Post-Newtonian Equations of Motion for Inertial Guided Space APT Systems

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Abstract: The equations of motion derived in this paper are aimed to determine according to the post-Newtonian framework the relative motion of Earth satellites with respect to inertial guided space Acquisition, Pointing and Tracking systems. The equations are suitable for satellites that are even far from the systems. The tool used to derive them is Synge's world-function for the Earth surrounding space. Hence the equations are written in terms of the quasi-Cartesian coordinates tied to the systems.

Key–Words: Earth satellites, Intersatellite laser links, Acquisition, Pointing and Tracking systems, Earth post-Newtonian framework

1 Introduction

The post-Newtonian framework of the Earth surrounding space is the framework that actually meets the present needs in accurate Positioning and Navigation [1]-[4]. In fact, this is the framework used to synchronize the atomic clocks on board the GPS satellites, so as to determine the round-trip times taken by the laser beams in SLR and LLR [5]-[8].

The emerging importance of space-based systems in locating radio transmitters, both on the Earth surface or in space, is also leading to build up the Geolocation models within this framework. The reason is that increasing accuracy in locating emitters is becoming a must, and, after all, the Geolocation problem can be posed, and solved, equally well than the Navigation problem. In fact, the Geolocation problem is mathematically the inverse of the Navigation problem [9]. Hence some post-Newtonian formulae related to Geolocation are considered to be standard, such as Soffel's frequency shift formula is considered by Montenbruck and Gill [10]-[11].

Likewise, the implementation of accurate Acquisition, Pointing, and Tracking (APT) systems is becoming a relevant task. In particular, systems with Laser technology merit more and more attention due to the fact that this technology has matured substantially in recent years [12]-[13].

However, the post-Newtonian framework is not used yet by the Sat-to-Sat laser communication systems, despite one important issue into the major tasks of engineering inertial guided laser terminals is to provide accurate tracking procedures for systems endowed with very narrow laser beams [14].

Now, since this fact is not due to the lack of accuracy of the APT hardware, we may reasonably conclude that it could be due to the difficulty for the Newtonian procedures to account in real time for the different curvatures of the Earth surrounding space at the positions of the targeted satellite and the APT system. That is, in Newtonian terms, it can be due to the difficulty to account in real time for the different tidal effects of the Earth on the respective orbital positions, particularly when the target is far from the system.

The post-Newtonian equations introduced below account for these small but important differences. Therefore, they can help increase the standard accuracy in the determination of the relative motion of the target, let us say S_2 from now on, with respect to the APT system, say S_1.

To derive the equations keeping consistency with previous works, the structure of the space-time about the Earth assumed in this paper is the same assumed by some authors and/or recommended in Geodesy and Geolocation (so as in other fields, such as in Electronic Warfare) [15]-[27]. The structure is the weak approximation to the Schwarzschild field generated by the Earth.

The equations are derived from Synge's equations of geodesics, which are written in terms of Fermi coordinates. Hence they involve Synge's world-function. Now, despite this function is a genuine relativistic tool, and so powerful that it is considered nowadays to be universal [28], it is not certainly a familiar tool. For this reason, we start showing the procedure followed to derive the equations by introducing this function, together with its most relevant properties (used in the paper) in Section 2. Then the equations are derived in Section 3. Finally, numerical simulations showing some typical relative trajectories, so as the validity of the equations, are shown in Section 4.

2 The World-function

The world-function is an old function. In fact, it was introduced into tensor calculus by Ruse [29], but it was only after Synge that it appeared as an outstanding tool to work with within space-time frameworks [30]. Since then it is known as Synge's world-function, and many relevant results have been obtained with it, among them those in [1].

To show this function, so as its properties (used in this paper), let us first consider, as an example, the form it takes for the 3-D Euclidean space, E_3, in Cartesian coordinates.

Let us then assume that x^{α_1} and x^{α_2} ($\alpha = 1, 2, 3$) are the Cartesian coordinates in E_3 of two spots P_1, P_2 supposedly occupied, not necessarily at the same instant, by two satellites that are moving in the Earth surrounding space. (To get as closer as possible to our problem, let us evoke it by denoting the satellites by S_1, S_2). Let us now assume that the geometry of the space about the Earth is Euclidean. Then the world-function $\Omega(P_1, P_2)$ is given by

$$\Omega(P_1, P_2) = \frac{1}{2}[(\Delta x^1)^2 + (\Delta x^2)^2 + (\Delta x^3)^2], \quad (1)$$

or, in compact form, by

$$\Omega(P_1, P_2) = \frac{1}{2}\Delta x^\alpha \Delta x^\alpha = \frac{1}{2}\delta_{\alpha\beta}\Delta x^\alpha \Delta x^\beta, \quad (2)$$

where $\Delta x^\alpha = x^{\alpha_2} - x^{\alpha_1}$. Thus, $\delta_{\alpha\beta}$ appears as what it actually is: the metric tensor of E_3 in Cartesian coordinates.

The most important characteristic of $\Omega(P_1, P_2)$ emerges from the most significant feature of Euclidean spaces: since there is only one straight line in E_3 joining P_1 and P_2, we have that if S_1, S_2 move along smooth paths, and P_1, P_2 are spots successively occupied by S_1, S_2 (not necessarily at the same instants), $\Omega(P_1, P_2)$ results in a smooth function, non-negative in this case, of the three coordinates of P_1 and of the three of P_2. Or shortly said, $\Omega(P_1, P_2)$ result to be a two-point smooth scalar function of P_1 and P_2, which are the end points of the segment P_1P_2, as suggested by the notation (Fig.1).

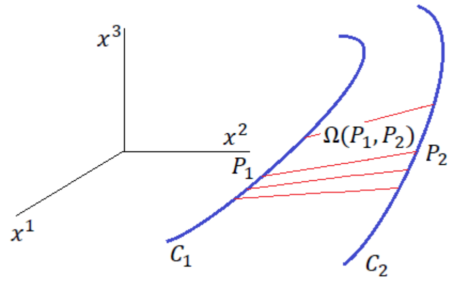

Figure 1: The world-function for the 3-D Euclidean space.

Let us now assume that the space-time about the Earth is flat, that is, that the structure of the space-time about the Earth is Minkowskian. This leads to assume that the 3-D space about the Earth is still Euclidean; also, that the model for the gravitational action of the Earth on the satellites is Newtonian, and finally, that the speed of any electromagnetic signal in vacuo is c, so that the metric tensor for this space-time in Earth Centered Inertial (ECI) coordinates, (x^α, ct), is $\eta_{ij} = diag(1, 1, 1, -1)$. (Latin indices range from 1 to 4). Then, unlike in (2), P_1 and P_2 are now events, and the world function relates events P_1 of S_1 to events P_2 of S_2 as for any other space-time, although this time according to the following rule: if (x^{α_1}, ct^1), (x^{α_2}, ct^2) are the ECI coordinates of P_1, P_2, then

$$\Omega(P_1, P_2) = \frac{1}{2}[\Delta x^\alpha \Delta x^\alpha - c^2 \Delta t^2], \quad (3)$$

where, as before, $\Delta x^\alpha = x^{\alpha_2} - x^{\alpha_1}$, and now $\Delta t = t^2 - t^1$. Or in compact form, to show the role of η_{ij},

$$\Omega(P_1, P_2) = \frac{1}{2}\eta_{ij}\Delta x^i \Delta x^j, \quad (4)$$

where $\Delta x^i = x^{i_2} - x^{i_1}$, with $x^{4_1} = ct^1$ and $x^{4_2} = ct^2$.

This function is illustrated in Fig.2, where L_1 represents the time history of events, or world line, of S_1, and L_2, the world line of S_2. Therefore, the projections of L_1 and L_2 onto the 3-D Euclidean space (spanned by the three space axis at the bottom of the picture) represent the trajectories of S_1 and S_2 in space.

Let us now note that the expression in (4) contains all the information on the intrinsic geometry of Minkowskian space. In fact, it contains the information in *finite* form. Thus we have for the classification of events in Minkowskian space that $\Omega(P_1, P_2)$ can be positive, negative, or null. In fact, if the geometry of the space-time about the Earth is assumed to be Minkowskian, then for a given event $P_1 \in S_1$ there are infinite events $P_2 \in S_2$ for which $\Omega(P_1, P_2) < 0$,

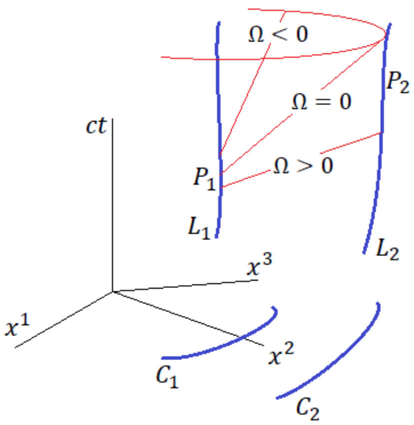

Figure 2: The world-function for Minkowskian space.

i.e. that can be experienced, at least in theory, by infinite travelers, after they have experienced P_1 (in this case P_1P_2 is known as a time-like segment); there are also infinite events of L_2 for which $\Omega(P_1, P_2) > 0$, i.e. that cannot be experienced by any traveler after it has experienced P_1 (in this case P_1P_2 is space-like), and finally, what is most important: there is only one event of L_2, say P_2, for which $\Omega(P_1, P_2) = 0$, so that P_2 can only be reached from P_1 by means of electromagnetic signals (whose world-lines are the same, and null, because the Minkowskian length of P_1P_2 is zero, according to (4)). This fact is important, since, after computing the coordinates of S_2 with respect to S_1 by means of two-way laser link series from S_1, it is what could allow us to conclude that the space about the Earth may not be Euclidean.

Analogously, for any curved space-time characterized by the metric tensor $g_{ij}(x^k)$ (signature $+, +, +, -$), and for any two events $P_1(x^{k_1})$, $P_2(x^{k_2})$, for which there is a unique geodesic $\Gamma_{P_1P_2}$ joining them, the world-function is defined by the line integral

$$\Omega(P_1, P_2) = \frac{1}{2}\int_0^1 g_{ij}U^iU^j d\omega, \qquad (5)$$

taken along $\Gamma_{P_1P_2}$, where $\Gamma_{P_1P_2}$ is given by $x^i = x^i(\omega)$, ω being an affine parameter satisfying $0 \leq \omega \leq 1$, so that $P_1 \equiv x^i(0)$, $P_2 \equiv x^i(1)$ and $U^i = dx^i/d\omega$. (As before, Latin indices range from 1 to 4, and Greek from 1 to 3).

It is then straightforward to deduce from the definition that if ds is the arc length element of $\Gamma_{P_1P_2}$, we have

$$\Omega(P_1, P_2) = \frac{1}{2}\epsilon\Big(\int_{P_1}^{P_2} ds\Big)^2, \qquad (6)$$

where $\epsilon = +1$, if $\Gamma_{P_1P_2}$ is space-like; $\epsilon = -1$, if $\Gamma_{P_1P_2}$ is time-like; and $\epsilon = 0$, if $\Gamma_{P_1P_2}$ is null; therefore, the world-function is, to within the factor ϵ, nothing else but half the square of the measure of the geodesic (assumed unique) that joins any two events in a given space-time. Hence, as a consequence, we have: (I) $\Omega(P_1, P_2)$ is single-valued, and does not depend on $\Gamma_{P_1P_2}$; it only depends on the eight coordinates x^{i_1}, x^{i_2}, of the events P_1, P_2, which are the end points of $\Gamma_{P_1P_2}$, as suggested by the notation; (II) $\Omega(P_1, P_2)$ is a two-point scalar function of x^{i_1} and x^{i_2}, i.e. it is invariant under coordinate transformations both at P_1 and P_2; (III) successive covariant derivatives of $\Omega(P_1, P_2)$ can be taken unambiguously with respect to the coordinates of P_1 and/or with respect to the coordinates of P_2 (following Synge, these derivatives will be indicated with simple subscripts, that is to say, without the usual stroke); (IV) the partial derivatives of $\Omega(P_1, P_2)$ with respect to P_1, i.e. $\Omega_{i_1}(P_1, P_2)$, are equal to $-U_{i_1}$, and analogously, $\Omega_{i_2}(P_1, P_2) = U_{i_2}$ (the minus sign in the first expression is consistent with the fact that Ω_{i_1} and Ω_{i_2} are the gradients of $\Omega(P_1, P_2)$ at the end points P_1, P_2). In particular, if $\Gamma_{P_1P_2}$ is not null ($\epsilon = \pm 1$), then $\Omega_{i_1}(P_1, P_2) = -L\lambda_{i_1}$ and $\Omega_{i_2}(P_1, P_2) = L\lambda_{i_2}$, where $L = \int_{P_1}^{P_2} ds$, and λ_{i_1}, λ_{i_2} are the unit tangent vectors to $\Gamma_{P_1P_2}$ at P_1 and P_2 respectively; (V) the norms of Ω_{i_1} and Ω_{i_2} are 2Ω, that is, $g^{i_1j_1}\Omega_{i_1}\Omega_{j_1} = g^{i_2j_2}\Omega_{i_2}\Omega_{j_2} = 2\Omega$, where $g^{i_1j_1}$ and $g^{i_2j_2}$ are the contra-variant metric tensors at P_1 and P_2 respectively; (VI) the second covariant derivatives $\Omega_{i_1j_1}(P_1, P_2)$ and $\Omega_{i_2j_2}(P_1, P_2)$ are equal to $\partial\Omega_{i_1}/\partial x^{j_1} - \Gamma^{a_1}_{i_1j_1}\Omega_{a_1}$ and to $\partial\Omega_{i_2}/\partial x^{j_2} - \Gamma^{a_2}_{i_2j_2}\Omega_{a_2}$ respectively, where the Christoffel symbols are taken at P_1 and P_2 respectively, as suggested by the notation; (VII) $\Omega_{i_1j_2}(P_1, P_2) = \partial\Omega_{i_1}/\partial x^{j_2}$, $\Omega_{i_2j_1}(P_1, P_2) = \partial\Omega_{i_2}/\partial x^{j_1}$, and $\Omega_{i_1j_2k_2}(P_1, P_2) = \partial\Omega_{i_1j_2}/\partial x^{k_2} - \Gamma^{a_2}_{j_2k_2}\Omega_{i_1a_2}$; (VIII) $\Omega^{i_1}(P_1, P_2) = g^{i_1j_1}\Omega_{j_1}$ and $\Omega^{i_2}(P_1, P_2) = g^{i_2j_2}\Omega_{j_2}$; (IX) in flat space-time with Cartesian coordinates x^i,

$$\Omega(P_1, P_2) = \frac{1}{2}\eta_{ij}(x^{i_2} - x^{i_1})(x^{j_2} - x^{j_1}), \qquad (7)$$

where $\eta_{ij} = diag(1, 1, 1, -1)$ and x^{i_1}, x^{i_2} are the coordinates of P_1 and P_2 respectively; (X) for quasi-Minkowskian metrics, i.e. for metrics having the form

$$g_{ij}(x^k) = \eta_{ij} + \gamma_{ij}(x^k), \qquad (8)$$

with $\gamma_{ij}(x^k) = \mathcal{O}(\varepsilon^2) \ll 1$, like for the metric about the Earth assumed in this paper, $\Omega(P_1, P_2)$ takes the form

$$\frac{1}{2}\eta_{ij}\Delta x^i\Delta x^j + \frac{1}{2}\Delta x^i\Delta x^j\int_C \gamma_{ij}d\omega + \mathcal{O}(\varepsilon^3), \qquad (9)$$

where $\Delta x^i = x^{i_2} - x^{i_1}$. Here C is the straight line joining P_1 and P_2, i.e. $x^i = (1-\omega)x^{i_1} + \omega x^{i_2}$

$(0 \leq \omega \leq 1)$, and ε is a small dimensionless parameter such that ε^2 is of the order of v^2 and U, where v is in our case the characteristic Classical 3-speed of the satellites in orbit about the Earth with respect to the Earth, and U the Newtonian potential of the Earth in the neighborhood of the Earth. Thus, the first part in (9) is the world-function in (7), and the remainder is $\mathcal{O}(\varepsilon^2)$ (note that we are now taking $c = G = 1$); and (XI) the world-functions in (7) and (9), so as their derivatives, can be expanded about P_1 and/or P_2 with the usual methods of approximation without abandoning the facilities of tensor calculus.

Since $-\Omega_{i_1}(P_1, P_2)$ is the Minkowskian 4-position vector of P_2 with respect to P_1 for the particular case in (7), and analogously, $-\Omega_{i_2}(P_1, P_2)$ is the 4-position vector of P_1 with respect to P_2, we are entitled to consider $-\Omega_{i_1}$ and $-\Omega_{i_2}$ as 4-position vectors for the more general case in (9). In fact, when P_1 and P_2 occur in the vicinity of the Earth, the heads of $-\Omega_{i_1}$ and $-\Omega_{i_2}$ can be thought as "not too far apart" from P_2 and P_1 respectively, provided that "far apart" is meant in the 4-Euclidean sense, that is to say, with the 4-Euclidean topology associated to Minkowskian space. Likewise, the second-order covariant derivatives yield relative velocities, and the third-order derivatives, relative accelerations [31].

3 The Equations of Motion

Let E be a space-time with metric $g_{ij}(x^k)$, i.e. $g_{ij}(x^\alpha, t)$, and world-function $\Omega(x^{k_1}, x^{k_2})$. Let $(\lambda^{k_1}_{(\alpha)}(s_1), \lambda^{k_1}_{(4)}(s_1))$ be an orthogonal tetrad of unit vectors Fermi-transported along a time-like (base) world line, let us say $M_1(x^{k_1}(s_1))$, with $\lambda^{k_1}_{(4)}(s_1) = A^{k_1}(s_1) = dx^{k_1}/ds_1$, where s_1 is the proper time of C_1, i.e. of the object whose world-line is M_1, so that $A^{4_1}(s_1) = \lambda^{4_1}_{(4)}(s_1) = dt/ds_1$; let $P_2(x^{k_2})$ be an arbitrary event in a time-like geodesic, say $M_2(x^{k_2}(s_2))$, where s_2 is the proper time of C_2, the object whose world-line is M_2; and let $(X^{(\alpha)}, s_1) = (X_{(\alpha)}, s_1)$ be the Fermi coordinates of $P_2(x^{k_2})$ with respect to C_1 (see e.g. [30]). If $b_1(s_1)$, the first curvature of M_1, is null for all s_1; if $\Omega_{i_1j_1l_1}$ and $\Omega_{i_1j_1l_2}$ are the third-order covariant derivatives of $\Omega(x^{k_1}, x^{k_2})$ taken as indicated by the indices, i.e. with respect to x^{i_1}, x^{j_1} and x^{l_1}, in the first case, and with respect to x^{l_2} for the third derivative in the second case; if, furthermore, $H^{k_2} = A^{k_2}(ds_2/ds_1)$ with $A^{k_2} = dx^{k_2}/ds_2$, and finally, if

$$dL_{(\alpha)}/ds_1 = \chi L_{(\alpha)} + \Omega_{i_1j_1l_2}\lambda^{i_1}_{(\alpha)}A^{j_1}H^{l_2}$$
$$+\Omega_{i_1j_2l_2}\lambda^{i_1}_{(a)}H^{j_2}H^{l_2}, \qquad (10)$$

with $L_{(\alpha)} = \Omega_{i_1j_2}\lambda^{i_1}_{(a)}H^{j_2}$, where $\Omega_{i_1j_2}$ are the second-order covariant derivatives of $\Omega(x^{k_1}, x^{k_2})$, first with respect to x^{i_1}, and then with respect to x^{j_2}; $\chi = (d^2s_2/ds_1^2)/\,(ds_2/ds_1)$, and $\Omega_{i_1j_1l_2}$, $\Omega_{i_1j_2l_2}$ are the third-order covariant derivatives whose interpretation is similar to those of the previous derivatives, then Synge's equations for C_2 in terms of the Fermi coordinates associated to M_1 read [30]

$$\frac{d^2X_{(\alpha)}}{ds_1^2} = -\Omega_{i_1j_1l_1}\lambda^{i_1}_{(\alpha)}A^{j_1}A^{l_1}$$
$$-\Omega_{i_1j_1l_2}\lambda^{i_1}_{(\alpha)}A^{j_1}H^{l_2} - \frac{dL_{(\alpha)}}{ds_1}. \qquad (11)$$

According to Synge the calculations to integrate Equations (11) become unmanageable. However, they become much simpler, and probably useful, if the following assumptions are considered to be reasonable in order to determine the relative motion of S_2 with respect to S_1: i) the structure of the space-time about the Earth is that of the post-Newtonian approximation of the Earth Schwarzschild field. In ECI coordinates the metric of this field is

$$g_{ij} = \eta_{ij} + \gamma_{ij}, \qquad (12)$$

where

$$\gamma_{\alpha\beta} = \frac{2m}{r}\frac{x^\alpha x^\beta}{r^2} + \mathcal{O}(\varepsilon^2),$$
$$\gamma_{\alpha 4} = \mathcal{O}(\varepsilon^3), \gamma_{44} = \frac{2m}{r} + \mathcal{O}(\varepsilon^2), \qquad (13)$$

m being the mass of the Earth measured in seconds $(c = G = 1)$ and $r^2 = x^\alpha x^\alpha$; ii) $ds_2/ds_1 = 1$ approx.; and iii) L_2 is nearly parallel to L_1 (physically this means that the relative speed of S_2 with respect to S_1 is small as compared to c); for in that case (i) $\lambda^{k_1}_{(\alpha)}(s_1)$ becomes an inertial guided system co-moving with S_1, i.e.,

$$\lambda^{\mu_1}_{(\alpha)}(s_1) = \delta^\mu_\alpha, \lambda^{4_1}_{(\alpha)}(s_1) = v^{\alpha_1}, \lambda^{\mu_1}_{(4)}(s_1) = v^{\mu_1},$$
$$\lambda^{4_1}_{(4)}(s_1) = 1 + \frac{1}{2}\gamma_{44}(x^{\alpha_1}) + \frac{1}{2}(v_1)^2, \qquad (14)$$

where $v^{\alpha_1} = A^{\alpha_1} = dx^{\alpha_1}/ds_1$ and $(v_1)^2 = v^{\alpha_1}v^{\alpha_1}$, and (ii) the space Fermi coordinates of P_2 with respect to $\lambda^{i_1}_{(\alpha)}(s_1)$, $X^{(\alpha_2)}$, become the quasi-Cartesian coordinates of S_2 with respect to S_1 at s_1. Further, if the coordinates of the events $P_2, P_{2'} \in L_2$ with respect to $\lambda^{i_1}_{(\alpha)}(s_1)$ and $\lambda^{i_1}_{(\alpha)}(s_1 + ds_1)$ are $X^{(i_2)} \equiv (X^{(\alpha_2)}, s_1)$ and $X^{(i_2)} + dX^{(i)} \equiv (X^{(\alpha_2)} + dX^{(\alpha)}, s_1 + ds_1)$ respectively, and P_1 is the foot at L_1 of the geodesic drawn from P_2 to cut orthogonally L_1 (or, in other words, if P_2 is in the instantaneous local space of

$P_1 \in L_1$), then the structure of the space-time as seen by S_1 at s_1 is given by

$$2\Omega(P_2, P_{2'}) = g_{(ij)} dX^{(i)} dX^{(j)}, \quad (15)$$

with

$$g_{(\alpha\beta)} = \delta_{\alpha\beta} + 2h_{(\alpha_1\beta_2)} + \mathcal{O}(\varepsilon^2),$$
$$g_{(\alpha 4)} = \mathcal{O}(\varepsilon^3),$$
$$g_{(44)} = -1 + 2h_{(4_1 4_1)} + 2h_{(4_1 4_2)} + \mathcal{O}(\varepsilon^2), \quad (16)$$

where

$$h_{(\alpha_1\beta_2)} = \frac{3}{2} X^{(\mu_2)} X^{(\nu_2)} \int_0^1 (1-u) u S_{(\alpha\beta\mu\nu)} du,$$
$$h_{(4_1 4_1)} = \frac{3}{2} X^{(\mu_2)} X^{(\nu_2)} \int_0^1 (1-u)^2 S_{(44\mu\nu)} du,$$
$$h_{(4_1 4_2)} = \frac{3}{2} X^{(\mu_2)} X^{(\nu_2)} \int_0^1 (1-u) u S_{(44\mu\nu)} du, \quad (17)$$

the integrals being taken along the straight line C described above ($x^i(u) = x^{i_1}(1-u) + x^{i_2}u$);

$$S_{(abcd)} = S_{(abcd)}(x^i(u))$$
$$= S_{jklm}(x^i(u))[\lambda^j_{(a)}\lambda^k_{(b)}\lambda^l_{(c)}\lambda^m_{(d)}](x^i(u)), \quad (18)$$

where $\lambda^i_{(a)}(x^i(u))$ are the tetrads obtained by parallel transport from $\lambda^{i_1}_{(a)}$ along C according to the metric (12), and

$$S_{jklm}(x^i(u)) = -\frac{1}{3}(R_{jlkm} + R_{jmkl})(x^i(u)), \quad (19)$$

where $R_{jklm}(x^i(u))$ is the Riemann tensor of (12) at $x^i(u)$.[1]

In fact, under these hypothesis Equations (11) become

$$\frac{d^2 X_{(\alpha_2)}}{ds_1^2} = -\Omega_{(\alpha_1 4_1 4_1)} - 2\Omega_{(\alpha_1 4_1 4_2)} - \Omega_{(\alpha_1 4_2 4_2)}, \quad (20)$$

where

$$\Omega_{(\alpha_1 4_1 4_1)} = -\Omega_{(\alpha_1 4_1 4_2)}$$
$$= -X^{(\gamma_2)} \int_0^1 (1-u)^2 R_{(\alpha 4 \gamma 4)} du,$$
$$\Omega_{(\alpha_1 4_2 4_2)} = 2X^{(\gamma_2)} \int_0^1 u^2 R_{(\alpha 4 \gamma 4)} du$$
$$-X^{(\mu_2)} X^{(\nu_2)} \int_0^1 (1-u) u^2 \frac{\partial R_{(\mu 4 \nu 4)}}{\partial x^\alpha} du, \quad (21)$$

and

$$R_{(\alpha 4 \gamma 4)} = -m \left(\frac{3x^\alpha(u) x^\gamma(u)}{r(u)^5} - \frac{\delta_{\alpha\gamma}}{r(u)^3} \right), \quad (22)$$

with $r(u)^2 = x^\delta(u) x^\delta(u)$. These are the post-Newtonian equations for the relative motion of S_2 with respect to S_1 that are valid even when S_2 is far from S_1.

In this regard, let us note that $h_{(\alpha_1\beta_2)}$ in (17) need not be used to derive (20), nor $\gamma_{\alpha\beta}$ in (13) to derive (22). Note, finally, that the equations (20) reduce to the equations of the geodesic deviation for nearby satellites. These equations are

$$\frac{d^2 X_{(\alpha)}}{ds_1^2} = -R_{(\alpha 4 \beta 4)} X^{(\beta)}, \quad (23)$$

with $R_{(\alpha 4 \beta 4)}$ evaluated at $x^{k_1}(s_1)$.

4 Numerical Simulations

To show the qualitative behavior of the solutions of (20), it is enough to assume that S_1 and S_2 are in coplanar circular orbits, taking care of not exceeding two time limits, which are unavoidable and characteristics of each simulation: the first is an upper bound that fix the mathematical validity of the equations in each case; the second is the upper bound from which the relative position of S_2 with respect to S_1 cannot be materialized by S_1. Otherwise, the fact is that the differences between the Newtonian relative motions and the solutions of (20) for arbitrary orbital motions, so as between the solutions of (23) and those of the linear approximation to (20), which are only valid for small differences of orbital radii and short time intervals, are significant enough, and straightforwardly analyzable. The reason is that, whatever the orbital elements are, those differences essentially depend on the semi-major axis and eccentricities of the orbits of S_1 and S_2.

Under the restrictions mentioned above, (23) becomes

$$\frac{d^2 X_{(1)}}{ds_1^2} = \frac{m}{r_1^3} \Big[\Big(3\cos^2 M_1 - 1\Big) X^{(1)}$$
$$+ (3\cos M_1 \sin M_1) X^{(2)} \Big],$$
$$\frac{d^2 X_{(2)}}{ds_1^2} = \frac{m}{r_1^3} \Big[(3\cos M_1 \sin M_1) X^{(1)}$$
$$+ \Big(3\sin^2 M_1 - 1\Big) X^{(2)} \Big]; \quad (24)$$

[1] The quasi-Cartesian angle directions do not differ from the Euclidean angles (see the components of $\lambda^{\mu_1}_{(\alpha)}$ in (14)). The quasi-Cartesian coordinates differ from the Cartesian coordinates only in that, instead of computing the range from S_1 to S_2 by means of the principal terms in (15), thus getting Cartesian coordinates, the range required to derive quasi-Cartesian coordinates is computed by means of (15)-(19).

the linear approximation to (20) for short time intervals and small radial distances is

$$\frac{d^2X_{(1)}}{ds_1^2} = \frac{m}{r_1^3}\Big(1-\frac{7}{4}\eta\Big)\Big[\Big(3\cos^2 M_1 - 1\Big)X^{(1)} + (3\cos M_1 \sin M_1)X^{(2)}\Big],$$

$$\frac{d^2X_{(2)}}{ds_1^2} = \frac{m}{r_1^3}\Big(1-\frac{7}{4}\eta\Big)\Big[(3\cos M_1 \sin M_1)X^{(1)} + \Big(3\sin^2 M_1 - 1\Big)X^{(2)}\Big], \qquad (25)$$

and the linear equations corresponding to (20) are

$$\frac{d^2X_{(1)}}{ds_1^2} = m\Bigg[\int_0^1 \Big[\Big(3r_1^2(1-u)^2\cos^2 M_1 + 6r_1r_2(1-u)u\cos M_1\cos M_2 + 3r_2^2u^2\cos^2 M_2\Big)\Big/\Big(r_1^2(1-u)^2 + 2r_1r_2(1-u)u \cos(M_1-M_2) + r_2^2u^2\Big)^{5/2} - 1\Big/\Big(r_1^2(1-u)^2 + 2r_1r_2(1-u)u\cos(M_1-M_2) + r_2^2u^2\Big)^{3/2}\Big](1-2u+3u^2)du\Bigg]X^{(1)}$$
$$+m\Bigg[\int_0^1 \Big[\Big(3r_1^2(1-u)^2\sin M_1\cos M_1 + 3r_1r_2(1-u)u\sin(M_1+M_2) + 3r_2^2u^2\sin M_2\cos M_2\Big)\Big/\Big(r_1^2(1-u)^2+2r_1r_2(1-u)u \cos(M_1-M_2) + r_2^2u^2\Big)^{5/2}\Big](1-2u+3u^2)du\Bigg]X^{(2)},$$

$$\frac{d^2X_{(2)}}{ds_1^2} = m\Bigg[\int_0^1 \Big[\Big(3r_1^2(1-u)^2\sin M_1\cos M_1 + 3r_1r_2(1-u)u\sin(M_1+M_2) + 3r_2^2u^2\sin M_2\cos M_2\Big)\Big/\Big(r_1^2(1-u)^2+2r_1r_2(1-u)u \cos(M_1-M_2) + r_2^2u^2\Big)^{5/2}\Big](1-2u+3u^2)du\Bigg]X^{(1)}$$
$$+m\Bigg[\int_0^1 \Big[\Big(3r_1^2(1-u)^2\sin^2 M_1 + 6r_1r_2(1-u)u\sin M_1\sin M_2 + 3r_2^2u^2\sin^2 M_2\Big)\Big/\Big(r_1^2(1-u)^2 + 2r_1r_2(1-u)u \cos(M_1-M_2) + r_2^2u^2\Big)^{5/2} - 1\Big/\Big(r_1^2(1-u)^2 + 2r_1r_2(1-u)u\cos(M_1-M_2) + r_2^2u^2\Big)^{3/2}\Big](1-2u+3u^2)du\Bigg]X^{(2)}, \qquad (26)$$

where $X^{(1)}$, $X^{(2)}$ are the plane orbital coordinates; $M_1 = M_1(s_1)$ is the mean anomaly of S_1 at s_1; $r_1^2 = x^{\delta_1}x^{\delta_1}$; $r_2^2 = x^{\delta_2}x^{\delta_2}$ as before, and $\eta = ((r_2 - r_1)/r_1) \ll 1$.

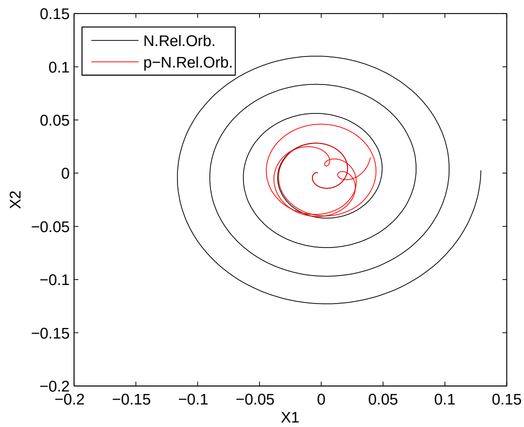

Figure 3: Relative orbits from S_1.

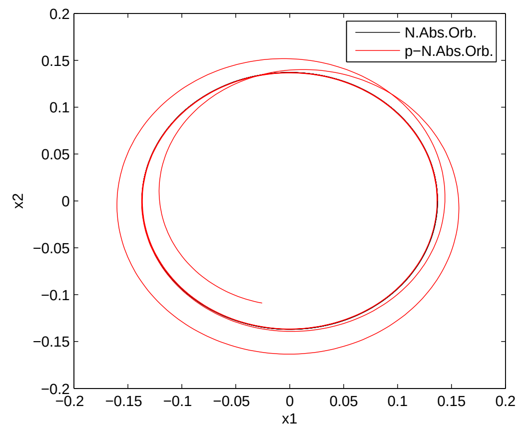

Figure 4: ECI centered orbits.

We note with respect to (25) that η does not depend on s_1. It can also be verified, as a matter of checking, that if $\eta = 0$, and the initial condition are $X_{(1)_0} = X_{(2)_0} = 0$, $(dX_{(1)}/ds_1)_0 = (dX_{(2)}/ds_1)_0 = 0$, then $X_{(1)}(s_1) = X_{(2)}(s_1) = 0$, as expected.

Finally, we note that when S_1 and S_2 are in opposition with respect to the Earth, then the line integrals in (26) become singular, since then $\cos(M_1 - M_2) = -1$, and there is a value of u ($u = r_1/(r_1 + r_2)$) for which the denominators in the integrands are zero.

Therefore, from the mathematical point of view the equations in (26) are valid until the instant at which that configuration is reached. But this implies that these equations are always applicable, since fortunately that limit is far beyond the physical limits mentioned above, which are due to the Earth size (as is known, these limits correspond to the time intervals within which S_2 is in the line of sight of S_1, so that the largest limit is reached when S_1 is geostationary).

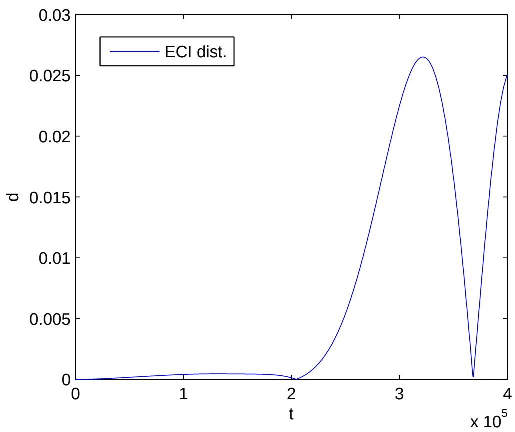

Figure 5: ECI distance from S2-New. Orb. to S2-p-New. Orb.

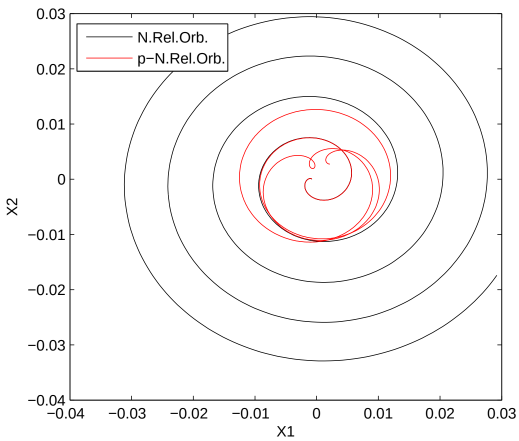

Figure 6: Relative orbits from S_1.

Since $c = G = 1$ in all the expressions and equations derived in the paper, the data involved in the simulations corresponding to Figs. 3-8 have been introduced in seconds. In particular, m has been assumed to amount $1.479 \cdot 10^{-11}$ sec. Figs. 3-5 have been generated by means of (26) for $r_1 = 14.002 \cdot 10^{-2}$ sec, $X_0^1 = r_2 - r_1 = -3 \cdot 10^{-3}$ sec, and $X_0^2 = 0$ sec, for the time interval $[0, 400000]$ sec. Figs. 6-8 have been

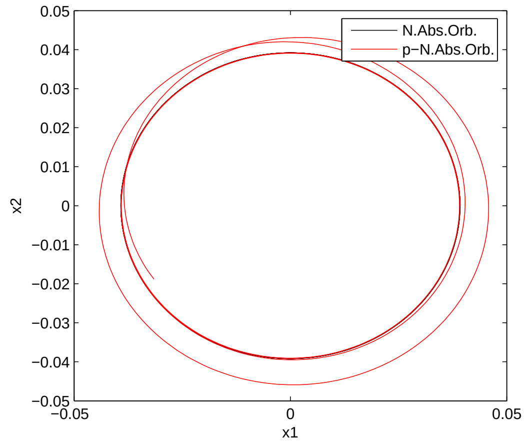

Figure 7: ECI centered orbits.

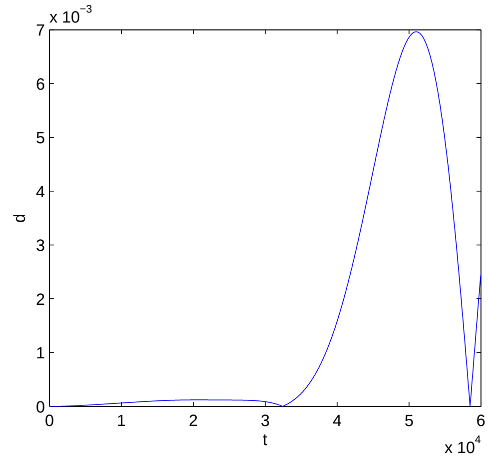

Figure 8: ECI distance from S2-New. Orb. to S2-p-New. Orb.

generated for $r_1 = 4.0 \cdot 10^{-2}$ sec, $X_0^1 = r_2 - r_1 = -8 \cdot 10^{-4}$ sec, and $X_0^2 = 0$ sec, for the time interval $[0, 50000]$ sec. Figs. 3 and 6 show the Newtonian and post-Newtonian relative orbits of S_2 with respect to S_1. The respective ECI orbits are shown in Figs. 4, 7, and to generate them, the transformations from the inertial local system given in (14) have been used to the respective order of approximation. In comparing Figs. 3, 6 with Figs. 4, 7 it can be seen, particularly when these last are sequentially plotted, that the small loops in Figs. 3, 6 correspond to delays and advances of the post-Newtonian motion of S_2 with respect to the Newtonian prediction. Finally, it can be deduced from Figs. 4, 7 and Figs. 5, 8 the following fact: the integrals in (21) manifest themselves in the oscillatory motion, i.e. in the tidal motion, of the post-Newtonian ECI orbit of S_2 about its Newtonian orbit.

5 Conclusion

The calculations to integrate (20) are certainly manageable, and so, feasible to increment the accuracy of the APT laser systems, provided that, to keep consistency, the initial data are obtained by means of post-Newtonian Geolocation formulae or by close tracking. In fact, the main characteristic of (20) is that they include the Earth tidal effects and reduce to the equations in (26), (25), (24), and (23) successively when S_2 is respectively closer and closer, up to be, finally, nearby S_1. This is the reason we can expected that the differences between the Newtonian and post-Newtonian predictions for S_1 may be very large according to (20), even up to tens of meters, as Figs. 5, 8 suggest; therefore, numerical integration of (20) and error analysis are required to provide accurate quantitative predictions.

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