The solvability of finite groups with four conjugacy class sizes

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Abstract: Let G be a finite group and let G^* be the set of elements of primary and biprimary orders of G. We show that when the conjugacy class sizes of G^* are $\{1, m, n, mn\}$, then G is solvable.

Key–Words: Conjugacy class sizes; Finite groups; Nilpotent groups; Solvable groups; Sylow p-subgroup.

1 Introduction

A well-established research area in finite group theory consists in exploring the relationship between the structure of a group G and the set of its conjugacy class sizes of the elements of G. The best known instance of this is Itô′s result in [1]:

Theorem 1 *[1, Theorem 1] If the sizes of the conjugacy classes of a group G are $\{1, m\}$, then G is nilpotent, $m = p^a$ for some prime p and $G = P \times A$, with P a Sylow p-subgroup of G and $A \subseteq Z(G)$.*

There exist several other results studying the solvability of a group under some arithmetical conditions on its conjugacy class sizes. For example, In [2] Beltrán and Felipe prove the following two results:

Theorem 2 *[2, Theorem 7] Let G be a group and suppose that the conjugacy class sizes of elements of G are exactly $\{1, p^a, n, p^a n\}$ with $(p, n) = 1$ and $a \geq 0$. Then G is solvable.*

Theorem 3 *[3, Theorem A] Let G be a group and suppose that the conjugacy class sizes of G are exactly $\{1, m, n, mn\}$ with $(m, n) = 1$, then G is solvable.*

On the other hand, some other authors replace conditions for all conjugacy classes by conditions referring to only some conjugacy classes in order to investigate the nilpotency or solvability of a finite group. We say that a group element has primary, biprimary or triprimary order respectively if its order is divisible by at most one, two or three primes. In [3] E. Alemany, A. Beltrán and M. J. Felipe extend Itô′s result by replacing conditions for all conjugacy classes by conditions referring to only some conjugacy classes. For instance, the conjugacy classes of elements of primary orders of G.

Theorem 4 *[3, Corollary B] Let G be a finite group and suppose that G has exactly two class sizes of elements of prime power order, 1 and m. Then m is a prime power and G is nilpotent. Moreover, $G = Q \times A$, with Q a Sylow q-subgroup of G and $A \leq Z(G)$.*

Recently, in [4] Kong focuses his attention on conjugacy classes sizes of all elements of primary, biprimary and triprimary orders of G and obtain a complete extension of Theorem 2. He proves:

Theorem 5 *[4, Theorem A] Let G be a group and let G^* be the set of elements of primary, biprimary and triprimary orders of G. Suppose that the conjugacy class sizes of G^* are exactly $\{1, p^a, n, p^a n\}$ with $(p, n) = 1$ and $a \geq 0$, then G is solvable.*

In the present paper, we will continue to focus our attention on conjugacy classes sizes of all elements of primary and biprimary orders of G and also obtain a complete extension of the Theorem 3 of Beltrán and Felipe. Our main result is the following:

Theorem 6 *Let G be a group and let G^* be the set of elements of primary and biprimary orders of G. Suppose that the conjugacy class sizes of G^* are $\{1, m, n, mn\}$ with $(m, n) = 1$, then G is solvable.*

In order to prove Theorem 6, we will first obtain the solvability of G when one of the class sizes m or n is a prime power, that is, when for instance $m = p^a$ for some prime p. The proof of Theorem 6 will consist then of proving that one of the two class sizes m or n is a prime power.

Theorem 7 *Let G be a group and let G^* be the set of elements of primary and biprimary orders of G.*

Suppose that the conjugacy class sizes of G^* are $\{1, p^a, n, p^a n\}$ with $(p, n) = 1$ and $a \geq 0$, then G is solvable.

There is an evident question about solvability arises when we eliminate the coprime hypothesis and we are interested in studying which arithmetical conditions on groups with four class sizes of elements of primary and biprimary orders yield their solvability. For p-solvable group, in [5] Kong eliminates the coprime hypothesis $(p, n) = 1$ and get the following main result:

Theorem 8 *[5, Theorem C] Let G be a p-solvable group and let G^* be the set of elements of primary and biprimary orders of G. Suppose that the conjugacy class sizes of G^* are $\{1, p^a, n, p^a n\}$, where p divides the positive integer n and p^a does not divide n, then G is up to central factors a $\{p, q\}$-group with p and q two distinct primes. In particular, G is solvable.*

At last, with the aid of Theorem 8, we get the following main result:

Theorem 9 *Let G be a finite p-solvable group and let G^* be the set of elements of primary and biprimary orders of G. Suppose that the conjugacy class sizes of G^* are $\{1, m, n, mn\}$, where m, n are positive integers which do not divide one to another, then G is up to central factors a $\{p, q\}$-group with p and q two distinct primes. In particular, G is solvable.*

Remark 10 *It is known that there seems to exist certain parallelism between the results obtained on the group structure from the set of its conjugacy class sizes and the results obtained from the set of its character degrees. In [6], groups whose character degrees are $\{1, m, n, mn\}$ are proved to be solvable. Our main result in this paper shows the same conclusion when the set of class sizes of G^* is $\{1, m, n, mn\}$ with $(m, n) = 1$. In fact, we believe that the result is true for arbitrary integers m and n, but we have not been able to prove it with the techniques we employ here. Recently in [13, 14], we use this techniques to study the nilpotency of a finite group.*

Throughout this paper all groups are finite. If G is a group, then x^G denotes the conjugacy class containing x and $|x^G|$ the size of x^G. Following Baer [7], we call $\mathrm{Ind}_G(x) = |x^G| = |G : C_G(x)|$, the index of x in G. The rest of our notation and terminology are standard.

2 Preliminary results

In this section, we state the necessary results for the proof of our main theorem.

Lemma 11 *[8, Theorem 5] Let G be a finite group and p a prime divisor of $|G|$. Then G has no $p'-$element of prime power order and index divisible by p if and only if G is a direct product of a p-group and a p'-group.*

Lemma 12 *[9, Lemma 5] Let G be a group. A prime p does not divide any conjugacy class length of any element of prime power order of G if and only if G has a central Sylow p-subgroup.*

Lemma 13 *[3, Lemma 6] Suppose that the three smallest non-trivial indices of elements of a group G are $a < b < c$, with (a,b)=1 and $a^2 < c$. Then the set $\{g \in G |\, |g^G| = 1 \text{ or } a\}$ is a normal subgroup of G.*

The following is the Thompson′s $A \times B$ Lemma.

Lemma 14 *[10] Let AB be a finite group represented as a group of automorphisms of a p-group G with $[A, B] = 1 = [A, C_G(B)]$, B a p-group and $A = O^p(A)$. Then $[A, G] = 1$.*

In order to prove the following Lemma, we need one application of the Classification of the Finite Simple Groups.

Lemma 15 *[11] Let G be a transitive permutation group on a set Ω with $|\Omega| > 1$. Then there exist a prime p and an element $x \in G$ of order a power of p such that x acts without fixed points on Ω.*

Lemma 16 *Let G be a π-separable group. where π is a non-empty subset of $\pi(G)$. Then the conjugacy class size of any π-element x of primary order in G is a π-number if and only if $G = O_\pi(G) \times O_{\pi'}(G)$.*

Proof: The converse direction is easy and so it is sufficient to prove the direct sense.

Let Ω be the set of all Hall π'-subgroups of G. Then Ω is non-empty as G is π-separable. Moreover, G acts transitively on Ω. By Lemma 15, for some prime p , there exists an p-element $g \in G$ that acts without fixed points on Ω. Suppose that $p \in \pi'$. Since G is π-separable, $g \in H_i$ for some $H_i \in \Omega$, and so $H_i^g = H_i$. This contradiction gives $r \in \pi$. Moreover, there exists some $H_j \in \Omega$ such that $H_j \leq C_G(g)$, which implies that $H_j^g = H_j$, also a contradiction.

Consequently, $H = O_{\pi'}(G) \trianglelefteq G$. Moreover, $H \leq C_G(x)$ for every π-element x of primary order, which implies $x \in C_G(H)$. Therefore, $G = O_\pi(G) \times O_{\pi'}(G)$. □

Lemma 17 *[3, Theorem A] Suppose that G is a finite p-solvable group and that 1 and m are the conjugacy class sizes of p'-elements of prime power order. Then $m = p^a q^b$, with q a prime distinct from p and a, $b \geq 0$. If $b = 0$, then G has abelian p-complement. If $b \neq 0$, then $G = PQ \times A$, with P and Q a Sylow $p-$subgroup and a Sylow $q-$subgroup of G, respectively, and $A \leq Z(G)$. Furthermore, if $a = 0$, then $G = P \times Q \times A$.*

Lemma 18 *[12, Proposition 1] Let G be a finite group with a subgroup A_0 such that A_0 is a characteristic subgroup of A, a subgroup of G, such that every element of A_0 has centralizer A or G. Let π be the set of primes dividing $|A_0/A_0 \cap Z(G)|$ and assume $|\pi| > 1$. Then either*
(i) $N_G(A)/A$ is a π'-group or
(ii)$|N_G(A)/A| = p$ for some $p \in \pi$.

3 Proofs of main results

Proof of Theorem 7 We finish the proof by the following several steps:

Step 1. *We may suppose that there are no p-element of index p^a. Consequently, there exists some p'-element of index p^a.*

Assume that x is a p-element of index p^a and take any p'-element y of primary order of $C_G(x)$. The fact that $C_G(xy) = C_G(x) \cap C_G(y) \subseteq C_G(x)$ implies that y has index 1 or n in $C_G(x)$, and so in particular, we can apply Lemma 11 to obtain that $C_G(x) = C_G(x)_p \times C_G(x)_{p'}$. If $C_G(x)_{p'}$ is not abelian, then class sizes of primary elements of such a p'-subgroup are exactly the two numbers 1 and n in $C_G(x)_{p'}$. Note that n must occur. It follows Theorem 4 that n is a prime power and therefore, G is solvable by Lemma **??** and Burnside′s $p^a q^b$-Theorem, we are done.

Now we prove that $C_G(x)_{p'}$ is not abelian. Otherwise, we have that $C_G(x)_{p'}$ has a p-complement H, which also a p-complement of G. Moreover, $H \nleq Z(G)$. Let $v \in H - Z(G)$ be a primary element. Then $|v^G| = p^a$ and $C_G(x) = C_G(v)$. For any $w \in C_G(x)_p$, we have that $|w^G| = 1$ or p^a. Since $|C_G(x) : C_G(x) \cap C_G(w)| = |C_G(v) : C_G(v) \cap C_G(w)| = 1$ or n and $C_G(x)_{p'} \leq C_G(w)$, we have that $C_G(x) = C_G(w)$. Hence $C_G(x)$ is abelian. Let $y \in G$ be a primary element of index n. By conjugation, there is some $g \in G$ such that $x^{g^{-1}} \in C_G(y)$, that is, $y^g \in C_G(x)$. Hence $C_G(x) \leq C_G(y^g)$. It follows that $|y^G|||x^G|$, a contradiction.

In order to apply this step it is enough to consider the decomposition of an element of index p^a as a product of a p-element by a p'-element.

Step 2. *We may suppose that there are no p'-element of index n. As a result, there exist p-elements of index n.*

Assume that $y \in G^*$ is a p'-element of index n. By considering the primary decomposition and the hypotheses we can further assume y to be a q-element for some prime $q \neq p$. Take any q'-element x of primary order of $C_G(y)$. Note that $C_G(xy) = C_G(x) \cap C_G(y)$ and that $|C_G(y) : C_G(x) \cap C_G(y)|$ must be equal to 1 or p^a. Thus any q'-element of primary order of $C_G(y)$ has index 1 or p^a in $C_G(y)$, and in particular, this index is a q'-number. By applying Lemma 11, we obtain that $C_G(y) = Q_y \times A$, with Q_y a q-group and A a q'-subgroup. Notice that the class size in A of any element of primary order of A is 1 or p^a, and so by Theorem 4 we can write $A = P \times B$, with $P \in \mathrm{Syl}_p(G)$ and B a $\{p, q\}'$-subgroup. Now we choose a p'-element of primary order, say t, of index p^a. As y is a q-element and t has index p^a, we can suppose without loss that $y \in C_G(t)$, so that $t \in C_G(y)$. But this is a contradiction since the p'-elements of primary order of $C_G(y)$ are centralized by P and t has index p^a.

We obtain the second statement with an argument as in the above step.

Step 3. *If x is a p-element of index $p^a n$, then $C_G(x) = P_x \times V_x$ with P_x a p-group and V_x an abelian p'-group such that $V_x \nsubseteq Z(G)$. If y is a p'-element of index $p^a n$, then $C_G(y) = P_y \times V_y$ with P_y an abelian p-group such that $P_y \nsubseteq Z(G)$ and V_y a p'-group.*

Let x be a p-element of index $p^a n$ and let $y \in C_G(x)$ be any p'-element of primary order. Then we have that $C_G(xy) = C_G(x) \cap C_G(y) \subseteq C_G(x)$. As $p^a n$ is the largest class size of G^*, we get that $C_G(xy) = C_G(x)$, Thus $C_G(x) \subseteq C_G(y)$. This implies that $y \in Z(C_G(x))$, hence we can write $C_G(x) = P_x \times V_x$ with P_x a p-group and V_x an abelian p'-group. It remains to show that V_x cannot be central in G.

Assume that $V_x \subseteq Z(G)$. Then we have that $V_x = Z(G)_{p'}$ and $|G : Z(G)|_{p'} = n$. Choose z a non-central p-element of G, which must have index n or $p^a n$ by Step 1. In every case, we can get that $Z(G)_{p'}$ is a p-complement of $C_G(z)$. This implies that if we choose any non-central p'-element w of primary order of G, then any p-element $C_G(w)$ must be central in G. This means that $Z(G)_p$ is a Sylow p-subgroup of $C_G(w)$. As w has index p^a or $p^a n$, we have that $|G : Z(G)|_p = p^a$. This yields $|G : Z(G)| = |G : Z(G)|_p |G : Z(G)|_{p'} = p^a n$, which contradicts the existence in G^* of elements of index $p^a n$. Hence, the first assertion of the step is proved.

The second part of this step can be proved by reasoning in a similar way with a p'-element of index $p^a n$.

Step 4. *If $p^a > n$, we define $L_p = \{x|\ x$ is p-element and $|x^G| = 1$ or $n\}$. Then L_p is an abelian normal p-subgroup of G. If $p^a < n$, for any prime q dividing n, we define $L_q = \{x|\ x$ is q-element and $|x^G| = 1$ or $p^a\}$. Then L_q is an abelian normal q-subgroup of G.*

Suppose that $p^a > n$. Applying Lemma 13, we obtain that the set $W = \{x||x^G| = 1 \text{ or } n\}$ is a normal subgroup of G. Now, if x is any element in G^* of index n and factorize $x = x_p x_{p'}$, with x_p and $x_{p'}$ a p-element and a p'-element of primary order, respectively, it follows that $x_{p'}$ must be central by Step 1, whence $x \in L_p \times Z(G)_{p'}$. Therefore, $W = L_p \times Z(G)_{p'}$ and L_p is also a normal p-subgroup of G, as we wanted to prove.

Finally, we see that L_p is abelian. If we take any $y \in L_p$, then $|L_p : C_{L_p}(y)|$ divides $(|L_p|, n) = 1$. Consequently, L_p is abelian.

Similarly, if $p^a < n$, then again by Lemma 13, the set $W' = \{x||x^G| = 1 \text{ or } p^a\}$ is a normal subgroup of G, and the argument to show that L_q is an abelian normal q-subgroup of G is similar.

Now we write $L_{p'} = \{x|\ x \text{ is } p'\text{-element of } G^*$ and $|x^G| = 1 \text{ or } p^a\}$. Notice that $L_{p'}$ is the direct product of the subgroups L_q for all primes q dividing n, and consequently, $L_{p'}$ is an abelian normal subgroup of G.

As a consequence of Step 4, we will distinguish two cases: $p^a > n$ and $p^a < n$.

Case 1. $p^a > n$.

Step 5. *L_p is an abelian normal Sylow p-subgroup of G. In particular, G is p-solvable.*

In order to prove that L_p is a Sylow p-subgroup of G, because of Step 1 it is enough to show that there are no p-elements of index $p^a n$. Suppose that z is a p-element of index $p^a n$. By Step 3, we may write $C_G(z) = P_z \times V_z$, with V_z a non-central abelian p'-group and P_z a p-group. If $t \in V_z$, it is clear that $C_G(z) \subseteq C_G(t)$, so that in particular $C_{L_p}(z) \subseteq C_{L_p}(t)$. By applying Lemma 14, we get $t \in M = C_G(L_p)$, and thus $V_z \subseteq M$. On the other hand, by Step 2 we know that any non-central element $t \in V_z$ of primary or biprimary order has index p^a or $p^a n$, so that $|C_G(t) : C_G(z)|$ must be 1 or n. This proves that $L_p \subseteq C_G(z)$ and it follows that L_p centralizes every p-element of index $p^a n$. Also any p-element of index n trivially centralizes L_p as it is abelian. We conclude then that any p-element of G lies in M, whence $|G : M|$ is a p'-number. Furthermore, since $L_p \subseteq M \subseteq C_G(k)$ for any non-central $k \in L_p$, which has index n, we have that n must divide $|G : M|$. Considering the equality

$$|G : M||M : V_z| = |G : C_G(z)||C_G(z) : V_z|,$$

together with the properties noted above, we conclude that V_z is a p-complement of M.

Let x be a p-element of G, which we know lies in M. If x has index 1 or n, then it certainly follows that $x \in Z(M)$. If x has index $p^a n$, then by Step 3, we write $C_G(x) = P_x \times V_x$ with V_x a non-central abelian p'-group and P_x a p-group. As we have seen above, V_x is a p-complement of M, and in particular $V_x \subseteq C_M(x)$ and $|M : C_M(x)|$ is a p-number. Thus the index of any p-element of M is a p-number. Moreover M is solvable by Wielandt′s theorem, as it possesses abelian p-complements. Hence, by applying Lemma 16, we can write $M = P \times V_z$, where $P \in \mathrm{Syl}_p(G)$. In particular, P is normal in G, but if we choose some non-central element $y \in V_z$ of primary or biprimary order, then we have $P \subseteq C_G(y)$, so that y has index n, which contradicts Step 2.

Step 6. *G is solvable.*

Let us consider a p-complement H of G, write $G = L_p H$, $\overline{G} = G/L_p \cong H$ and use bars to work in $\overline{G}$. For any $\overline{x} \in \overline{G}$ we can choose x to be a p'-element of primary order of H. If we take $\overline{y} \in C_{\overline{G}}(\overline{x})$, we can also assume without loss that $y \in H$. Since $[\overline{x}, \overline{y}] = 1$ we have $[x, y] \in L_p \cap H = 1$, and so $y \in C_G(x)$. This shows that $C_{\overline{G}}(\overline{x}) = \overline{C_G(x)}$.

On the other hand, we know that x has index p^a or $p^a n$ and this is equal to

$$|G : C_G(x)L_p||C_G(x)L_p : C_G(x)|.$$

The first index is a p'-number since L_p is a Sylow p-subgroup and the second one is a p-number, so that $|G : C_G(x)L_p|$ is equal to 1 or n. Therefore, the equality of centralizers obtained above shows that $\overline{x}$ has index 1 or n in $\overline{G}$. By applying Theorem 4, we conclude that $\overline{G}$ is nilpotent, and as a consequence, G is solvable.

Case 2. $p^a < n$.

Step 7. *For any prime q dividing n, if L_q is not central in G, then L_q is an abelian normal Sylow q-subgroup of G.*

We will assume that L_q is not central and not a Sylow q-subgroup of G for some prime q dividing n. This means that there is a q-element, say w, of index $p^a n$. By Step 3 we can write $C_G(w) = P_w \times V_w$ with P_w a non-central abelian p-group and V_w a p'-subgroup. For each $u \in P_w$ we have $C_G(w) \subseteq$

$C_G(u)$ and in particular $C_{L_q}(w) \subseteq C_{L_q}(u)$, so that by Thompson′s theorem we conclude that $u \in N = C_G(L_q)$ and hence $P_w \subseteq N$.

Now for the rest of this step, since L_q is not central in G, we can fix a q-element y of index p^a, notice that $N \subseteq C_G(y)$. As w has index $p^a n$, it easily follows that $|C_G(y) : N|$ and $N : P_w|$ are both p'-numbers, so that $P_w \in \mathrm{Syl}_p(N)$ and $P_w \in \mathrm{Syl}_p(C_G(y))$. This property holds not only for w but also for any p'-element of index $p^a n$.

We show now that any q-element of G lies in N. Certainly $L_q \subseteq N$. We choose a q-element z from L_q (and thus of index $p^a n$), we will prove that it lies in N. By Step 3, we can write $C_G(z) = P_z \times V_z$, where P_z is a non-central p-group and V_z is a p'-subgroup. By the above paragraph we know that $P_z \subseteq N$. For any $u \in P_z$ we have $C_G(z) \subseteq C_G(u)$, and since u has index n or np^a by Step 1, it follows that $|C_G(u) : C_G(z)|$ is 1 or p^a. Since $P_z \subseteq N$ we have $L_q \subseteq C_G(u)$, so that $L_q \subseteq C_G(z)$. Then $z \in N$, as we wanted to prove.

Now let any element $t \in C_G(y)$ and consider the primary decomposition $t = t_q t_{q_1} \cdots t_{q_s}$, where $q, q_i \in \pi(G)$, $i = 1, \cdots, s$. Then for every t_{q_i}, we have $C_G(t_{q_i}) \cap C_G(y) = C_G(t_{q_i} y) \subseteq C_G(y)$ and $|C_G(y) : C_G(t_{q_i} y)|$ must be equal to 1 or n. Since $P_w \in \mathrm{Syl}_p(C_G(y))$, we deduce that there exists some $g \in C_G(y)$ such that $P_w^g \subseteq C_G(t_{q_i} y) \subseteq C_G(t_{q_i})$. Consequently, $t_{q_i} \in C_G(P_w^g)$. Now we distinguish two cases for t_q. Suppose first that $t_q \in L_q$, so that $P_w^g \subseteq N \subseteq C_G(t_q)$. In this case we conclude that $t \in C_G(P_w^g)$. In the other case, that is, when $t_q \notin L_q$, so that t_q has index $p^a n$, as in the first paragraph we can write $C_G(t_q) = P_{t_q} \times V_{t_q}$, where P_{t_q} is abelian and non-central in G. The property proved above for w also holds for t_q, so that P_{t_q} is a Sylow p-subgroup of $C_G(y)$ and as a result $P_{t_q} = P_w^g$ for some $g \in C_G(y)$. Since P_{t_q} is central in $C_G(t_q)$, we conclude that $t \in C_G(P_w^g)$ too. These properties yield that

$$C_G(y) = \bigcup_{g \in C_G(y)} C_{C_G(y)}(P_w)^g,$$

As $C_G(y)$ cannot be union of proper conjugate subgroups of $C_G(y)$, we deduce that P_w must be central in $C_G(y)$. But we know that P_w is not central in G, so that if we choose some non-central element $u \in C_G(y)$, we have $C_G(y) \subseteq C_G(u)$, which implies that u is a p-element of index p^a, contradicting Step 1.

Step 8. *G is solvable.*

Choose any prime q dividing n and take a fixed q-element y of index p^a and $P_y \in \mathrm{Syl}_p(C_G(y))$. First we notice that any q'-element of primary order, say x, of $C_G(y)$ has index 1 or n in $C_G(y)$, so that there exists some $g \in C_G(y)$ such that $P_y^g \subseteq C_G(x)$. Now, for any $t \in C_G(y)$ and $t = t_q t_{q_1} \cdots t_{q_s}$ with the usual notation. By the above property we have $t_{q_i} \in C_G(P_y^g)$ for some $g \in C_G(y)$. On the other hand, certainly $t_q \in L_q$, whence we can write

$$C_G(y) = \bigcup_{g \in C_G(y)} C_{C_G(y)}(P_y)^g L_q.$$

This forces that $C_G(y) = C_{C_G(y)}(P_y) L_q$ and as a consequence, $|G : C_{C_G(y)}(P_y)|$ is a $\{p, q\}$-number. Now, if there exists some non-central $u \in P_y$ which has index n or np^a, we conclude that n is a q-power and this forces G to be solvable by Lemma 12 and Burnside′s $p^a q^b$-Theorem.

So, finally we suppose that P_y is central in G and write $C_G(y) = P_y \times V_y$ where V_y is a p'-subgroup. As y has index p^a, we have $L_{p'} \subseteq V_y$. If the equality holds, then G has a normal abelian p-complement, and so in particular G is solvable and the proof is finished again. Therefore, we assume that $V_y \neq L_{p'}$, so that some element $t \in V_y$ of primary or biprimary order has index $p^a n$. By Step 3, we know that $C_G(t) = P_t \times V_t$, where P_t a non-central p-group. As $y \in V_t$ we have $P_t \subseteq P_y$ and this is a contradiction. □

Now, we are ready to prove our Theorem 6.

Proof of Theorem 6 We denote by π the set of primes dividing m and π' the set of primes dividing n. By Lemma 12, we can certainly assume that $\pi(G) = \pi \cup \pi'$. In order to apply Theorem 7, we will show that either m or n is a prime power. In fact, we will prove that if $n < m$ then there exists a prime $q \in \pi$ such that the set of q-elements in G of index 1 or n is an abelian normal Sylow q-subgroup of G. This is the key to the proof that m must be a power of q. We finish the proof by the following several steps:

Step 1. *We can assume that G has no π-elements of index m and no π'-elements of index n.*

Let x be a π-element of index m. Notice that by the primary decomposition we can assume without loss of generality that x is a p-element for some prime $p \in \pi$ such that $|x^G| = m$. Now, if y is a p'-element of primary order of $C_G(x)$, then as $C_G(xy) = C_G(x) \cap C_G(y) \subseteq C_G(x)$, we have that y has necessarily index 1 or n in $C_G(x)$, which is a p'-number. By Lemma 11, $C_G(x)$ can be written as a direct product of a p-subgroup and a p'-subgroup, so $C_G(x) = C_G(x)_p \times C_G(x)_{p'}$. By reasoning in a similar way in Step 1 of the proof of Theorem 7 we can prove that $C_G(x)_{p'}$ is not abelian. Thus, the class sizes of elements of primary orders of $C_G(x)_{p'}$ are 1 or n. By Theorem 4, we have that n is a prime power and consequently, G is solvable by Theorem 7 and the theorem is proved.

The second assertion holds because the hypotheses are symmetric in m and n.

Step 2. *If x is a π-element of index mn, then $C_G(x) = U_x \times V_x$ with U_x a π-group and V_x an abelian π'-group such that $V_x \nsubseteq Z(G)$. Similarly, If y is a π'-element of index mn, then $C_G(y) = U_y \times V_y$ with U_y an abelian π-group such that $U_y \nsubseteq Z(G)$ and V_y a π'-group.*

Suppose that x is a π-element in G^* of index mn. Certainly we can assume by considering the primary decomposition that x is a p-element for some prime $p \in \pi$. Let $y \in C_G(x)$ be any p'-element of primary order. Then we have that $C_G(xy) = C_G(x) \cap C_G(y) \subseteq C_G(x)$. As mn is the largest class size of G^*, we get that $C_G(xy) = C_G(x)$, Thus $C_G(x) \subseteq C_G(y)$. This implies that $y \in Z(C_G(x))$, hence we can write $C_G(x) = U_x \times V_x$ with U_x a π-group and V_x an abelian π'-group. It remains to show that V_x cannot be central in G.

Assume that $V_x \subseteq Z(G)$. Then we have that $V_x = Z(G)_{\pi'}$ and $|G : Z(G)|_{\pi'} = n$. Choose z a non-central π-element of primary order of G, which must have index n or mn by Step 1. In every case, we can get that $Z(G)_{\pi'}$ is a π-complement of $C_G(z)$. This implies that if we choose any non-central π'-element w of primary order of G, then any π-element of primary order of $C_G(w)$ must be central in G. This means that $Z(G)_\pi$ is a Hall π-subgroup of $C_G(w)$. As w has index m or mn, we have that $|G : Z(G)|_\pi = m$. This yields $|G : Z(G)| = |G : Z(G)|_\pi |G : Z(G)|_{\pi'} = mn$, which contradicts the existence in G^* of elements of index mn. Hence, the first assertion of the step is proved.

The second part of this step can be proved by reasoning in a similar way with a π'-element of index mn.

From now on we will assume that $n < m$.

Step 3. *For a prime $q \in \pi$, we define $L_q = \{x | \ x$ is q-element and $|x^G| = 1$ or $n\}$. Then L_q is an abelian normal q-subgroup of G.*

Applying Lemma 13, we obtain that the set $W = \{x||x^G| = 1 \text{ or } n\}$ is a normal subgroup of G. Now, if x is any element in G^* of index n and factorize $x = x_q x_{q'}$, with x_q and $x_{q'}$ a q-element and a q'-element of primary order, respectively, it follows that $x_{q'}$ must be central by Step 1, whence $x \in L_q \times Z(G)_{q'}$. Therefore, $W = L_q \times Z(G)_{q'}$ and L_q is also a normal q-subgroup of G, as we wanted to prove.

Finally, we see that L_q is abelian. If we take any $y \in L_q$, then $|L_q : C_{L_q}(y)|$ divides $(|L_q|, n) = 1$. Consequently, L_q is abelian.

Now we write $L_\pi = \{x| \ x$ is π-element of G^* and $|x^G| = 1$ or $n\}$. Notice that L_π is the direct product of the subgroups L_q for all primes $q \in \pi$, and consequently, L_π is an abelian normal subgroup of G.

Step 4. *Let $q \in \pi$. If L_q is not central in G, then L_q is an abelian normal Sylow q-subgroup of G.*

We will assume that L_q is not central in G and that L_q is not Sylow q-subgroup of G and work to get a contradiction. In this case, we may choose a q-element w of index mn and by Step 2, write $C_G(w) = U_w \times V_w$, with V_w a noncentral abelian π'-group and U_w a π-group. If $u \in V_w$, it is clear that $C_G(w) \subseteq C_G(u)$, so in particular $C_{L_q}(w) \subseteq C_{L_q}(u)$. By applying Lemma 14, we get $u \in M = C_G(L_q)$ and therefore, $V_w \subseteq M$.

For the rest of this step we fix some non-central element $y \in L_q$, so that $M \subseteq C_G(y)$. As w has index mn and y has index n, it is easy to see that $|C_G(y) : M|$ and $|M : V_w|$ are both π-numbers, and so V_w is a Hall π'-subgroup of M and $C_G(y)$.

We show now that any q-element of G lies in M. It is trivial that $L_q \subseteq M$, and so we consider an element $z \notin L_q$, thus z must have index mn. By Step 2, we may write $C_G(z) = U_z \times V_z$, where U_z is a π-group and V_z a non-central abelian π'-subgroup, and arguing with z as we did above with w, we obtain that $V_z \subseteq M$. Suppose that $u \in V_z$ in G^* and u is non-central, then $C_G(z) \subseteq C_G(u)$. In addition, by Step 1, u has index m or mn, so that $|C_G(u) : C_G(z)|$ is equal to 1 or n. Since $V_z \subseteq M$ we have $L_q \subseteq C_G(u)$, so that $L_q \subseteq C_G(z)$ and consequently $z \in M$, as we wanted to prove.

Now let any element $t \in C_G(y)$ and consider the primary decomposition $t = t_q t_{q_1} \cdots t_{q_s}$, where $q, q_i \in \pi(G)$, $i = 1, \cdots, s$. Then for every t_{q_i}, we have $C_G(t_{q_i}) \cap C_G(y) = C_G(t_{q_i} y) \subseteq C_G(y)$ and $|C_G(y) : C_G(t_{q_i} y)|$ must be equal to 1 or m because y has index n. Since V_w is a non-central π-complement of $C_G(y)$ for some $r \in \pi'$ we can choose a non-central Sylow r-subgroup R of $C_G(y)$ which is also a Sylow r-subgroup of M. Then there exists some $g \in C_G(y)$ such that $R^g \subseteq C_G(t_{q_i} y) \subseteq C_G(t_{q_i})$, so that $t_{q_i} \in C_G(R^g)$. Now we distinguish two cases for t_q. Suppose first that $t_q \in L_q$, so that $R^g \subseteq M \subseteq C_G(t_q)$, we conclude that $t \in C_G(R^g)$.

In the other case, that is, when t_q is not in L_q and hence has index mn, we can again write $C_G(t_q) = U_{t_q} \times V_{t_q}$, where V_{t_q} is an abelian non-central π'-subgroup and U_{t_q} is a π-group. Furthermore, notice that the property of w given at the beginning of this step holds for t_q, that is, V_{t_q} is also a π-complement of M and of $C_G(y)$. As we know that M has non-central π-complements and non-central Sylow r-subgroups, we may consider the Sylow r-subgroup R_1 of V_{t_q}, which is not central. Then $t_{q_i} \in C_G(t_q) \subseteq C_G(R_1)$ and trivially $t_q \in C_G(R_1)$, whence $t \in C_G(R_1)$. Since $R_1 = R^g$ for some $g \in C_G(y)$ the above remarks combine to yield the equality

$$C_G(y) = \bigcup_{g \in C_G(y)} C_{C_G(y)}(R)^g,$$

which implies that R must be central in $C_G(y)$. But we know that R is not central in G, and so if we take some non-central $u \in R$, we have $C_G(y) \subseteq C_G(R) \subseteq C_G(u)$. This provides a π'-element u of index n, contradicting Step 1.

Step 5. *G is solvable.*

By considering the decomposition of any element of index n and taking into account Step 1, we see immediately that L_π is non-trivial. Thus we may choose a prime $q \in \pi$ and fix a q-element $y \in L_\pi$ of index n. In particular, we are assuming that L_q is not central in G, and thus L_q is a Sylow q-subgroup of G by Step 4. It is easy to check that any q'-element of primary order of $C_G(y)$ has index 1 or m in $C_G(y)$. We will assume first that there exists a non-central Sylow r-subgroup R of $C_G(y)$ for some prime $r \in \pi'$. If w is a q'-element of primary order of $C_G(y)$, then there exists some $g \in C_G(y)$ such that $R^g \subseteq C_G(w)$, that is, $w \in C_{C_G(y)}(R)^g$. Thus, if we consider the $\{q, q'\}$-decomposition of any element of $C_G(y)$, taking into account that L_q is a Sylow q-subgroup of G, we have

$$\begin{aligned} C_G(y) &= \bigcup_{g \in C_G(y)} C_{C_G(y)}(R)^g L_q \\ &= \bigcup_{g \in C_G(y)} (C_{C_G(y)}(R) L_q)^g. \end{aligned}$$

This implies that $C_G(y) = C_{C_G(y)}(R)L_q$, and accordingly, $|C_G(y) : C_{C_G(y)}(R)|$ is a q-number. Now, we take some non-central $u \in R$, which has index m or mn. Observe that $C_{C_G(y)}(R) \subseteq C_G(u) \cap C_G(y) = C_G(uy) \subseteq C_G(y)$, so that uy has index n or mn. The first case leads to $C_G(y) \subseteq C_G(u)$, which is a contradiction, and so uy has index mn and it follows that m is a q-power. By Theorem 7, we obtain that G is solvable and the theorem is proved.

Therefore, we will assume that for each prime $r \in \pi'$ every Sylow r-subgroup of $C_G(y)$ is central in G and we will obtain a contradiction. In this case, we have $C_G(y) = S \times Z(G)_{\pi'}$ for some π-subgroup S, and also $|G : Z(G)|_{\pi'} = n$. If there exists a π-element in G^* of index mn, then the decomposition of its centralizer given by Step 2 easily leads to a contradiction. Thus we assume that there are no π-elements in G^* of index mn, and consequently that $S = L_\pi$. Furthermore, this implies that any π-element of index n has the same centralizer, that is, $L_\pi \times Z(G)_{\pi'}$. Now take an element w in G^* of index mn and consider the factorization $w = w_\pi w_{\pi'}$, where w_π and $w_{\pi'}$ are elements of primary orders. We know that w_π has index 1 or n. If w_π has index n, by the above comments $w_{\pi'}$ must be central, contradicting the fact that w has index mn. Finally, if w_π is central in G, then $C_G(w) = C_G(w_{\pi'})$ and by Step 2 we can write $C_G(w_{\pi'}) = U_{w_{\pi'}} \times V_{w_{\pi'}}$ with $U_{w_{\pi'}}$ a non-central abelian π-subgroup and $V_{w_{\pi'}}$ a π'-subgroup. If $t \in U_{w_{\pi'}}$ in G^* is not central in G, then $C_G(w) \subseteq C_G(t)$. But t must have index n because we are assuming that there are no π-elements in G^* of index mn. It follows that $C_G(t) = L_\pi \times Z(G)_{\pi'}$, whence $w_{\pi'}$ is central and this leads to the final contradiction. Now the theorem is proved. □

At last we prove our Theorem 9.

Proof of Theorem 9 We can certainly assume that $\pi(G) = \pi(m) \bigcup \pi(n)$ by Lemma 12. The proof is divided into several steps.

Step 1. *If x is an element in G^* such that $|x^G| = m$, then $C_G(x)$ is maximal among all centralizers in G^*. Also, either $C_G(x)$ is abelian or $n = p^a q^b$ for some primes p and q, and $C_G(x) = P_x Q_x \times T_x$, where P_x and Q_x are p- and q-subgroups respectively, and T_x is an abelian $\{p, q\}'$-group. The same properties can be assumed for all elements of index n.*

Suppose that x in G^* such that $|x^G| = m$, and hence it can be assumed to be a p-element for some prime p. Notice that $C_G(x)$ is a maximal subgroup among all centralizers in G^*. Since every p'-element of primary or biprimary order of $C_G(x)$ has index 1 or n in $C_G(x)$ we have two possibilities as Kong and Liu have already explained in the proof of Theorem 8: either $C_G(x) = P_x \times H_x$ with H_x an abelian p'-subgroup or $n = p^a q^b$ for some prime q and $C_G(x) = P_x Q_x \times T_x$ as described in the statement. We will show that the first possibility also yields to the statement of this step. If $H_x \subseteq Z(G)$, then $|G : Z(G)| = mp^d$ and since any class size in G^* divides this index, we deduce that n is a power of p, so our claim is proved. Therefore, for some prime q we can take some noncentral q-element $t \in H_x$ and certainly $C_G(x) = C_G(t)$. But this implies again that any q'-element of primary or biprimary order of $C_G(t)$ has index 1 or n. Hence, we have either $C_G(t) = Q_t \times K_t$ with K_t an abelian q-complement, and accordingly $C_G(x)$ is abelian, or n is again a product of two prime powers, as wanted.

Now if y in G^* has index n then $C_G(y)$ is maximal among all centralizers as well and the assertion in the statement of the other property for $C_G(y)$ follows exactly as for $C_G(x)$.

Step 2. *We can assume that there exists some element in G^* of index m or n such that its centralizer is not abelian. We can fix a q-element x of in-*

dex m such that $C_G(x)$ is not abelian and $C_G(x) = P_xQ_x \times Z_{\{p,q\}'}$, where P_x and Q_x are p- and q-subgroups and such that x is a q-element. Moreover, $n = p^aq^b$.

Suppose that all elements of index m and n have abelian centralizer and put $X = C_G(x)$ and $Y = C_G(y)$ for some elements x and y in G^* of index m and n respectively. If $u \in X \cap Y$, then $X, Y \leq C_G(u)$, whence $|G : C_G(u)| < \min\{m, n\}$ and so $u \in Z$. Therefore, $X \cap Y = Z$. Then

$$|X/Z| = |X/X \cap Y| = |XY|/|Y| \leq |G : Y| = n$$

and so

$$|G/Z| = |G : X||X/Z| \leq mn.$$

But this certainly contradicts the fact that G^* has elements of index mn.

Thus, we will assume that there exists some element in G^* of index m or n whose centralizer is not abelian. By the symmetry of the hypothesis, we can fix some x in G^* of index m. By Step 1, we have $C_G(x) = P_xQ_x \times T_x$, where P_x and Q_x are p- and q-subgroups, and T_x is an abelian $\{p, q\}'$-group. Furthermore, $n = p^aq^b$. We prove that T_x must be central in G. In fact, if there exists some noncentral r-element $t \in T_x$ for some prime $r \neq q, p$ then, since $C_G(x) \subseteq C_G(t)$, they are equal by the maximality. Also, every r'-element of $C_G(t)$ has index 1 or n. Since r is not a divisor of n, by Lemma 17 we obtain that n is either a power of p or q, and then the proof of the theorem finishes by Theorem 8. Furthermore, by using the maximality of $C_G(x)$ in G^*, we can also assume without loss that x is has prime power order, and without loss, for instance, that it is a q-element.

Step 3. *If z is an element in G^* such taht $|z^G| = mn$, then $C_G(z) = P_zQ_z \times Z_{\{p,q\}'}$, where P_z is a p-subgroup and Q_z is a q-subgroup of $C_G(z)$ both noncentral in G, and $Z_{\{p,q\}'}$ denotes the $\{p, q\}$-complement of Z. As a consequence, we have that both p and q divide $|C_G(y)/Z|$ for every element y in G^* of index n.*

Let z be an element in G^* of index mn. If r divides $|C_G(z)/Z|$ and $r \neq p, q$, then $|G/Z|_r > (mn)_r \geq m_r$, but $m_r = |G/Z|_r$ by Step 2. Thus we can write $C_G(z) = P_zQ_z \times Z_{\{p,q\}'}$, with P_z a p-subgroup and Q_z a q-subgroup of $C_G(z)$.

We will prove now that both P_z and Q_z cannot be central in G. Suppose that Q_z is central in G. Since p divides $|C_G(x)/Z|$, where x is the element fixed in Step 2, there exists some p-element in $C_G(x)\backslash Z$. Let us take any noncentral p-element, say w, in $C_G(x)$. Notice that wx cannot have index mn, otherwise $|C_G(wx)/Z| = |C_G(z)/Z|$ is a p-power and this is not possible because $x \in C_G(wx)\backslash Z$. Thus $C_G(wx) = C_G(x) = C_G(w)$, and every q'-element, so in particular every p-element, of $C_G(x)$ is central in $C_G(x)$. Hence $C_G(x) = P_x \times Q_x \times Z_{\{p,q\}'}$ with P_x abelian. Likewise, if t is a q-element of $C_G(w)$, similarly we get $C_G(tw) = C_G(w) = C_G(t)$ and $C_G(w) = C_G(x)$ has an abelian Sylow q-subgroup too. This shows that $C_G(x)$ is abelian, which is a contradiction.

Observe that P_x is not central, otherwise Step 2 would imply that n is a prime power and the theorem is true by Theorem C. Now let t be a p-element in $C_G(x)\backslash Z$. If P_z is central in G then, arguing as in the above paragraph, we have that xt cannot have index mn and so $C_G(xt) = C_G(t) = C_G(x)$. We deduce again that $C_G(t) = C_G(x)$ is abelian, a contradiction, so the step is proved. As we know that n and m are not coprime numbers, from now on we will assume that p is a common prime of m and n (we argue similarly if the common prime is q).

Step 4. *G is a $\{p, q\}$-group. In particular, G is solvable.*

Let y be an element in G^* of index n. We are going to show that we can assume that $C_G(y)$ is not abelian. Suppose that $C_G(y)$ is abelian. First, we will assume that $C_G(y) = N_G(C_G(y))$ and we will get a contradiction. We know that p divides $|C_G(y)/Z|$ by Step 3. Moreover, as $C_G(y)$ is abelian and by the maximality of this centralizer, there exists some p-element $t \in C_G(y)$ such that $C_G(t) = C_G(y)$. Since p divides m and n, we have that the Sylow p-subgroups of G are not abelian. Let P be a Sylow p-subgroup of G such that $t \in P$ and notice that $Z(P) = Z_p$. We can take $\overline{z} \in Z(P/Z_p)$ and then $\overline{z}\overline{t} = \overline{t}\overline{z}$. Whence $z \in N_G(C_G(t)) = C_G(t)$. Thus, $z \in C_G(y)$. By the maximality of $C_G(y)$ and using the fact that $C_G(y)$ is abelian again, we conclude that $C_G(y) = C_G(z)$. However, $P \in N_G(C_G(z)) = C_G(z)$, but this is not possible, as wanted.

Thus, by arguing as in the above paragraph, we can take a p-element $z \in N_G(C_G(y))\backslash C_G(y)$, whence p divides $|N_G(C_G(y))/C_G(y)|$. However, by using Lemma 18 and taking $A = A_0 = C_G(y)$, we obtain $|N_G(C_G(y))/C_G(y)| = p$.

On the other hand, we know that p and q both divide $|C_G(y)/Z|$ by Step 3. Since q divides n, if Q_y is the Sylow q-subgroup of $C_G(y)$, then $Q_y < Q$ for some Sylow q-subgroup Q of G. If $v \in N_Q(Q_y)\backslash Q_y$, then $Q_y \subseteq C_G(y) \cap C_G(y^v)$. As $C_G(y)$ is abelian, we deduce that $C_G(y) = C_G(yv) = C_G(y)^v$. This means that v is a q-element lying in $N_G(C_G(y))\backslash C_G(y)$, but this cannot occur.

Therefore, $C_G(y)$ is not abelian, as wanted. By using Steps 1 and 3, we conclude that $m = p^cq^d$.

Consequently, m and n are $\{p, q\}$-numbers and G is a $\{p, q\}$-group. In particular, G is solvable. $\square$

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