The linear k-arboricity of the Mycielski graph $M(K_n)$

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Abstract: A linear k-forest of an undirected graph G is a subgraph of G whose components are paths with lengths at most k. The linear k-arboricity Of G, denoted by $la_k(G)$, is the minimum number of linear k-forests needed to partition the edge set $E(G)$ of G. In case that the lengths of paths are not restricted, we then have the linear arboricity of G, denoted by $la(G)$. In this paper, the exact values of the linear 3-arboricity and the linear arboricity of the Mycielski graph $M(K_n)$, and the linear k-arboricity of the Mycielski graph $M(K_n)$ when n is even and $k \geq 5$, are obtained.

Key–Words: Linear k-forest; linear k-arboricity; Mycielski graph; bipartite difference

1 Introduction

All graphs considered in this paper are finite, undirected, loopless and without multiple edges. For a positive integer k and a real number x, let

$$[k] = \{1, 2, \cdots, k\},$$

$\lfloor x \rfloor$ and $\lceil x \rceil$ denote the smallest integer not less than x and the largest integer not greater than x, respectively. For integers $a \leq b$, let $[a, b]$ denote the integer set

$$\{a, a + 1, \cdots, b\}.$$

We refer to [24] for terminology in graph theory.

In recent years, many parameters and graph classes were studied. For example, in [28], Zuo showed that a Conjecture holds for all unicyclic graphs and all bicyclic graphs, in [25], Xue, Zuo et al. studied the hamiltonicity and path t-coloring of Sierpiński-like graphs; In [29], Jin and Zuo gave the further ordering bicyclic graphs with respect to the Laplacian spectra radius; In [30], Lai et al. gave a survey for the more recent developments of the research on supereulerian graphs and the related problems; In [31], Jiang and Zhang studied Randomly M_t-decomposable multigraphs and M_2-equipackable multigraphs; and in [32], Zuo et al. studied the equitable colorings of Cartesian product graphs of wheels with complete bipartite graphs.

A decomposition of a graph is a list of subgraphs such that each edge appears in exactly one subgraph in the list. If a graph G has a decomposition $G_1, G_2, \cdots, G_d$, then we say that $G_1, G_2, \cdots, G_d$ decompose G, or G can be decomposed into $G_1, G_2, \cdots, G_d$. Furthermore, a linear k-forest is a forest whose components are paths of lengths at most k. The linear k-arboricity of a graph G, denoted by $la_k(G)$, is the least number of linear k-forests needed to decompose G.

An independent set in a graph is a set of pairwise nonadjacent vertices. A complete graph is a simple graph in which each pair of distinct vertices is joined by an edge. We denote one complete graph on n vertices by K_n. A bipartite graph is one graph whose vertex set can be partitioned into two subsets X and Y so that each edge has one end in X and the other end in Y; such a partition (X, Y) is called a bipartition of the graph. A complete bipartite graph is a simple bipartite graph with bipartition (X, Y) in which each vertex of X is joined to each vertex of Y; if

$$\mid X \mid = m, \quad \mid Y \mid = n,$$

such a graph is denoted by $K_{m,n}$, which is called balanced complete bipartite graph if $m = n$.

The notion of linear k-arboricity of a graph was first introduced by Habib and Peroche [16]. It is a natural generalization of edge coloring. Clearly, a linear 1-forest is induced by a matching, and $la_1(G)$ is the edge chromatic number, or chromatic index, $\chi'(G)$ of a graph. Moreover, the linear k-arboricity $la_k(G)$ is also a refinement of the ordinary linear arboricity $la(G)$ (or $la_\infty(G)$) which is the case when every component of each forest is a path with no length constraint.

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In a search for triangle-free graphs with arbitrarily large chromatic numbers, Mycielski developed an interesting graph transformation as follows: For a graph G with vertex set $V(G) = V$ and edge set $E(G) = E$, the Mycielskian of G is the graph $M(G)$ with vertex set

$$V \cup V' \cup \{w\},$$

where $V' = \{x' | x \in V\}$, and edge set

$$E \cup \{xy' | xy \in E\} \cup \{y'w | y' \in V'\}.$$

The vertex x' is called the $twin$ of the vertex x (and x is also called the twin of x'), and the vertex w is called the root of $M(G)$. If there is no ambiguity we shall always use w as the root of $M(G)$.

In 1982, Habib and Peroche [15] proposed the following conjecture for an upper bound on $la_k(G)$.

Conjecture 1. If G is a graph with maximum degree $\Delta(G)$ and $k \geq 2$, then

$$la_k(G) \leq \begin{cases} \lceil \frac{\Delta(G)\cdot|V(G)|}{2\lfloor \frac{k\cdot|V(G)|}{k+1} \rfloor} \rceil, \\ when \quad \Delta(G) =| V(G) | -1, \\ \\ \lceil \frac{\Delta(G)\cdot|V(G)|+1}{2\lfloor \frac{k\cdot|V(G)|}{k+1} \rfloor} \rceil, \\ when \quad \Delta(G) <| V(G) | -1. \end{cases}$$

For $k =| V(G) | -1$, it is the Akiyama's conjecture [1].

Conjecture 2. [1] $\quad la(G) \leq \lceil \frac{(\Delta(G)+1)}{2} \rceil$.

So far, quite a few results on the verification of Conjecture 1 have been obtained in the literature, especially for graphs with particular structures, such as trees [8, 9, 16], cubic graphs [6, 19, 23], regular graphs [3, 4], planar graphs [20], balanced complete bipartite graphs [12, 14, 13], balanced complete multipartite graphs [27] and complete graphs [8, 11, 12, 26, 13]. The linear 2-arboricity, the linear 3-arboricity, and the lower bound of linear k-arboricity of balanced complete bipartite graph were obtained in [14, 13, 12], respectively. In [17, 18, 25, 28], the exact value of the linear 6-arboricity and 8-arboricity of the complete bipartite graph $K_{m,n}$, the linear $(n-1)$-arboricity of balanced complete multipartite graphs $K_{n(m)}$, Hamming graphs K_n^m, the Cartesian product of K_n with $K_{n,n}$, and the Cartesian product graphs C_{nt}^m were obtained. The circular chromatic numbers of Mycielski's graphs was obtained in [10].

In this paper, our attention focuss on determining the linear 3-arboricity and the linear arboricity of the Mycielski graph $M(K_n)$, as well as the linear k-arboricity ($k \geq 5$) of the Mycielski graph $M(K_n)$ when n is even.

2 Some basic lemmas

Lemma 1. *For any graph G, positive integers m and n, if $m > n$, then*

$$\chi'(G) \geq la_n(G) \geq la_m(G) \geq la(G).$$

Lemma 2. *If H is a subgraph of G, then $la_k(G) \geq la_k(H)$.*

As for a lower bound on $la_k(G)$, since any vertex in a linear k-forest has degree at most 2 and a linear k-forest in a graph G has at most

$$\lfloor \frac{k\cdot \mid V(G) \mid}{k+1} \rfloor$$

edges, the following result is obvious.

Lemma 3. *For any connected graph G with maximum degree $\triangle(G)$, we have*

$$la_k(G) \geq \max\left\{ \lceil \frac{\Delta(G)}{2} \rceil, \lceil \frac{|E(G)|}{\lfloor \frac{k|V(G)|}{k+1} \rfloor} \rceil \right\}.$$

Lemma 4. *[12] For $n \geq 3$, the complete graph K_n is decomposable into edge-disjoint Hamilton cycles if and only if n is odd. For $n \geq 2$, the complete graph K_n is decomposable into edge-disjoint Hamilton paths if and only if n is even.*

Lemma 5. *[12] Let $V(K_{2n}) = \{v_0, v_1, \cdots, v_{2n-1}\}$. For $0 \leq i \leq n-1$, put*

$$P_i = v_{0+i}v_{1+i}v_{2n-1+i}v_{2+i}$$

$$v_{2n-2+i} \cdots v_{n+1+i}v_{n+i}$$

where the subscripts of v_j are taken modulo $2n$. Then P_i, $i = 0, 1, 2, \cdots, n-1$, are disjoint Hamilton paths of complete graph K_{2n}.

Lemma 6. *[12] Let $n = 2k+1$, $n \geq 3$, and*

$$V(K_n) = \{v_0, v_1, \cdots, v_{2k-1}, u\}.$$

Then K_n can be decomposed into k edge-disjoint Hamilton cycles

$$C_i = uv_{0+i}v_{1+i}v_{2k-1+i}v_{2+i}v_{2k-2+i}$$

$$\cdots v_{k+1+i}v_{k+i}u$$

for $0 \leq i \leq k-1$, where the subscripts of v_j are taken modulo $2k$.

The following result came from [21], for the sake of the completeness, we give the proof here.

Lemma 7. *[21] The complete graph K_t is Hamilton cycle decomposable.*

Proof. The result is trivially true for $t = 1, 2$. Let $t = 2m + 1 \geq 3$, and let the vertices of K_t be $v_0, v_1, v_2, \cdots, v_{2m}$. Let H be the Hamilton cycle of K_t, and be given by

$$v_0v_1v_2v_{2m}v_3v_{2m-1}v_4\cdots$$

$$v_{m+3}v_mv_{m+2}v_{m+1}v_0.$$

Let σ be the permutation

$$(v_0)(v_1v_2v_3\cdots v_{2m-1}v_{2m}).$$

Then

$$H(=\sigma^0(H)), \sigma^1(H), \sigma^2(H), \cdots, \sigma^{m-1}(H)$$

is a Hamilton cycle decomposition of K_t.

Let $t = 2m \geq 4$. Let the vertices of K_t be $v_0, v_1, \cdots, v_{2m-1}$. Let H be the Hamilton cycle of K_t:

$$v_0v_1v_2v_{2m-1}v_3v_{2m-2}v_4\cdots$$

$$v_{m-1}v_{m+2}v_mv_{m+1}v_0$$

and let σ be the permutation

$$(v_0)(v_1v_2v_3\cdots v_{2m-2}v_{2m-1}).$$

Then

$$H(=\sigma^0(H)), \sigma^1(H), \sigma^2(H), \cdots, \sigma^{m-2}(H)$$

are $m-1$ edge-disjoint Hamilton cycles of K_t. The remaining edges

$$v_0v_m, v_1v_{2m-1}, v_2v_{2m-2},$$

$$v_3v_{2m-3}, \cdots, v_{m-1}v_{m+1}$$

form a 1-factor of K_t. □

It is well known that the following result holds.

Lemma 8. $\chi'(K_{2n}) = \chi'(K_{2n-1}) = 2n - 1$.

3 Main results

Before state our result, we introduce a notion bipartite difference. Let G be a bipartite graph, and V_1, V_2 be its bipartite sets with

$$V_1 = \{u_{10}, u_{11}, \cdots, u_{1(r-1)}\},$$

and

$$V_2 = \{u_{20}, u_{21}, \cdots, u_{2(s-1)}\}.$$

Suppose that $|V_2| = s \geq |V_1| = r$. For the edge $u_{1p}u_{2q}$ in $G(V_1, V_2)$, the value $(q - p)(mod\ s)$ is called the bipartite difference of the edge $u_{1p}u_{2q}$.

It is easy to find that, an edge set which consisted by the edges in $G(V_1, V_2)$ with the same bipartite difference must be a matching. In fact, if $G(V_1, V_2)$ is a balanced complete bipartite graph $K_{n,n}$, then such a matching is a perfect matching. Furthermore, we can decompose the edges of $K_{n,n}$ into n pairwise disjoint perfect matchings $M_0, M_1, \cdots, M_{n-1}$ such that M_i is exactly the set of edges of bipartite difference i in $K_{n,n}$ for $i = 0, 1, \cdots, n - 1$.

Theorem 9. $\chi'(M(K_n)) = \Delta + 1 = 2n - 1$.

Proof. Let the vertex set and edge set of the complete graph K_n be

$$V(K_n) = \{v_i \mid i \in [1, n]\},$$

and

$$E(K_n) = \{v_iv_j \mid i, j \in [1, n], i \neq j\},$$

respectively. Then by the definition of Mycielski graph, the vertex set and edge set of $M(K_n)$ are

$$V(M(K_n)) = \{v_i, u_i \mid i \in [1, n]\} \cup \{w\},$$

and

$$E(M(K_n)) = E(K_n) \cup \{wu_i \mid i \in [1, n]\}$$
$$\cup \{u_iv_j \mid i, j \in [1, n], i \neq j\},$$

respectively, where u_i is the *twin* of v_i for $i \in [1, n]$.

Now we consider the edge set

$$\{u_iv_j \mid i, j \in [1, n], i \neq j\}$$

first. By the definition of Mycielski graph, it is easy to find that the subgraph which induced by the edge set

$$\{u_iv_j \mid i, j \in [1, n], i \neq j\}$$

can be considered as a subgraph which is induced by a complete bipartite graph $K_{n,n}$ get rid of a perfect matching M_0, denoted by $K_{n,n} \setminus M_0$. It is clear that $K_{n,n} \setminus M_0$ can be decomposed into $n-1$ disjoint perfect matchings, denoted by $M_1, M_2, \cdots, M_{n-1}$, respectively, where

$$M_\alpha = \{v_iu_{i+\alpha(mod\,n)} \mid i \in [1, n]\}$$

for $\alpha \in [0, n-1]$. Then we can use α to color M_α for $\alpha \in [1, n-1]$, and use at least $n-1$ colors to color the edges

$$\{u_iv_j \mid i, j \in [1, n], i \neq j\}.$$

By Lemma 8, we can use at least $n-1$ colors to color $E(K_n)$. By the fact that $d(w)=n$, we can use at least another n colors which are different from the $n-1$ colors that colored the edges

$$\{u_iv_j \mid i,j\in[1,n], i\neq j\}$$

to color wu_i for $i\in[1,n]$. Thus, $\chi'(M(K_n))\geq 2n-1$.

On the other hand, we can use n colors, say $1,2,\cdots,n$, to color wu_i for $i\in[1,n]$. By the above fact, we can use color $n+\alpha$ to color M_α for $\alpha\in[1,n-1]$. By Lemma 8, we can use n colors, say $1,2,\cdots,n$, to color $E(K_n)$, thus $\chi'(M(K_n))\leq 2n-1$.

Thus, we have obtained that $\chi'(M(K_n))=\Delta+1=2n-1$. □

Theorem 10. *$la_3(M(K_n))=n$.*

Proof. Similarly as in Theorem 9, let the vertex set and edge set of the complete graph K_n be $V(K_n)=\{v_i \mid i\in[1,n]\}$, and $E(K_n)=\{v_iv_j \mid i,j\in[1,n], i\neq j\}$, respectively. Then the vertex set and edge set of $M(K_n)$ are

$$V(M(K_n))=\{v_i,u_i \mid i\in[1,n]\}\cup\{w\},$$

and

$$E(M(K_n))=E(K_n)\cup\{wu_i \mid i\in[1,n]\}$$
$$\cup\{u_iv_j \mid i\in[1,n], j\in[1,n], i\neq j\},$$

respectively, where u_i is the *twin* of v_i ($i\in[1,n]$).

We consider two cases according to the parity of n.

Case 1. n is odd.

Let $n=2m+1$. By the fact that

$$\chi'(K_n)=n,$$

we can use n colors, say $1,2,\cdots,n$, to color $E(K_n)$.

In the following, we show that there exists an edge coloring of K_n such that for any two vertices of $V(K_n)$ the color sets appear on the edges which are adjacent with them are different. We can consider the vertices of K_n as the vertices of an n-regular polygon, label them by $1,2,\cdots,n$ ordered, and label the edges by the labels of the vertices in the n-regular polygon which are parallel with them. Then we can consider the labels of the edges are just their coloring, and it is easy to find that this is a proper edge coloring of K_n, and for any two vertices of $V(K_n)$ the color sets appear on the edges which are adjacent with them are different.

Then by the fact that $\chi'(K_n)=n$, the above coloring is just a normal edge coloring. Since $d_{K_n}(v_i)=n-1$ for any vertex v_i, there exists just one color that does not appear on the edges which are adjacent with v_i, where $i\in[1,n]$. If color j does not appear on the edges which are adjacent with v_i, then we can denote v_i as v_{2j} where $2j$ is taken modulo n and $mod\ (2j)\in[1,n]$, since n is odd. Accordingly, u_i is denoted by u_{2j} for every $i\in[1,n]$.

Now we color the edge set

$$\{u_iv_j \mid i,j\in[1,n], i\neq j\}.$$

It is easy to find that the subgraph which induced by the edge set

$$\{u_iv_j \mid i,j\in[1,n], i\neq j\}$$

can be regarded as a subgraph which induced by a complete bipartite graph $K_{n,n}$ get rid of a perfect matching M_0, denoted by $K_{n,n}\setminus M_0$. It is easy to find that $K_{n,n}\setminus M_0$ can be decomposed into $n-1$ disjoint perfect matchings, denoted by $M_1,M_2,\cdots,M_{n-1}$, where

$$M_\alpha=\{u_iv_{i+\alpha(mod\,n)} \mid i\in[1,n]\}$$

for $\alpha\in[0,n-1]$, then we can use α to color M_α for $\alpha\in[1,n-1]$.

By the definition of Mycielski graph, the degree of the vertex w in $M(K_n)$ is n, then we can use i to color the edge wu_i for $i\in[1,n]$.

Thus, by the edge coloring of K_n, because wu_1 and u_1v_2 are colored by 1, there does not exist an edge v_2v_j for $j\in[1,n]$ with color 1. Similarly, since wu_i and u_iv_{2i} are colored by i, there does not exist an edge $v_{2i}v_j$ with color i for $i\in[2,n-1]$. Hence it is easy to find that every component of the subgraph which induced by the edges with the same color is just a path with length no more than three. Thus we have $la_3(M(K_n))\leq n$ immediately. On the other hand, by Lemma 3, we have

$$la_3(M(K_n))\geq\lceil\frac{|E(G)|}{\lfloor\frac{k|V(G)|}{k+1}\rfloor}\rceil$$
$$\geq\lceil\frac{n(3n-1)}{2\lfloor\frac{3(2n+1)}{4}\rfloor}\rceil\geq\lceil\frac{(2m+1)(6m+2)}{2\lfloor\frac{3(4m+3)}{4}\rfloor}\rceil$$
$$=2m+1=n.$$

Thus, $la_3(M(K_n))=n$, and the result is proved. □

Case 2. n is even.

Let $n=2m$, by the fact that $\chi'(K_n)=n-1$, then we can use $n-1$ colors, say $1,2,\cdots,n-1$, to color the $E(K_n)$.

In the following, we give an edge coloring of K_n with $n-1$ colors. We can consider the $n-1$ vertices of K_n as the vertices of a $(n-1)$-regular polygon, and label them by

$$1, 2, \cdots, n-1$$

ordered, and label the edges by the labels of the vertices of the $(n-1)$-regular polygon which are parallel with them. Then we put the last vertex of K_n in the center of the $(n-1)$-regular polygon, denoted by v, and it is easy to find that it is adjacent to the other $n-1$ vertices of K_n, then we label the edge which connect v and the vertex which label with i by i. Thus we can consider the labels of the edges are just their coloring. It is easy to find that this is a proper edge coloring of K_n.

Now we color the edge set

$$\{u_iv_j \mid i, j \in [1, n], i \neq j\}.$$

It is easy to find that the subgraph which induced by the edge set $\{u_iv_j \mid i, j \in [1, n], i \neq j\}$ can be regarded as a subgraph which induced by a complete bipartite graph $K_{n,n}$ get rid of a perfect matching M_0, denoted by $K_{n,n} \setminus M_0$. It is easy to find that $K_{n,n} \setminus M_0$ can be decomposed into $n-1$ disjoint perfect matchings, denoted by $M_1, M_2, \cdots, M_{n-1}$, respectively, where

$$M_\alpha = \{u_iv_{i+\alpha(mod n)} \mid i \in [1, n]\}$$

for $\alpha \in [0, n-1]$. Then we can use j to color M_j for $j \in [1, n-1]$.

By the definition of Mycielski graph, the degree of the vertex w in $M(K_n)$ is n, then we can use i to color wu_i for $i \in [1, n/2]$, use $j+1$ to color wu_j for $j \in [n/2+1, n-1]$, and use n to color wu_n.

At last, we recolor some edges. We can use color n to color the edges

$$u_1v_2, u_2v_4, \cdots, u_{\frac{n}{2}}v_n, u_{\frac{n}{2}+1}v_3,$$

$$u_{\frac{n}{2}+2}v_5, u_{\frac{n}{2}+3}v_7, \cdots, u_{n-2}v_{n-3}.$$

Because wu_1 has been colored by 1, there does not exist an edge u_1v_k with color 1 for $k \in [2, n]$. Since wu_i is colored by i, there does not exist an edge u_iv_k with color $i \in [2, \frac{n}{2}]$, for $k \in [1, n] \setminus \{i\}$. Because $wu_{\frac{n}{2}+1}$ is colored by $\frac{n}{2}+2$, there does not exist an edge $u_{\frac{n}{2}+1}v_k$ with $\frac{n}{2}+2$ for $k \neq n/2+1$. Since wu_j is colored by $j+1$, there does not exist an edge u_jv_k with color $j+1$, for

$$j \in [\frac{n}{2}+2, n-1],$$

and any $k \in [1, n] \setminus \{j\}$. So it is easy to find that every component of the subgraph which induced by the edges with the same color is just a path with length no more than three. Thus we have $la_3(M(K_n)) \leq n$ immediately. On the other hand, by Lemma 3, we have

$$la_3(M(K_n)) \geq \lceil \frac{|E(G)|}{\lfloor \frac{k|V(G)|}{k+1} \rfloor} \rceil$$

$$\geq \lceil \frac{n(3n-1)}{2\lfloor \frac{3(2n+1)}{4} \rfloor} \rceil \geq \lceil \frac{(2m)(6m-1)}{2\lfloor \frac{3(4m+1)}{4} \rfloor} \rceil$$

$$= 2m = n.$$

Thus, $la_3(M(K_n)) = n$, and the result is proved. □

Theorem 11. $la(M(K_n)) = n-1$.

Proof. Similarly as in Theorem 9, let the vertex set and edge set of the complete graph K_n be

$$V(K_n) = \{v_i \mid i \in [1, n]\},$$

and

$$E(K_n) = \{v_iv_j \mid i, j \in [1, n], i \neq j\}.$$

Then the vertex set and edge set of $M(K_n)$ are, respectively,

$$V(M(K_n)) = \{v_i, u_i \mid i \in [1, n]\} \cup \{w\},$$

and

$$E(M(K_n)) = E(K_n) \cup \{wu_i | i \in [1, n]\}$$

$$\cup \{u_iv_j \mid i \in [1, n], j \in [1, n], i \neq j\},$$

where u_i is the *twin* of v_i for $i \in [1, n]$.

We consider two cases according to the parity of n.

Case 1. $n = 2m$ is even.

By Lemma 7, we know that the edge set of the complete graph K_n can be decomposed into $m-1$ disjoint Hamilton cycles

$$H_k = v_{2m}v_{1+k}v_{2+k}v_{2m-1+k}v_{3+k}v_{2m-2+k} \cdots$$

$$v_{m+2+k}v_{m+k}v_{m+1+k}v_{2m},$$

for $0 \leq k \leq m-2$, and a 1-factor

$$F = \{v_0v_m, v_1v_{2m-1}, v_2v_{2m-2}, v_3v_{2m-3},$$

$$\cdots, v_{m-1}v_{m+1}\},$$

where the subscripts of v_j are taken modulo $2m-1$ and $mod\ j \in [1, 2m-1]$ in H_k except the terminal and end vertex. Clearly, every even cycle H_k can be decomposed into two 1-factors:

$$\{v_{2m}v_{1+k}, v_{2+k}v_{2m-1+k}, v_{3+k}v_{2m-2+k},$$

$$\cdots, v_{m+k}v_{m+1+k}\}$$

and

$$\{v_{1+k}v_{2+k}, v_{2m-1+k}v_{3+k}, \cdots,$$
$$v_{m+2+k}v_{m+k}, v_{m+1+k}v_{2m}\},$$

for $0 \leq k \leq m-2$. Thus, $E(K_n)$ can be decomposed into $2(m-1)+1 = n-1$ 1-factors, and we can color a 1-factor by one color, and color F by color 1, so we can color this $(n-1)$ 1-factors by $n-1$ colors, say $1, 2, \cdots, n-1$.

Now we color the edge set

$$\{u_iv_j \mid i, j \in [1, n], i \neq j\}.$$

The subgraph which induced by the edge set

$$\{u_iv_j \mid i, j \in [1, n], i \neq j\}$$

can be considered as a subgraph which induced by a complete bipartite graph $K_{n,n}$ get rid of a perfect matching M_0, denoted by $K_{n,n} \setminus M_0$. It is clear that $K_{n,n} \setminus M_0$ can be decomposed into $n-1$ disjoint perfect matchings, denoted by $M_1, M_2, \cdots, M_{n-1}$, respectively, where

$$M_\alpha = \{u_iv_{i+\alpha(mod\ n)} \mid i \in [1, n]\}$$

for $\alpha \in [0, n-1]$, then we can use color i to color M_i for $i \in [1, n-1]$.

According to the definition of Mycielski graph, the degree of the vertex w in $M(K_n)$ is n, then we can use i to color wu_i for $i \in [1, n-1]$, and use 1 to color wu_n.

Since $wu_1, wu_n, u_1v_2, u_nv_1$ are colored by 1, and the color of v_1v_2 is not 1, it is easy to find that every component of the subgraph which induced by the edges with the same color is just a path. Thus we have $la(M(K_n)) \leq n-1$ immediately.

On the other hand, by Lemma 3, we have

$$la(M(K_n)) \geq \lceil \frac{\Delta(M(K_n))}{2} \rceil$$
$$= \lceil \frac{2(n-1)}{2} \rceil = n-1.$$

Hence we obtain that $la(M(K_n)) = n-1$..

Case 2. $n = 2m+1$ is odd.

Subcase 2.1. m is odd.

It is obvious that the complete graph K_{2m+1}, with

$$V(K_{2m+1}) = \{v_1, v_2, \cdots, v_{2m}, v_{2m+1}\},$$

can be decomposed into m edge-disjoint Hamilton cycles

$$C_i = v_{2m+1}v_{1+i}v_{2+i}v_{2m+i}v_{3+i}v_{2m-1+i}$$
$$\cdots v_{m+2+i}v_{m+1+i}v_{2m+1}$$

for $0 \leq i \leq m-1$, where the subscripts of v_j are taken modulo $2m+1$ and the subscripts of v_j belong to $[1, 2m]$ except v_{2m+1}.

Next, we take away the $(m+1)$-th edge from each Hamilton cycle C_i for $i \in [0, m-1]$. After taking away the $(m+1)$-th edge from each Hamilton cycle C_i $(i \in [0, m-1])$, we have m Hamilton paths and the edges we taken away are

$$v_1v_{m+1}, v_2v_{m+2}, v_3v_{m+3}, \cdots, v_mv_{2m}.$$

If the $(m+1)$-th edge of a Hamilton cycle is v_1v_{m+1}, then after taking away the edge v_1v_{m+1} from this Hamilton cycle, we can color it by color 1. Similarly, if the $(m+1)th$ edge of a Hamilton cycle is v_iv_{m+i}, then after taking away the edge v_iv_{m+i} from this Hamilton cycle, we can color it by color i for $i \in [2, m]$.

Now we color the edge set

$$\{u_iv_j \mid i, j \in [1, n], i \neq j\}.$$

The subgraph which induced by the edge set

$$\{u_iv_j \mid i, j \in [1, n], i \neq j\}$$

can be viewed as a subgraph which induced by a complete bipartite graph $K_{n,n}$ get rid of a perfect matching M_0, denoted by $K_{n,n} \setminus M_0$. It is easy to find that $K_{n,n} \setminus M_0$ can be decomposed into $n-1$ disjoint perfect matchings, denoted by $M_1, M_2, \cdots, M_{n-1}$, respectively, where

$$M_\alpha = \{u_iv_{i-\alpha(modn)} \mid i \in [1, n]\},$$

for $\alpha \in [0, n-1]$, so these edges can be decomposed into $M_1, M_2, \cdots, M_{2m}$.

It is easy to see that the edges of M_1 and M_{m+1} can just form a cycle, after taking away two edges u_2v_1 and $u_{m+2}v_{m+1}$ in this cycle, we can color other edges by one color, say $m+1$. It is clear that we can give the edge v_1v_{m+1} color $m+1$. Similarly, the edges of M_2 and M_{m+2} can just form a cycle, after taking away two edges

$$u_4v_2, u_{m+4}v_{m+2}$$

from this cycle, we can color other edges by one color, say $m+2$. It is obvious that we can give the edge v_2v_{m+2} color $m+2$. So the edges of M_i and M_{m+i} can just form a cycle, after taking away $u_{2i}v_i, u_{m+2i}v_{m+i}$ from this cycle, we can color other edges by $m+i$. It is easy to find that we can give the edge v_iv_{m+i} the color $m+i$ for $i \in [3, m]$. Thus we can color u_2v_1 and $u_{m+2}v_{m+1}$ by 1, color u_4v_2

and $u_{m+4}v_{m+2}$ by 2, and color $u_{2i}v_i$ and $u_{m+2i}v_{m+i}$ by i for $i \in [3, m]$. By the fact that m is odd, $2i$ is even and $m+2$ is odd, we can color wu_2 and wu_1 by 1, and color wu_{2i} and wu_{2i-1} by i for $i \in [2, \frac{m+1}{2}]$. We can color wu_{m+2} by $m+1$, color wu_{m+3} and wu_{m+4} by $\frac{m+3}{2}$, and color wu_{2j} and wu_{2j+1} by j for $j \in [\frac{m+5}{2}, m]$, where the subscripts are all taken modulo $2m+1$ and $mod\ j \in [1, 2m+1]$.

It is easy to find that every component of the subgraph which induced by the edges with the same color is just a path. Thus we have

$$la(M(K_n)) \leq n-1$$

immediately. On the other hand, by Lemma 3, we have

$$la(M(K_n)) \geq \lceil \frac{\Delta(M(K_n))}{2} \rceil$$

$$= \lceil \frac{2(n-1)}{2} \rceil = n-1.$$

Hence $la(M(K_n)) = n-1$.

Subcase 2.2. m is even.

Clearly, the complete graph K_{2m+1}, with

$$V(K_{2m+1}) = \{v_1, v_2, \cdots, v_{2m}, v_{2m+1}\},$$

can be decomposed into m edge-disjoint Hamilton cycles

$$C_i = v_{2m+1}v_{1+i}v_{2+i}v_{2m+i}v_{3+i}v_{2m-1+i}$$

$$\cdots v_{m+2+i}v_{m+1+i}v_{2m+1}$$

for $0 \leq i \leq m-1$, where the subscripts of v_j are taken modulo $2m+1$ and the subscripts of v_j belong to $[1, 2m]$ except v_{2m+1}.

Next, after taking away the $(m+1)$-th edge from each Hamilton cycle C_i for $i \in [0, m-1]$, we obtain m Hamilton paths and all the edges we taken away are

$$v_1v_{m+1}, v_2v_{m+2}, v_3v_{m+3}, \cdots,\ and\ v_mv_{2m}.$$

If the $(m+1)th$ edge of a Hamilton cycle is v_1v_{m+1}, then after taking away v_1v_{m+1} from this Hamilton cycle, we can color it by color 1. Similarly, if the $(m+1)th$ edge of a Hamilton cycle is v_iv_{m+i}, then after taking away v_iv_{m+i} from this Hamilton cycle, we can color it by color i for $i \in [2, m]$.

Now we color the edge set

$$\{u_iv_j \mid i, j \in [1, n], i \neq j\}.$$

The subgraph which induced by the edge set

$$\{u_iv_j \mid i, j \in [1, n], i \neq j\}$$

can be regarded as a subgraph which induced by a complete bipartite graph $K_{n,n}$ get rid of a perfect matching M_0, denote it by $K_{n,n} \setminus M_0$. It is obvious that $K_{n,n} \setminus M_0$ can be decomposed into $n-1$ disjoint perfect matchings, denote them by $M_1, M_2, \cdots, M_{n-1}$, where

$$M_\alpha = \{u_iv_{i-\alpha(mod n)} \mid i \in [1, n]\}$$

for $\alpha \in [0, n-1]$. Thus these edges can be decomposed into $M_1, M_2, \cdots, M_{2m}$.

It is easy to find that the edges of M_1 and M_{m+1} can just form a cycle. After taking away edges

$$u_2v_1, u_{m+2}v_{m+1}$$

from this cycle, we can color other edges by one color, say $m+1$. It is clear that we can give the edge v_1v_{m+1} color $m+1$. The edges of M_2 and M_{m+2} can just form a cycle, after taking away edges

$$u_4v_2, u_{m+4}v_{m+2}$$

from this cycle, we can color other edges by $m+2$. It is easy to find that we can give the edge v_2v_{m+2} color $m+2$. So the edges of M_i and M_{m+i} can just form a cycle for every $i \in [3, m-1]$, after taking away edges

$$u_{2i}v_i, u_{m+2i}v_{m+i}$$

from this cycle, we can color other edges by $m+i$, and it is easy to find that we can give the edge v_iv_{m+i} color $m+i$ for $i \in [3, m-1]$. Hence the edges of M_m and M_{2m} can just form a cycle, after taking away edges

$$u_{2m}v_m, u_{m-1}v_{2m}$$

from this cycle, we can color other edges by $2m$. It is easy to find that we can give the edge v_mv_{2m} color $2m$.

Thus we can color edges u_2v_1 and $u_{m+2}v_{m+1}$ by 1, color u_4v_2 and $u_{m+4}v_{m+2}$ by 2, color $u_{2i}v_i$ and $u_{m+2i}v_{m+i}$ by color i for $i \in [3, m-1]$, and color $u_{2m}v_m$ and $u_{m-1}v_{2m}$ by m.

By the fact that m is even, $2i$ is even and $m-1$ is odd, we can color wu_2 and wu_1 by 1, color wu_{2i} and wu_{2i-1} by i for $i \in [2, \frac{m-2}{2}]$, color wu_{m-1} by $2m$, color wu_m and wu_{m+1} by $\frac{m}{2}$, and color wu_{2j} and wu_{2j+1} by color j for $j \in [\frac{m+2}{2}, m]$.

It is easy to find that every component of the subgraph which induced by the edges with the same color is just a path. Thus we have $la(M(K_n)) \leq n-1$, immediately. On the other hand, by Lemma 3, we have

$$la(M(K_n)) \geq \lceil \frac{\Delta(M(K_n))}{2} \rceil$$

$$= \lceil \frac{2(n-1)}{2} \rceil = n - 1.$$

Hence we obtain that $la(M(K_n)) = n - 1$. □

Theorem 12. $la_k(M(K_n)) = n - 1$*, when* n *is even and* $k \geq 5$.

Proof. By the proof of the Theorem 11 in the case when n is even, it is easy to find that every component of the subgraph which induced by the edges with the same color is just a path with length no more than 5, so $la_k(M(K_n)) \leq n - 1$. By Lemma 3, we have

$$la_k(M(K_n)) \geq \lceil \frac{\Delta(M(K_n))}{2} \rceil$$

$$= \lceil \frac{2(n-1)}{2} \rceil = n - 1,$$

hence, $la_k(M(K_n)) = n - 1$ when n is even and $k \geq 5$. □

Acknowledgements: The research is supported by NSFC for youth with code 61103073.

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