, let $S^*(\alpha)$, $C(\alpha)$, $K(\beta,\alpha)$ and $K^*(\beta,\alpha)$ denote, respectively, the well-known subclasses of $\mathbb{A}$ consisting of univalent functions which are starlike of order α, convex of order α, close-to-convex order β and type α and quasi-convex of order β and type α (see [23], [28], [32], [34], [38], [40], [43], and [44] etc.).

Let M be the class of all functions φ which are analytic and univalent in U and for which $\varphi(U)$ is convex with $\varphi(0)=1$ and $Re\{\varphi(z)\}>0;\ z\in U$.

We begin with recalling the principle of subordination between analytic functions.

Definition 1. For two functions $f(z)$ and $g(z)$, analytic in U, we say that $f(z)$ is subordinate to $g(z)$ in U, written $f\prec g$ or $f(z)\prec g(z)$, if there exists an analytic function $w(z)$ in U, satisfying the following conditions: $w(0)=0$ and $|w(z)|<1$, such that $f(z)=g(w(z))$.

In particular, If $g(z)$ is univalent in U, then $f\prec g$, if and only if (see [31] and [6]) $f(0)=g(0)$ and $f(U)\subset g(U)$.

Definition 2. Making use of Definition 1, several authors have investigated the subclasses $S^*(\alpha;\varphi)$, $C(\alpha;\varphi)$, $K(\beta,\alpha;\psi,\varphi)$ and $K^*(\beta,\alpha;\psi,\varphi)$ of the class $\mathbb{A}$ for $0\leq\alpha,\beta<1$ and $\varphi,\psi\in M$, which are defined as follows (see [9], [10], [11], [20], and [27]):

$$S^*(\alpha;\varphi)=\left\{f: f(z)\in\mathbb{A} \text{ and } \frac{1}{1-\alpha}\left(\frac{zf'(z)}{f(z)}-\alpha\right)\prec\varphi(z)\quad (0\leq\alpha<1,\varphi\in M, z\in U)\right\},$$

$$C(\alpha;\varphi)=\left\{f: f(z)\in\mathbb{A} \text{ and } \frac{1}{1-\alpha}\left(1+\frac{zf''(z)}{f'(z)}-\alpha\right)\prec\varphi(z)\quad (0\leq\alpha<1,\varphi\in M, z\in U)\right\},$$

$$K(\beta,\alpha;\psi,\varphi)=\left\{f: f(z)\in\mathbb{A} \text{ and } \exists\, g(z)\in S^*(\alpha,\varphi); \frac{1}{1-\beta}\left(\frac{zf'(z)}{g(z)}-\beta\right)\prec\psi(z)\left(0\leq\alpha,\beta<1,\psi\in M, z\in U\right)\right\},$$

and

$$K^*(\beta,\alpha;\psi,\varphi)=\left\{f: f(z)\in\mathbb{A} \text{ and } \exists\, g(z)\in C(\alpha,\varphi); \frac{1}{1-\beta}\left(\frac{(zf'(z))'}{g'(z)}-\beta\right)\prec\psi(z)\left(0\leq\alpha,\beta<1,\psi\in M,\ z\in U\right)\right\}.$$

In particular, for $\varphi(z)=\psi(z)=(1+z)/(1-z)$, we obtain the familiar classes $S^*(\alpha), C(\alpha), K(\beta,\alpha)$ and $K^*(\beta,\alpha)$, respectively.

Furthermore, if we set $\alpha=0$ and $\varphi(z)=\psi(z)=(1+Az)/(1-Bz)$ $(-1\le B<A\le 1)$, we obtain the following function classes:

$S^*\left(0,\frac{1+Az}{1-Bz}\right)=S^*(A,B)$ and $C\left(0,\frac{1+Az}{1-Bz}\right)=C(A,B)$, which were introduced by Janowski [18] (see also [17]).

Following the recent work of El-Ashwah and Aouf [14] and [13, with $p=1$], for $m\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$, $\mathbb{N}=\{1,2,3,...\}$, $\lambda>0, l>-1$ and for function $f(z)\in\mathbb{A}$ given by (1.1), the integral operator $L^m_{\lambda,l}:\mathbb{A}\to\mathbb{A}$ is defined as follows:

$$L^m_{\lambda,l}f(z)=\begin{cases} f(z), & m=0,\\ \frac{l+1}{\lambda}z^{1-\frac{l+1}{\lambda}}\int_0^z t^{\frac{l+1}{\lambda}-2}L^{m-1}_{\lambda,l}f(t)\,dt, & m=1,2,\ldots. \end{cases} \quad (2)$$

It is clear from (1.2) that:

$$L^m_{\lambda,l}f(z)=z+\sum_{n=2}^{\infty}\left(\frac{l+1}{l+\lambda(n-1)+1}\right)^m a_n z^n \quad (3)$$

Also, for $\mu>0$ and $a,c\in\mathbb{C}$, are such that $\mathrm{Re}\{c-a\}\ge 0$, $\mathrm{Re}\{a\}>-\mu$ Raina and Sharma [39] defined the integral operator $J^{a,c}_{\mu}:\mathbb{A}\to\mathbb{A}$, as follows:

$$J^{a,c}_{\mu}f(z)=\begin{cases} f(z); & a=c,\\ \frac{\Gamma(c+\mu)}{\Gamma(a+\mu)}\frac{1}{\Gamma(c-a)}\int_0^1(1-t)^{c-a-1}t^{a-1}f(zt^{\mu})dt; & \mathrm{Re}\{c-a\}>0. \end{cases} \quad (4)$$

For $f(z)$ defined by (1.1), it is easily from (1.4) that:

$$J^{a,c}_{\mu}f(z)=z+\frac{\Gamma(c+\mu)}{\Gamma(a+\mu)}\sum_{n=2}^{\infty}\frac{\Gamma(a+n\mu)}{\Gamma(c+n\mu)}a_n z^n. \quad (5)$$

$(\mu>0; a,c\in\mathbb{C}; \mathrm{Re}\{c-a\}\ge 0; \mathrm{Re}\{a\}>-\mu)$

By combining the two linear operators $L^m_{\lambda,l}$ and $J^{a,c}_{\mu}$, we define the generalized operator

$$I^m_{\lambda,l}(a,c,\mu):\ \mathbb{A}\to\mathbb{A},$$

is defined for the purpose of this paper as following:

$$I^m_{\lambda,l}(a,c,\mu)f(z)=L^m_{\lambda,l}\left(J^{a,c}_{\mu}f(z)\right)=J^{a,c}_{\mu}\left(L^m_{\lambda,l}f(z)\right), \quad (6)$$

which can be easily expressed as follows:

$$I^m_{\lambda,l}(a,c,\mu)f(z)=z+\frac{\Gamma(c+\mu)}{\Gamma(a+\mu)}\sum_{n=2}^{\infty}\frac{\Gamma(a+n\mu)}{\Gamma(c+n\mu)}\left(\frac{l+1}{l+\lambda(n-1)+1}\right)^m a_n z^n, \quad (7)$$

$(\mu>0; a,c\in\mathbb{C}; \mathrm{Re}\{c-a\}\ge 0; \mathrm{Re}\{a\}>-\mu; \lambda>0; l>-1; m\in\mathbb{N}_0)$.

In view of (1.3), (1.5) and (1.6), it is clear that:

$I^0_{\lambda,l}(a,c,\mu)f(z)=J^{a,c}_{\mu}f(z)$ and $I^m_{\lambda,l}(a,a,\mu)f(z)=L^m_{\lambda,l}f(z)$.

The importance of the operator $I^m_{\lambda,l}(a,c,\mu)$ comes from its generalization of a lot of previous operators, as follows:

(i) $I^m_{\lambda,l}(\nu-1,0,1)f(z)=I^m_{\lambda,l,\nu}f(z)$ $(\lambda>0; l>-1; \nu>0; m\in\mathbb{N}_0)$ (see Aouf and El-Ashwah [2]);

(ii) $I^s_{1,l}(\nu-1,0,1)f(z)=I^s_{l,\nu}f(z)$ $(l>-1; \nu>0; s\in\mathbb{R})$ (see Cho and Kim [9]);

(iii) $I^m_{\lambda,0}(\nu-1,0,1)f(z)=I^m_{\lambda,\nu}f(z)$ $(\lambda>0; \nu>0; m\in\mathbb{Z})$ (see Aouf et al. [4]);

(iv) $I^{-n}_{\lambda,l}(a,a,\mu)f(z)=I^n(\lambda,l)f(z)$ $(\lambda>0; l>-1; n\in\mathbb{N}_0)$ (see Catas [8]);

(v) $I^m_{\lambda,l}(a,a,\mu)f(z)=J^m(\lambda,l)f(z)$ $(\lambda>0; l>-1; m\in\mathbb{N}_0)$ (see El-Ashwah and Aouf [14]);

(vi) $I^{-n}_{\lambda,0}(a,a,\mu)f(z)=I^n_{\lambda}f(z)$ $(\lambda>0; n\in\mathbb{Z})$ (see Patel [37]);

(vii) $I^{\nu}_{1,\alpha-1}(a,a,\mu)f(z)=L^{\nu}_{\alpha}f(z)$ $(\nu>0; \alpha>0)$ (see Komatu [21], see also Aouf [1]);

(viii) $I^{\sigma}_{1,1}(a,a,\mu)f(z)=L^{\sigma}f(z)$ $(\sigma>0)$ (see Jung et al. [19], see also Liu [24]);

(ix) $I^{\beta}_{1,1}(a,a,\mu)f(z)=L^{\beta}f(z)$ $(\beta\in\mathbb{Z})$ (see Uralegaddi and Somanatha [46], Flett [15]);

(x) $I^n_{1,0}(a,a,\mu)f(z)=I^n f(z)$ and $I^{-n}_{1,0}(a,a,\mu)f(z)=D^n f(z)$ $(n\in\mathbb{N}_0)$ (see Salagean [42]);

(xi) $I^{\nu}_{1,l}(a,a,\mu)f(z)=P^{\nu}_l f(z)$ $(\nu>0; l>-1)$ (see Gao et al. [16]);

(xii) $I^1_{1,\sigma}(a,a,\mu)f(z)=L_{\sigma}f(z)$ $(\sigma>0)$ (see Owa and Srivastava [36] and Srivastava and Owa [45]);

(xiii) $I^0_{\lambda,l}(\beta,\alpha+\beta-\gamma+1,1)f(z)=\Re^{\alpha,\gamma}_{\beta}f(z)$ $(\gamma>0; \alpha\ge\gamma-1; \beta>-1)$ (see Aouf et al. [3]);

(xiv) $I^0_{\lambda,l}(\beta,\alpha+\beta,1)f(z)=Q^{\alpha}_{\beta}f(z)$ $(\alpha\ge 0; \beta>-1)$ (see Liu and Owa [25], see also Jung et al. [19] and Li [22]);

(xv) $I_{\lambda,l}^{0}(a-1,c-1,1)f(z)=L(a,c)f(z)\left(a,c\in\mathbb{C}\setminus\mathbb{Z}_0^-,\ \mathbb{Z}_0^-=\{0,-1,-2,...\}\right)$ (see Carlson and Shaffer [7]);
(xvi) $I_{\lambda,l}^{0}(\nu-1,\eta,1)f(z)=I_{\eta,\nu}f(z)\left(\nu>0;\eta>-1\right)$ (see Choi et al. [11]);
(xvii) $I_{\lambda,l}^{0}(\alpha,0,1)f(z)=D^{\alpha}f(z)\left(\alpha>-1\right)$ (see Ruscheweyh [41]);
(xviii) $I_{\lambda,l}^{0}(1,n,1)f(z)=D^{n}f(z)\left(n\in\mathbb{N}_0\right)$ (see Noor [33] and Noor and Noor [35]).

Thus, the new results obtained in this paper can ensure the results obtained in the earlier works also introduce new results of the other well-known operators as special choices of the parameters $a,c,\mu,m,l,\lambda,\varphi,$ and ψ.

Using (1.7), we can obtain the following recurrence relations, which are needed for our proofs in following two sections:

$$z\left(I_{\lambda,l}^{m+1}(a,c,\mu)f(z)\right)'=\frac{1+l}{\lambda}I_{\lambda,l}^{m}(a,c,\mu)f(z)-\frac{1+l-\lambda}{\lambda}I_{\lambda,l}^{m+1}(a,c,\mu)f(z), \quad (8)$$

$$z\left(I_{\lambda,l}^{m}(a,c,\mu)f(z)\right)'=\frac{a+\mu}{\mu}I_{\lambda,l}^{m}(a+1,c,\mu)f(z)-\frac{a}{\mu}I_{\lambda,l}^{m}(a,c,\mu)f(z). \quad (9)$$

Definition 3. For $\mu>0,a,c\in\mathbb{C};\operatorname{Re}\{c-a\}\geq 0,$ $\operatorname{Re}\{a\}>-\mu,\lambda>0,l>-1,0\leq\alpha,\beta<1,m\in\mathbb{N}_0$ and the operator $I_{\lambda,l}^{m}(a,c,\mu)f(z)$ defined by (1.12), we introduce the following subclasses of the normalized analytic functions class $\mathbb{A}$, as follows:

$$S_{\lambda,l}^{*m}(\alpha;a,c,\mu;\varphi)=\left\{f:f(z)\in\mathbb{A}\text{ and }I_{\lambda,l}^{m}(a,c,\mu)f(z)\in S^{*}(\alpha,\varphi)\right\},$$

$$C_{\lambda,l}^{m}(\alpha;a,c,\mu;\varphi)=\left\{f:f(z)\in\mathbb{A}\text{ and }I_{\lambda,l}^{m}(a,c,\mu)f(z)\in C(\alpha;\varphi)\right\},$$

$$K_{\lambda,l}^{m}(\beta,\alpha;a,c,\mu;\psi,\varphi)=\left\{f:f(z)\in\mathbb{A}\text{ and }I_{\lambda,l}^{m}(a,c,\mu)f(z)\in K(\beta,\alpha;\psi,\varphi)\right\},$$

and

$$K_{\lambda,l}^{*m}(\beta,\alpha;a,c,\mu;\psi,\varphi)=\left\{f:f(z)\in\mathbb{A}\text{ and }I_{\lambda,l}^{m}(a,c,\mu)f(z)\in K^{*}(\beta,\alpha;\psi,\varphi)\right\}.$$

For the subclasses defined above, we note that:

$$f(z)\in C_{\lambda,l}^{m}(\alpha;a,c,\mu;\varphi)\Leftrightarrow zf'(z)\in S_{\lambda,l}^{*m}(\alpha;a,c,\mu;\varphi), \quad (10)$$

and

$$f(z)\in K_{\lambda,l}^{*m}(\beta,\alpha;a,c,\mu;\psi,\varphi)\Leftrightarrow zf'(z)\in K_{\lambda,l}^{m}(\beta,\alpha;a,c,\mu;\psi,\varphi). \quad (11)$$

Remark 1. If we set $a=c$ in Definition 1, we obtain the following subclasses of $\mathbb{A}$:

$$S_{\lambda,l}^{*m}(\alpha;\varphi)=\left\{f\ :\ f(z)\in\mathbb{A}\text{ and }L_{\lambda,l}^{m}f(z)\in S^{*}(\alpha,\varphi)\right\},$$

$$C_{\lambda,l}^{m}(\alpha;\varphi)=\left\{f\ :\ f(z)\in\mathbb{A}\text{ and }L_{\lambda,l}^{m}f(z)\in C(\alpha;\varphi)\right\},$$

$$K_{\lambda,l}^{m}(\beta,\alpha;\psi,\varphi)=\left\{f\ :\ f(z)\in\mathbb{A}\text{ and }L_{\lambda,l}^{m}f(z)\in K(\beta,\alpha;\psi,\varphi)\right\},$$

$$K_{\lambda,l}^{*m}(\beta,\alpha;\psi,\varphi)=\left\{f\ :\ f(z)\in\mathbb{A}\text{ and }L_{\lambda,l}^{m}f(z)\in K^{*}(\beta,\alpha;\psi,\varphi)\right\}.$$

Where $L_{\lambda,l}^{m}f(z)$ is defined by (1.7).

Remark 2. If we set $m=0$ in Definition 1, we obtain the following subclasses of $\mathbb{A}$:

$$S^{*}(\alpha;a,c,\mu;\varphi)=\left\{f\ :\ f(z)\in\mathbb{A}\text{ and }J_{\mu}^{a,c}f(z)\in S^{*}(\alpha;\varphi)\right\},$$

$$C(\alpha;a,c,\mu;\varphi)=\left\{f\ :\ f(z)\in\mathbb{A}\text{ and }J_{\mu}^{a,c}f(z)\in C(\alpha;\varphi)\right\},$$

$$K(\beta,\alpha;a,c,\mu;\psi,\varphi)=\left\{f\ :\ f(z)\in\mathbb{A}\text{ and }J_{\mu}^{a,c}f(z)\in K(\beta,\alpha;\psi,\varphi)\right\},$$

$$K^{*}(\beta,\alpha;a,c,\mu;\psi,\varphi)=\left\{f\ :\ f(z)\in\mathbb{A}\text{ and }J_{\mu}^{a,c}f(z)\in K^{*}(\beta,\alpha;\psi,\varphi)\right\}.$$

Where $J_{\mu}^{a,c}f(z)$ is defined by (1.10).

In order to introduce our main results, we shall need the following lemmas.

Lemma 1 (see [12]). *Let h be a convex univalent function in U with $h(0)=1$ and $\operatorname{Re}\{\mu h(z)+\nu\}>0$ $(\mu,\nu\in\mathbb{C})$. If p is an analytic function in U with $p(0)=1$, then*

$$p(z)+\frac{zp'(z)}{\mu p(z)+\nu}\prec h(z);\quad z\in U,$$

implies that

$$p(z)\prec h(z);\quad z\in U.$$

Lemma 2 (see [29] and [30]). *Let h be a convex function in U* with $h(0)=1$. *Suppose also that w be an analytic function in U with $\operatorname{Re}\{w(z)\}\geq 0$ $\left(z\in U\right)$. If p is an analytic function in U with $p(0)=1$, then*

$$p(z)+w(z)zp'(z)\prec h(z);\quad z\in U,$$

implies that

$$p(z)\prec h(z);\quad z\in U.$$

2 Inclusion Relationships

Unless otherwise mentioned, we shall assume throughout the paper that $\mu>0,$ $a,c\in\mathbb{C},$

$\mathrm{Re}\{c-a\}\geq 0$, $\mathrm{Re}\{a\}>-\mu$, $\lambda>0$, $l>-1$, $0\leq\alpha,\beta<1$, $m\in\mathbb{N}_0$, $f(z)\in\mathbb{A}$ and $\varphi(z),\psi(z)\in M$. In this section, we give several inclusion relationships for analytic function classes, which are associated with the generalized operator $I_{\lambda,l}^{m}(a,c,\mu)$ defined by (1.12).

Theorem 1. *Let* $\max_{z\in U}\mathrm{Re}\{\varphi(z)\}<\min\left\{\frac{\mathrm{Re}\left\{\frac{a}{\mu}\right\}+\alpha}{\alpha-1},\frac{\frac{1+l-\lambda}{\lambda}+\alpha}{\alpha-1}\right\}.$

Then

$$S_{\lambda,l}^{*m}(\alpha;a+1,c,\mu;\varphi)\subset S_{\lambda,l}^{*m}(\alpha;a,c,\mu;\varphi)\subset S_{\lambda,l}^{*m+1}(\alpha;a,c,\mu;\varphi). \quad (12)$$

Proof. We begin with proving that

$$S_{\lambda,l}^{*m}(\alpha;a+1,c,\mu;\varphi)\subset S_{\lambda,l}^{*m}(\alpha;a,c,\mu;\varphi). \quad (13)$$

Let $f(z)\in S_{\lambda,l}^{*m}(\alpha;a+1,c,\mu;\varphi)$ and set

$$\frac{1}{1-\alpha}\left(\frac{z\left(I_{\lambda,l}^{m}(a,c,\mu)f(z)\right)'}{I_{\lambda,l}^{m}(a,c,\mu)f(z)}-\alpha\right)=p(z), \quad (14)$$

where $p(z)=1+p_1z+p_2z^2+\ldots$ is analytic in U and $p(z)\neq 0$ for all $z\in U$. Applying (9) and (14), we obtain

$$\frac{a+\mu}{\mu}\frac{I_{\lambda,l}^{m}(a+1,c,\mu)f(z)}{I_{\lambda,l}^{m}(a,c,\mu)f(z)}=(1-\alpha)p(z)+\frac{a}{\mu}+\alpha. \quad (15)$$

By using the logarithmic differentiation on both side of (15), we obtain

$$\frac{z\left(I_{\lambda,l}^{m}(a+1,c,\mu)f(z)\right)'}{I_{\lambda,l}^{m}(a+1,c,\mu)f(z)}=\frac{z\left(I_{\lambda,l}^{m}(a,c,\mu)f(z)\right)'}{I_{\lambda,l}^{m}(a,c,\mu)f(z)}+\frac{(1-\alpha)zp'(z)}{(1-\alpha)p(z)+\frac{a}{\mu}+\alpha},$$

by using (14) again, we have

$$\frac{1}{1-\alpha}\left(\frac{z\left(I_{\lambda,l}^{m}(a+1,c,\mu)f(z)\right)'}{I_{\lambda,l}^{m}(a+1,c,\mu)f(z)}-\alpha\right)=p(z)+\frac{zp'(z)}{(1-\alpha)p(z)+\frac{a}{\mu}+\alpha}. \quad (16)$$

Since $\mathrm{Re}\{\varphi(z)\}<\frac{\mathrm{Re}\left\{\frac{a}{\mu}\right\}+\alpha}{\alpha-1}$ for all $z\in U$ and $f(z)\in S_{\lambda,l}^{*m}(\alpha;a+1,c,\mu;\varphi)$, from (16) we see that

$$\mathrm{Re}\left\{(1-\alpha)\varphi(z)+\frac{a}{\mu}+\alpha\right\}>0 \quad (z\in U),$$

and

$$p(z)+\frac{zp'(z)}{(1-\alpha)p(z)+\frac{a}{\mu}+\alpha}\prec\varphi(z) \quad (z\in U).$$

Thus, by using Lemma 1 and (14), we observe that

$$p(z)\prec\phi(z) \quad (z\in U),$$

which implies that

$$f(z)\in S_{\lambda,l}^{*m}(\alpha;a,c,\mu;\varphi),$$

which proves the first inclusion relationship (13). Now, we prove the second inclusion relationship, asserted as following

$$S_{\lambda,l}^{*m}(\alpha;a,c,\mu;\varphi)\subset S_{\lambda,l}^{*m+1}(\alpha;a,c,\mu;\varphi). \quad (17)$$

Let $f(z)\in S_{\lambda,l}^{*m}(\alpha;a,c,\mu;\varphi)$ and set

$$\frac{1}{1-\alpha}\left(\frac{z\left(I_{\lambda,l}^{m+1}(a,c,\mu)f(z)\right)'}{I_{\lambda,l}^{m+1}(a,c,\mu)f(z)}-\alpha\right)=q(z), \quad (18)$$

where $q(z)=1+q_1z+q_2z^2+\ldots$ is analytic in U and $q(z)\neq 0$ for all $z\in U$. Then, by using arguments similar to those detailed above with (8), it follows that

$$q(z)\prec\varphi(z) \quad (z\in U),$$

which implies that

$$f(z)\in S_{\lambda,l}^{*m+1}(\alpha;a,c,\mu;\varphi),$$

which proves the second inclusion relationship (17). Combining the inclusion relationships (13) and (17), we complete the proof of Theorem 1.

Theorem 2. *Let* $\max_{z\in U}\mathrm{Re}\{\varphi(z)\}<\min\left\{\frac{\mathrm{Re}\left\{\frac{a}{\mu}\right\}+\alpha}{\alpha-1},\frac{\frac{1+l-\lambda}{\lambda}+\alpha}{\alpha-1}\right\}.$

Then

$$C_{\lambda,l}^{m}(\alpha;a+1,c,\mu;\varphi)\subset C_{\lambda,l}^{m}(\alpha;a,c,\mu;\varphi)\subset C_{\lambda,l}^{m+1}(\alpha;a,c,\mu;\varphi). \quad (19)$$

Proof. Applying (10) and Theorem 1, we observe that

$$\begin{aligned}
&f(z)\in C_{\lambda,l}^{m}(\alpha;a+1,c,\mu;\varphi)\\
&\Leftrightarrow I_{\lambda,l}^{m}(a+1,c,\mu)f(z)\in C(\alpha;\varphi)\\
&\Leftrightarrow z(I_{\lambda,l}^{m}(a+1,c,\mu)f(z))'\in S^{*}(\alpha;\varphi)\\
&\Leftrightarrow I_{\lambda,l}^{m}(a+1,c,\mu)\left(zf'(z)\right)\in S^{*}(\alpha;\varphi)\\
&\Leftrightarrow zf'(z)\in S_{\lambda,l}^{*m}(\alpha;a+1,c,\mu;\varphi)\\
&\Rightarrow zf'(z)\in S_{\lambda,l}^{*m}(\alpha;a,c,\mu;\varphi)\\
&\Leftrightarrow I_{\lambda,l}^{m}(a,c,\mu)\left(zf'(z)\right)\in S^{*}(\alpha;\varphi)\\
&\Leftrightarrow z(I_{\lambda,l}^{m}(a,c,\mu)f(z))'\in S^{*}(\alpha;\varphi)\\
&\Leftrightarrow I_{\lambda,l}^{m}(a,c,\mu)f(z)\in C(\alpha;\varphi)\\
&\Leftrightarrow f(z)\in C_{\lambda,l}^{m}(\alpha;a,c,\mu;\varphi),
\end{aligned}$$

and

$$f(z)\in C_{\lambda,l}^{m}(\alpha;a,c,\mu;\varphi)$$
$$\Leftrightarrow zf^{'}(z)\in S_{\lambda,l}^{*m}(\alpha;a,c,\mu;\varphi)$$
$$\Rightarrow zf^{'}(z)\in S_{\lambda,l}^{*m+1}(\alpha;a,c,\mu;\varphi)$$
$$\Leftrightarrow z(I_{\lambda,l}^{m+1}(a,c,\mu)f(z))^{'}\in S^{*}(\alpha;\varphi)$$
$$\Leftrightarrow I_{\lambda,l}^{m+1}(a,c,\mu)f(z)\in C(\alpha;\varphi)$$
$$\Leftrightarrow f(z)\in C_{\lambda,l}^{m+1}(\alpha;a,c,\mu;\varphi).$$

Which evidently proves Theorem 2.

Theorem 3. *Let* $\max_{z\in U}\operatorname{Re}\{\varphi(z)\}<\min\left\{\frac{\operatorname{Re}\left\{\frac{a}{\mu}\right\}+\alpha}{\alpha-1},\frac{\frac{1+l-\lambda}{\lambda}+\alpha}{\alpha-1}\right\}.$ *Then*

$$K_{\lambda,l}^{m}(\beta,\alpha;a+1,c,\mu;\psi,\varphi)\subset K_{\lambda,l}^{m}(\beta,\alpha;a,c,\mu;\psi,\varphi)\subset K_{\lambda,l}^{m+1}(\beta,\alpha;a,c,\mu;\psi,\varphi). \tag{20}$$

Proof. We begin with proving that

$$K_{\lambda,l}^{m}(\beta,\alpha;a+1,c,\mu;\psi,\varphi)\subset K_{\lambda,l}^{m}(\beta,\alpha;a,c,\mu;\psi,\varphi). \tag{21}$$

Let $f(z)\in K_{\lambda,l}^{m}(\beta,\alpha;a+1,c,\mu;\psi,\varphi)$. Then, there exists a function $r(z)\in S^{*}(\alpha;\varphi)$ such that

$$\frac{1}{1-\beta}\left(\frac{z\left(I_{\lambda,l}^{m}(a+1,c,\mu)f(z)\right)^{'}}{r(z)}-\beta\right)\prec\psi(z)\ \left(z\in U\right).$$

Choose the function $g(z)$ such that $I_{\lambda,l}^{m}(a+1,c,\mu)g(z)=r(z)$, so that we have $g(z)\in S_{\lambda,l}^{*m}(\alpha;a+1,c,\mu;\varphi)$ and

$$\frac{1}{1-\beta}\left(\frac{z\left(I_{\lambda,l}^{m}(a+1,c,\mu)f(z)\right)^{'}}{I_{\lambda,l}^{m}(a+1,c,\mu)g(z)}-\beta\right)\prec\psi(z)\ \left(z\in U\right). \tag{22}$$

Next, we set

$$\frac{1}{1-\beta}\left(\frac{z\left(I_{\lambda,l}^{m}(a,c,\mu)f(z)\right)^{'}}{I_{\lambda,l}^{m}(a,c,\mu)g(z)}-\beta\right)=p(z), \tag{23}$$

where $p(z)=1+p_1z+p_2z^2+...$ is analytic in U and $p(z)\neq 0$ for all $z\in U$. Thus, by using the identity (9), we obtain

$$\frac{1}{1-\beta}\left(\frac{z\left(I_{\lambda,l}^{m}(a+1,c,\mu)f(z)\right)^{'}}{I_{\lambda,l}^{m}(a+1,c,\mu)g(z)}-\beta\right)=\frac{1}{1-\beta}\left(\frac{I_{\lambda,l}^{m}(a+1,c,\mu)\left(zf^{'}(z)\right)}{I_{\lambda,l}^{m}(a+1,c,\mu)g(z)}-\beta\right)$$

$$=\frac{1}{1-\beta}\left(\frac{z\left(I_{\lambda,l}^{m}(a,c,\mu)\left(zf^{'}(z)\right)\right)^{'}+\frac{a}{\mu}I_{\lambda,l}^{m}(a,c,\mu)\left(zf^{'}(z)\right)}{z\left(I_{\lambda,l}^{m}(a,c,\mu)g(z)\right)^{'}+\frac{a}{\mu}I_{\lambda,l}^{m}(a,c,\mu)g(z)}-\beta\right)$$

$$=\frac{1}{1-\beta}\left(\frac{\frac{z\left(I_{\lambda,l}^{m}(a,c,\mu)\left(zf^{'}(z)\right)\right)^{'}}{I_{\lambda,l}^{m}(a,c,\mu)g(z)}+\frac{a}{\mu}\frac{I_{\lambda,l}^{m}(a,c,\mu)\left(zf^{'}(z)\right)}{I_{\lambda,l}^{m}(a,c,\mu)g(z)}}{\frac{z\left(I_{\lambda,l}^{m}(a,c,\mu)g(z)\right)^{'}}{I_{\lambda,l}^{m}(a,c,\mu)g(z)}+\frac{a}{\mu}}-\beta\right). \tag{24}$$

Moreover, since

$$g(z)\in S_{\lambda,l}^{*m}(\alpha;a+1,c,\mu;\varphi)\subset S_{\lambda,l}^{*m}(\alpha;a,c,\mu;\varphi),$$

by using Theorem 1, we can put

$$\frac{1}{1-\alpha}\left(\frac{z\left(I_{\lambda,l}^{m}(a,c,\mu)g(z)\right)^{'}}{I_{\lambda,l}^{m}(a,c,\mu)g(z)}-\alpha\right)=G(z), \tag{25}$$

where $G(z)\prec\varphi(z)\quad\left(z\in U\right)$. Then, by virtue of (23) and (24), we observe that

$$I_{\lambda,l}^{m}(a,c,\mu)\left(zf^{'}(z)\right)=\left[\left(1-\beta\right)p(z)+\beta\right]\left(I_{\lambda,l}^{m}(a,c,\mu)g(z)\right) \tag{26}$$

and

$$\frac{1}{1-\beta}\left(\frac{z\left(I_{\lambda,l}^{m}(a+1,c,\mu)f(z)\right)^{'}}{I_{\lambda,l}^{m}(a+1,c,\mu)g(z)}-\beta\right)=\frac{1}{1-\beta}\left(\frac{\frac{z\left(I_{\lambda,l}^{m}(a,c,\mu)\left(zf^{'}(z)\right)\right)^{'}}{I_{\lambda,l}^{m}(a,c,\mu)g(z)}+\frac{a}{\mu}\left[\left(1-\beta\right)p(z)+\beta\right]}{\left[\left(1-\alpha\right)G(z)+\alpha\right]+\frac{a}{\mu}}-\beta\right). \tag{27}$$

Upon differentiating both sides of (26), we have

$$\frac{z\left(I_{\lambda,l}^{m}(a,c,\mu)\left(zf^{'}(z)\right)\right)^{'}}{I_{\lambda,l}^{m}(a,c,\mu)g(z)}=\left(1-\beta\right)zp^{'}(z)+\left[\left(1-\beta\right)p(z)+\beta\right]\left[\left(1-\alpha\right)G(z)+\alpha\right] \tag{28}$$

Making use of (22), (27), and (28), we get

$$\frac{1}{1-\beta}\left(\frac{z\left(I_{\lambda,l}^{m}(a+1,c,\mu)f(z)\right)^{'}}{I_{\lambda,l}^{m}(a+1,c,\mu)g(z)}-\beta\right)=p(z)+\frac{zp^{'}(z)}{\left(1-\alpha\right)G(z)+\alpha+\frac{a}{\mu}}\prec\psi(z)\ \left(z\in U\right). \tag{29}$$

Using $\max_{z\in U}\operatorname{Re}\{\varphi(z)\}<\frac{\operatorname{Re}\left\{\frac{a}{\mu}\right\}+\alpha}{\alpha-1}$ and $G(z)\prec\varphi(z)$ $\left(\varphi\in M,z\in U\right)$, then we have

$$\mathrm{Re}\left\{(1-\alpha)G(z)+\alpha+\frac{a}{\mu}\right\}>0\ \left(z\in U\right).$$

Hence, upon taking

$$w(z)=\frac{1}{(1-\alpha)G(z)+\alpha+\frac{a}{\mu}}$$

in (29), and applying Lemma 2, we obtain that

$$p(z)\prec\psi(z)\ \left(z\in U\right),$$

then, in view of (23) we deduce that $f(z)\in K_{\lambda,l}^{m}(\beta,\alpha;a,c,\mu;\psi,\varphi)$ which proves (21).For the second part, by using arguments similar to those detailed above with (8), thus we choose to omit the details. The proof of Theorem 3 is completed.

Theorem 4. *Let* $\max_{z\in U}\ \mathrm{Re}\{\varphi(z)\}<\min\left\{\frac{\mathrm{Re}\left\{\frac{a}{\mu}\right\}+\alpha}{\alpha-1},\frac{\frac{1+l-\lambda}{\lambda}+\alpha}{\alpha-1}\right\}.$

Then

$$K_{\lambda,l}^{*m}(\beta,\alpha;a+1,c,\mu;\psi,\varphi)$$
$$\subset K_{\lambda,l}^{*m}(\beta,\alpha;a,c,\mu;\psi,\varphi)$$
$$\subset K_{\lambda,l}^{*m+1}(\beta,\alpha;a,c,\mu;\psi,\varphi). \qquad (30)$$

Proof. Just, as we derived Theorem 2 as a consequence of Theorem 1 by using the equivalence (10). Similarly, we can prove Theorem 4 as a consequence of Theorem 3 in conjunction with the equivalence (11). Therefore, again, we choose to omit the details involved.

Remark 3. (i) Taking $a=\nu-1\left(\nu>0\right)$, $c=0$ and $\mu=1$ in Theorems 1-3, we obtain the results obtained by Aouf and El-Ashwah [2, Theorems 1-3];
(ii) Taking $m=s\left(s\in\mathbb{R}\right),\lambda=1,a=\nu-1\left(\nu>0\right)$, $c=0$ and $\mu=1$ in Theorems 1-3, we obtain the results obtained by Cho and Kim [9, Theorems 2.1-2.3];
(iii) Taking $l=0,a=\nu-1\left(\nu>0\right),c=0$, and $\mu=1$ in Theorems 1-3, we obtain the results obtained by Aouf et al. [4, Theorems 1-3].

Taking $a=c$ in Theorems 1-4, we obtain the following corollary.

Corollary 1. For the subclasses $S_{\lambda,l}^{*m}(\alpha;\varphi)$, $C_{\lambda,l}^{m}(\alpha;\varphi)$, $K_{\lambda,l}^{m}(\beta,\alpha;\psi,\varphi)$ and $K_{\lambda,l}^{*m}(\beta,\alpha;\psi,\varphi)$ defined in Remark 1, we have the following inclusion relations.

$$S_{\lambda,l}^{*m}(\alpha;\varphi)\subset S_{\lambda,l}^{*m+1}(\alpha;\varphi),$$
$$C_{\lambda,l}^{m}(\alpha;\varphi)\subset C_{\lambda,l}^{m+1}(\alpha;\varphi),$$
$$K_{\lambda,l}^{m}(\beta,\alpha;\psi,\varphi)\subset K_{\lambda,l}^{m+1}(\beta,\alpha;\psi,\varphi),$$
$$K_{\lambda,l}^{*m}(\beta,\alpha;\psi,\varphi)\subset K_{\lambda,l}^{*m+1}(\beta,\alpha;\psi,\varphi).$$

Remark 4. (i) Taking $\lambda=1,m=\mu\left(\mu>0\right),l=a-1\left(a>0\right)$ and $\varphi(z)=\psi(z)=\frac{1+z}{1-z}$ in Corollary 1, we obtain the results obtained by Aouf [1, Theorems 1-4];
(ii) Taking $\lambda=l=1,m=\sigma\left(\sigma>0\right)$ and $\varphi(z)=\psi(z)=\frac{1+z}{1-z}$ in Corollary 1, we obtain the results obtained by Liu [24, Theorems 1-4].

Taking $m=0$ in Theorems 1-4, we obtain the following corollary.

Corollary 2. For the subclasses $S^{*}(\alpha;a,c,\mu;\varphi)$, $C(\alpha;a,c,\mu;\varphi)$, $K(\beta,\alpha;a,c,\mu;\psi,\varphi)$ and $K^{*}(\beta,\alpha;a,c,\mu;\psi,\varphi)$ defined in Remark 2, we have the following inclusion relations.

$$S^{*}(\alpha;a+1,c,\mu;\varphi)\subset S^{*}(\alpha;a,c,\mu;\varphi),$$
$$C(\alpha;a+1,c,\mu;\varphi)\subset C(\alpha;a,c,\mu;\varphi),$$
$$K(\beta,\alpha;a+1,c,\mu;\psi,\varphi)\subset K(\beta,\alpha;a,c,\mu;\psi,\varphi),$$
$$K^{*}(\beta,\alpha;a+1,c,\mu;\psi,\varphi)\subset K^{*}(\beta,\alpha;a,c,\mu;\psi,\varphi).$$

Remark 5. Taking $\alpha=\beta=0$, $a=\nu-1\left(\nu>0\right)$, $c=\lambda\left(\lambda>-1\right)$ and $\mu=1$ in Corollary 2, we obtain the results obtained by Choi et al. [11, Theorems 1-3].

3 Integral-Preserving Properties

Now, we recall the definition of the generalized Bernardi-Libera-Livingston integral operator L_{σ} : $\mathbb{A}\to\mathbb{A}$, as following (see [36]):

$$L_{\sigma}f(z)=\frac{\sigma+1}{z^{\sigma}}\int_{0}^{z}t^{\sigma-1}f(t)dt\ \ (\sigma>-1,\ f(z)\in\mathbb{A}).\ (31)$$

The operator $L_{\sigma}f(z)\left(\sigma\in\mathbb{N}\right)$ was introduced by Bernardi [5]. In particular, the operator $L_{1}f(z)$ was studied earlier by Libera [23] and

Livingston [26]. Using (7) and (31), it is clear that $L_\sigma f(z)$ satisfies the following relationship:

$$z\left(I_{\lambda,l}^{m}(a,c,\mu)L_\sigma f(z)\right)' = \left(\sigma+1\right)I_{\lambda,l}^{m}(a,c,\mu)f(z)-\sigma I_{\lambda,l}^{m}(a,c,\mu)L_\sigma f(z). \quad (32)$$

Now, we begin the Integral-preserving property involving the integral operator L_σ by the following theorem.

Theorem 5. *Let* $\max_{z\in U} \operatorname{Re}\{\varphi(z)\} < \frac{\alpha+\sigma}{\alpha-1}$. *If* $f(z)\in S_{\lambda,l}^{*m}(\alpha;a,c,\mu;\varphi)$, *then*

$$L_\sigma f(z)\in S_{\lambda,l}^{*m}(\alpha;a,c,\mu;\varphi).$$

proof. Let $f(z)\in S_{\lambda,l}^{*m}(\alpha;a,c,\mu;\varphi)$ and set

$$\frac{1}{1-\alpha}\left(\frac{z\left(I_{\lambda,l}^{m}(a,c,\mu)L_\sigma f(z)\right)'}{I_{\lambda,l}^{m}(a,c,\mu)L_\sigma f(z)}-\alpha\right)=p(z), \quad (33)$$

where $p(z)=1+p_1z+p_2z^2+...$ is analytic in U and $p(z)\neq 0$ for all $z\in U$. By applying (32) and (33), we have

$$\left(\sigma+1\right)\frac{I_{\lambda,l}^{m}(a,c,\mu)f(z)}{I_{\lambda,l}^{m}(a,c,\mu)L_\sigma f(z)}=\left(1-\alpha\right)p(z)+\alpha+\sigma. \quad (34)$$

By using the logarithmic differentiation on both side of (34), we have

$$\frac{1}{1-\alpha}\left(\frac{z\left(I_{\lambda,l}^{m}(a,c,\mu)f(z)\right)'}{I_{\lambda,l}^{m}(a,c,\mu)f(z)}-\alpha\right) = p(z)+\frac{zp'(z)}{\left(1-\alpha\right)p(z)+\alpha+\sigma}. \quad (35)$$

Since $\max_{z\in U} \operatorname{Re}\{\varphi(z)\} < \frac{\alpha+\sigma}{\alpha-1}$ and $f(z)\in S_{\lambda,l}^{*m}(\alpha;a,c,\mu;\varphi)$, from (35), we have

$$\operatorname{Re}\left\{\left(1-\alpha\right)\varphi(z)+\alpha+\sigma\right\}>0$$

and

$$p(z)+\frac{zp'(z)}{\left(1-\alpha\right)p(z)+\alpha+\sigma}\prec\varphi(z) \quad (z\in U).$$

Hence, by Using Lemma 1, we obtain

$$p(z)\prec\varphi(z) \quad (z\in U),$$

then, in view of (33) we deduce that $L_\sigma f(z)\in S_{\lambda,l}^{*m}(\alpha;a,c,\mu;\varphi)$, which completes the proof of Theorem 5.

Taking $a=c$ in Theorem 5, we obtain the following corollary.

Corollary 3. *Let* $\max_{z\in U} \operatorname{Re}\{\varphi(z)\} < \frac{\alpha+\sigma}{\alpha-1}$. *If* $f(z)\in S_{\lambda,l}^{*m}(\alpha;\varphi)$, *then* $L_\sigma f(z)\in S_{\lambda,l}^{*m}(\alpha;\varphi)$.

Remark 6. (i) Taking $\lambda=1$, $m=\nu\left(\nu>0\right)$, $l=a-1\left(a>0\right)$ and $\varphi(z)=\frac{1+z}{1-z}$ in Corollary 3, we obtain the results obtained by Aouf [1, Theorem 5];
(ii) Taking $\lambda=l=1$, $m=\sigma\left(\sigma>0\right)$ and $\varphi(z)=\frac{1+z}{1-z}$ in Corollary 3, we obtain the results obtained by Liu [24, Theorem 5].

Taking $m=0$ in Theorem 5, we obtain the following corollary.

Corollary 4. *Let* $\max_{z\in U} \operatorname{Re}\{\varphi(z)\} < \frac{\alpha+\sigma}{\alpha-1}$. *If* $f(z)\in S^*(\alpha;a,c,\mu;\varphi)$, *then* $L_\sigma f(z)\in S^*(\alpha;a,c,\mu;\varphi)$.

Remark 7. Taking $\alpha=0$, $a=\nu-1\left(\nu>0\right)$, $c=\lambda\left(\lambda>-1\right)$ and $\mu=1$ in Corollary 4, we obtain the results obtained by Choi et al. [11, Theorem 4].

The next Integral-preserving property involving the integral operator L_σ is given by the following theorem

Theorem 6. *Let* $\max_{z\in U} \operatorname{Re}\{\varphi(z)\} < \frac{\alpha+\sigma}{\alpha-1}$. *If* $f(z)\in C_{\lambda,l}^{m}(\alpha;a,c,\mu;\varphi)$, *then* $L_\sigma f(z)\in C_{\lambda,l}^{m}(\alpha;a,c,\mu;\varphi)$.

Proof. Applying (10) and Theorem 5, we observe that

$$\begin{aligned} f(z)&\in C_{\lambda,l}^{m}(\alpha;a,c,\mu;\varphi)\\ &\Leftrightarrow zf'(z)\in S_{\lambda,l}^{*m}(\alpha;a,c,\mu;\varphi)\\ &\Rightarrow L_\sigma\left(zf'(z)\right)\in S_{\lambda,l}^{*m}(\alpha;a,c,\mu;\varphi)\\ &\Leftrightarrow z\left(L_\sigma f(z)\right)'\in S_{\lambda,l}^{*m}(\alpha;a,c,\mu;\varphi)\\ &\Leftrightarrow L_\sigma f(z)\in C_{\lambda,l}^{m}(\alpha;a,c,\mu;\varphi). \end{aligned}$$

The proof of Theorem 6 is evidently completed.

Taking $a=c$ in Theorem 6, we obtain the following corollary.

Corollary 5. *Let* $\max_{z\in U} \operatorname{Re}\{\varphi(z)\} < \frac{\alpha+\sigma}{\alpha-1}$. *If* $f(z)\in C_{\lambda,l}^{m}(\alpha;\varphi)$, *then* $L_\sigma f(z)\in C_{\lambda,l}^{m}(\alpha;\varphi)$.

Remark 8. (i) Taking $\lambda=1,\ m=\mu(\mu>0)$, $l=a-1(a>0)$ and $\varphi(z)=\frac{1+z}{1-z}$ in Corollary 5, we obtain the results obtained by Aouf [1, Theorem 6];
(ii) Taking $\lambda=l=1, m=\sigma(\sigma>0)$ and $\varphi(z)=\frac{1+z}{1-z}$ in Corollary 5, we obtain the results obtained by Liu [24, Theorem 6].

Taking $m=0$ in Theorem 6, we obtain the following corollary.

Corollary 6. *Let* $\max_{z\in U} \operatorname{Re}\{\varphi(z)\} < \frac{\alpha+\sigma}{\alpha-1}$. *If* $f(z)\in C(\alpha;a,c,\mu;\varphi)$, *then* $L_\sigma f(z)\in C(\alpha;a,c,\mu;\varphi)$.

Remark 9. Taking $\alpha=0$, $a=\mu-1(\mu>0)$, $c=\lambda(\lambda>-1)$ and $\mu=1$ in Corollary 6, we obtain the results obtained by Choi et al. [11, Theorem 5].

Also, an Integral-preserving property involving the integral operator L_σ is given by the following theorem.

Theorem 7. *Let* $\max_{z\in U} \operatorname{Re}\{\varphi(z)\} < \frac{\alpha+\sigma}{\alpha-1}$. *If* $f(z)\in K_{\lambda,l}^{m}(\beta,\alpha;a,c,\mu;\psi,\varphi)$, *then*

$$L_\sigma f(z)\in K_{\lambda,l}^{m}(\beta,\alpha;a,c,\mu;\psi,\varphi).$$

Proof. Let $f(z)\in K_{\lambda,l}^{m}(\beta,\alpha;a,c,\mu;\psi,\varphi)$. Then, in view of (1.4), there exists a function $g(z)\in S_{\lambda,l}^{*m}(\alpha;a,c,\mu;\varphi)$ and

$$\frac{1}{1-\beta}\left(\frac{z\left(I_{\lambda,l}^{m}(a,c,\mu)f(z)\right)'}{I_{\lambda,l}^{m}(a,c,\mu)g(z)}-\beta\right)\prec\psi(z)\quad(z\in U).\tag{36}$$

Set

$$\frac{1}{1-\beta}\left(\frac{z\left(I_{\lambda,l}^{m}(a,c,\mu)L_\sigma f(z)\right)'}{I_{\lambda,l}^{m}(a,c,\mu)L_\sigma g(z)}-\beta\right)=p(z)\tag{37}$$

where $p(z)=1+p_1z+p_2z^2+\ldots$ is analytic in U and $p(z)\neq0$ for all $z\in U$. Applying (33), we obtain

$$\frac{1}{1-\beta}\left(\frac{z\left(I_{\lambda,l}^{m}(a,c,\mu)f(z)\right)'}{I_{\lambda,l}^{m}(a,c,\mu)g(z)}-\beta\right)$$

$$=\frac{1}{1-\beta}\left(\frac{I_{\lambda,l}^{m}(a,c,\mu)\left(zf'(z)\right)}{I_{\lambda,l}^{m}(a,c,\mu)g(z)}-\beta\right)$$

$$=\frac{1}{1-\beta}\left(\frac{z\left(I_{\lambda,l}^{m}(a,c,\mu)L_\sigma\left(zf'(z)\right)\right)'+\sigma\left(I_{\lambda,l}^{m}(a,c,\mu)L_\sigma\left(zf'(z)\right)\right)}{z\left(I_{\lambda,l}^{m}(a,c,\mu)L_\sigma g(z)\right)'+\sigma I_{\lambda,l}^{m}(a,c,\mu)L_\sigma g(z)}-\beta\right)$$

$$=\frac{1}{1-\beta}\left(\frac{\frac{z\left(I_{\lambda,l}^{m}(a,c,\mu)L_\sigma\left(zf'(z)\right)\right)'}{I_{\lambda,l}^{m}(a,c,A)L_\sigma g(z)}+\sigma\frac{\left(I_{\lambda,l}^{m}(a,c,\mu)L_\sigma\left(zf'(z)\right)\right)}{I_{\lambda,l}^{m}(a,c,A)L_\sigma g(z)}}{\frac{z\left(I_{\lambda,l}^{m}(a,c,A)L_\sigma g(z)\right)'}{I_{\lambda,l}^{m}(a,c,A)L_\sigma g(z)}+\sigma}-\beta\right).\tag{38}$$

Since $g(z)\in S_{\lambda,l}^{*m}(\alpha;a,c,\mu;\varphi)$, by using Theorem 5, we have $L_\sigma g(z)\in S_{\lambda,l}^{*m}(\alpha;a,c,\mu;\varphi)$, then we obtain

$$\frac{1}{1-\alpha}\left(\frac{z\left(I_{\lambda,l}^{m}(a,c,\mu)L_\sigma g(z)\right)'}{I_{\lambda,l}^{m}(a,c,\mu)L_\sigma g(z)}-\alpha\right)=H(z)\prec\varphi(z)\left(z\in U\right).\tag{39}$$

Then, by using the same techniques as in the proof of Theorem 3, we conclude from (36), (37), (38) and (39) that

$$\frac{1}{1-\beta}\left(\frac{z\left(I_{\lambda,l}^{m}(a,c,\mu)f(z)\right)'}{I_{\lambda,l}^{m}(a,c,\mu)g(z)}-\beta\right)$$

$$=p(z)+\frac{zp'(z)}{(1-\alpha)H(z)+\alpha+\sigma}\prec\psi(z)\quad\left(z\in U\right).\tag{40}$$

Hence, upon setting

$$\psi(z)=\frac{1}{(1-\alpha)H(z)+\alpha+\sigma},$$

in (40), in view of Lemma 2, we obtain

$$p(z)\prec\psi(z)\quad\left(z\in U\right),$$

which leads to

$$L_\sigma f(z)\in K_{\lambda,l}^{m}(\beta,\alpha;a,c,\mu;\psi,\varphi).$$

which completes the proof of Theorem 7.

Taking $a=c$ in Theorem 7, we obtain the following corollary.

Corollary 7. *Let* $\max_{z\in U} \operatorname{Re}\{\varphi(z)\} < \frac{\alpha+\sigma}{\alpha-1}$. *If* $f(z)\in K_{\lambda,l}^{m}(\beta,\alpha;\psi,\varphi)$, *then* $L_\sigma f(z)\in K_{\lambda,l}^{m}(\beta,\alpha;\psi,\varphi)$.

Remark 10. (i) Taking $\lambda=1,\ m=\mu(\mu>0)$, $l=a-1(a>0)$ and $\varphi(z)=\psi(z)=\frac{1+z}{1-z}$ in Corollary 7, we obtain the results obtained by Aouf [1, Theorem 7];

(ii) Taking $\lambda = l = 1, m = \sigma(\sigma > 0)$ and $\varphi(z) = \psi(z) = \frac{1+z}{1-z}$ in Corollary 7, we obtain the results obtained by Liu [24, Theorem 7].

Taking $m = 0$ in Theorem 7, we obtain the following corollary.

Corollary 8. *Let* $\max_{z \in U} \operatorname{Re}\{\varphi(z)\} < \frac{\alpha+\sigma}{\alpha-1}$. *If* $f(z) \in K(\beta, \alpha; a, c, \mu; \psi, \varphi)$, *then* $L_\sigma f(z) \in K(\beta, \alpha; a, c, \mu; \psi, \varphi)$.

Remark 11. Taking $\alpha = \beta = 0$, $a = \nu - 1(\nu > 0)$, $c = \lambda(\lambda > -1)$ and $\mu = 1$ in Corollary 6, we obtain the results obtained by Choi et al. [11, Theorem 6].

Theorem 8. *Let* $\max_{z \in U} \operatorname{Re}\{\varphi(z)\} < \frac{\alpha+\sigma}{\alpha-1}$. *If* $f(z) \in K_{\lambda,l}^{*m}(\beta, \alpha; a, c, \mu; \psi, \varphi)$, *then*

$$L_\sigma f(z) \in K_{\lambda,l}^{*m}(\beta, \alpha; a, c, \mu; \psi, \varphi).$$

Proof. Just as we derived Theorem 6 from Theorem 5 by using (10). Easily, we can deduce Theorem 8 from Theorem 7 by using (11). So we choose to omit the proof.

Remark 12. (i) Taking $m = s(s \in \mathbb{R})$, $\lambda = 1$, $a = \nu - 1(\nu > 0)$, $c = 0$ and $\mu = 1$ in Theorems 5-7, we obtain the results obtained by Cho and Kim [9, Theorems 3.1-3.3];
(ii) Taking $a = \nu - 1(\nu > 0)$, $c = 0$ and $\mu = 1$ in Theorems 5-7, we obtain the results obtained by Aouf and El-Ashwah [2, Theorems 4-6];
(iii) Taking $l = 0$, $a = \nu - 1(\nu > 0)$, $c = 0$ and $\mu = 1$ in Theorems 5-7, we obtain the results obtained by Aouf et al. [4, Theorems 4-6].

Taking $a = c$ in Theorem 8, we obtain the following corollary.

Corollary 9. *Let* $\max_{z \in U} \operatorname{Re}\{\varphi(z)\} < \frac{\alpha+\sigma}{\alpha-1}$. *If* $f(z) \in K_{\lambda,l}^{*m}(\beta, \alpha; \psi, \varphi)$, *then* $L_\sigma f(z) \in K_{\lambda,l}^{*m}(\beta, \alpha; \psi, \varphi)$.

Remark 13. (i) Taking $\lambda = 1$, $m = \mu(\mu > 0)$, $l = a - 1(a > 0)$ and $\varphi(z) = \psi(z) = \frac{1+z}{1-z}$ in Corollary 9, we obtain the results obtained by Aouf [1, Theorem 8];

(ii) Taking $\lambda = l = 1$, $m = \sigma(\sigma > 0)$ and $\varphi(z) = \psi(z) = \frac{1+z}{1-z}$ in Corollary 9, we obtain the results obtained by Liu [24, Theorem 8].

Taking $m = 0$ in Theorem 8, we obtain the following corollary.

Corollary 10. *Let* $\max_{z \in U} \operatorname{Re}\{\varphi(z)\} < \frac{\alpha+\sigma}{\alpha-1}$. *If* $f(z) \in K^*(\beta, \alpha; a, c, \mu; \psi, \varphi)$, *then* $L_\sigma f(z) \in K^*(\beta, \alpha; a, c, \mu; \psi, \varphi)$.

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