Resolution of Stokes Equations with the $C_{a,b}$ Boundary Condition Using Mixed Finite Element Method

JAOUAD EL-MEKKAOUI and AHMED ELKHALFI
Departement of Genie Mecanique
Sidi Mohammed ben abdellah University
Faculte des Sciences et Techniques B.P. 2202
-Route d’Imouzzer - Fès
MOROCCO
Jawad-mekkaoui@hotmail.com

ABDESLAM ELAKKAD
Centre Régional des Métiers d’Education et de Formation (CRMEF), Fès
B.P: 243 Sefrou,
MAROC.
elakkadabdeslam@yahoo.fr

Abstract: In this paper, we introduce the Stokes equations with a new boundary condition. In this context, we show the existence and uniqueness of the solution of the weak formulation associated with the proposed problem. To solve this problem, we use the discretization by mixed finite element method. In addition, two types of a posteriori error indicator are introduced and are shown to give global error estimates that are equivalent to the true error. In order to evaluate the performance of the method, the numerical results are compared with some previously published works and with others coming from commercial code like ADINA system.

Key–Words: Stokes equations, $C_{a,b}$ Boundary condition, Mixed finite element method, Residual error estimator, Adina system.

1 Introduction

This paper describes a numerical solution of Stocks equations with a new boundary generalizes the well known basis conditions, especially the Dirichlet and the Neumann conditions. So, we prove that the weak formulation of the proposed modeling has an unique solution. To calculate this latter, we use the discretization by mixed finite element method. Moreover, we propose two types of a posteriori error indicator which are shown to give global error estimates that are equivalent to the true error. To compare our solution with the some previously ones, as ADINA system, some numerical results are shown. In this modeling flow of porous media, it is essential to use a discretization method which satisfies the physics of the problem, i.e. conserve mass locally and preserve continuity of flux. The Raviart-Thomas Mixed Finite Element (MFE) method of lowest order satisfies these properties. Moreover, both the pressure and the velocity are approximated with the same order of convergence [4, 6]. The discretization of the velocity is based on the properties of Raviart-Thomas. Other works have been introduced by Brezzi, Fortin, Marini, Dougla and Robert [4, 5, 7]. This method was widely used for the prediction of the behavior of fluid in the hydrocarbons tank. A posteriori error analysis in problems related to fluid dynamics is a subject that has received a lot of attention during the last decades. In the conforming case there are several ways to define error estimators by using the residual equation. in particular, for the Stokes problem, M. Ainsworth, J. Oden [9], R.E. Bank, B.D. Welfert [10], C. Crestensen, S.A. Funken [11], D. Kay, D. Silvestre [12] and R. Verfurth [13], introduced several error estimators and provided that they are equivalent to the energy norm of the errors. Other works for the stationary Navier-Stokes problem have been introduced in [14, 17, 18, 20, 16]. This paper describes a numerical solution of Stokes equations with a boundary condition noted $C_{a,b}$. For the equations, we offer a choice of tow-dimensional domains on which the problem can be posed, along with boundary conditions and other aspects of the problem, and a choice of finite element discretization on a rectangular element mesh.

The plan of the paper is as follows. Section 2 presents the model problem used in this paper. The weak formulation is presented in section 3. In section 4, we show the existence and uniqueness of the solution. The discretization by mixed finite elements is described in section 5. Section 6 introduced two types of a posteriori error bounds of the computed solution. Numerical experiments carried out within the framework of this publication and their comparisons with other results are shown in Section 7.
2 Governing Equations

We will consider the model of viscous incompressible flow in an idealized, bounded, connected domain in \( \mathbb{R}^2 \).

\[
\begin{align*}
-\Delta \mathbf{u} + \mathbf{f} &= \nabla p \quad \text{in } \Omega, \\
-\nabla \cdot \mathbf{u} &= 0 \quad \text{in } \Omega.
\end{align*}
\] (1)

by the \( C_{a,b} \) boundary condition

\[
a \mathbf{u} + b(\nabla \mathbf{u} - p I) \mathbf{u} = \mathbf{0} \quad \text{in } \Gamma := \partial \Omega \tag{3}
\]

We also assume that \( \Omega \) has a polygonal boundary \( \Gamma : = \partial \Omega \), so \( \mathbf{u} \) is the usual outward-pointing normal. The vector field \( \mathbf{u} \) is the velocity of the flow and the scalar variable \( p \) represents the pressure.

Our mathematical model is the Stoks system with a new boundary condition noted \( C_{a,b} \), \( \nabla \) is the gradient, \( \nabla \cdot \) is the divergence and \( \nabla^2 \) is the Laplacian operator, \( \mathbf{f} \), \( \mathbf{u} \) and \( b \) are the polynomials such that, \( \mathbf{f} \) defined in \( \Omega \) and \( a, b \) nonzero defined on \( \partial \Omega \) verify:

There are two strictly positive constants \( \alpha_1 \) and \( \beta_1 \) such that:

\[
\alpha_1 \leq \frac{a(x)}{b(y)} \leq \beta_1 \text{ for all } x \in \Gamma. \tag{4}
\]

Remark. If \( a \) and \( b \) are two strictly positive constants such that \( a << b \) then \( C_{a,b} \) is the Neumann boundary condition and if \( b << a \) then \( C_{a,b} \) is the Dirichlet boundary condition. for that \( a \) is called the Dirichlet coefficient and \( b \) is the Neumann coefficient.

3 The weak formulation

We define the spaces:

\[
\begin{align*}
h^1 &= \{ \mathbf{u} : \Omega \rightarrow \mathbb{R}^2; \quad \frac{\partial \mathbf{u}}{\partial x}, \frac{\partial \mathbf{u}}{\partial y} \in L^2(\Omega) \} \tag{5} \\
H^1(\Omega) &= \{ h^1 \}^2 \tag{6} \\
L^0_0(\Omega) &= \{ q \in L^2(\Omega); \quad \int_{\Omega} q = 0 \} \tag{7}
\end{align*}
\]

The standard weak formulation of the Stokes flow problem (1) - (2)-(3) is the following:

Find \( \mathbf{u} \in H^1(\Omega) \) and \( p \in L^0_0(\Omega) \) such that:

\[
\begin{align*}
\int_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{v} + \int_{\Gamma_b} a \mathbf{u} \cdot \mathbf{n} \cdot \mathbf{v} \\
- \int_{\Omega} p \nabla \cdot \mathbf{u} = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} + \int_{\Gamma} \frac{1}{b} \mathbf{n} \cdot \mathbf{v} \\
\int_{\Omega} q \nabla \cdot \mathbf{u} = 0
\end{align*}
\] (8)

for all \( \mathbf{v} \in H^1(\Omega) \) and \( q \in L^2_0(\Omega) \).

Let the bilinear forms

\[
\begin{align*}
A(\mathbf{u}, \mathbf{v}) &= \int_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{v} + \int_{\Gamma_b} a \mathbf{u} \cdot \mathbf{n} \cdot \mathbf{v} \\
B(\mathbf{u}, q) &= -\int_{\Omega} q \nabla \cdot \mathbf{u}
\end{align*}
\] (9)

Given the functional \( L : H^1(\Omega) \rightarrow \mathbb{R} \)

\[
L(\mathbf{v}) = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} + \int_{\Gamma} \frac{1}{b} \mathbf{n} \cdot \mathbf{v} \tag{11}
\]

The underlying weak formulation (8) may be restated as:

Find \( \mathbf{u} \in H^1(\Omega) \) and \( p \in L^0_0(\Omega) \) such that:

\[
\begin{align*}
\left\{ \begin{array}{ll}
A(\mathbf{u}, \mathbf{v}) + B(\mathbf{u}, p) = L(\mathbf{v}) \\
B(\mathbf{u}, q) = 0
\end{array} \right.
\] (12)

for all \( \mathbf{v} \in H^1(\Omega) \) and \( q \in L^2_0(\Omega) \).

4 The existence and uniqueness of the solution

In this section we will study the existence and uniqueness of the solution of problem (12), for that we need the following results:

**Theorem 1** There are two strictly positive constants \( c_1 \) and \( c_2 \) such that:

\[
c_1 \| \mathbf{v} \|_{1,\Omega} \leq \| \mathbf{v} \|_{J,\Omega} \leq c_2 \| \mathbf{v} \|_{1,\Omega} \tag{13}
\]

where \( \| \mathbf{v} \|_{J,\Omega} = [A(\mathbf{v}, \mathbf{v})]^{\frac{1}{2}} \tag{14} \)

\[
\| \mathbf{v} \|_{1,\Omega} = \left[ \int_{\Omega} \nabla \mathbf{v} : \nabla \mathbf{v} + \int_{\Gamma} \frac{1}{b} \mathbf{n} \cdot \mathbf{v} \right]^{\frac{1}{2}} \leq \left[ \| \mathbf{v} \|^2_{1,\Omega} + \| \mathbf{v} \|^2_{0,\Gamma} \right]^{\frac{1}{2}} \tag{15}
\]

**Proof.** The mapping \( \gamma_0 : H^1(\Omega) \rightarrow L^2(\Gamma) \), such that \( \gamma_0(\mathbf{v}) = \mathbf{v} / \Gamma \), is continuous (see [2] theorem 1, 2), then there exists \( c > 0 \) such that:

\[
\| \mathbf{v} \|^2_{0,\Gamma} \leq c \| \mathbf{v} \|_{1,\Omega} \quad \forall \mathbf{v} \in H^1(\Omega).
\]

Using this result and (4) give, \( \| \mathbf{v} \|_{J,\Omega} \leq c_2 \| \mathbf{v} \|_{1,\Omega} \) for all \( \mathbf{v} \in H^1(\Omega) \) with \( c_2 = (\beta_1 c_1^2 + 1)^{\frac{1}{2}} \).

On the other hand, according to 5.55 in [1], there exists a constant \( \rho > 0 \) such that:

\[
\| \mathbf{v} \|^2_{0,\Gamma} \leq \rho (\| \nabla \mathbf{v} \|^2_{0,\Omega} + \| \mathbf{v} \|^2_{0,\Gamma}).
\]
Using (4) and \( \| \nabla \overline{\mathbf{v}} \|_{1,\Omega} \leq \| \overline{\mathbf{v}} \|_{2,\Omega} \), give
\[
c_1 \| \overline{\mathbf{v}} \|_{1,\Omega} \leq \| \overline{\mathbf{v}} \|_{J,\Omega} \quad \forall \overline{\mathbf{v}} \in H^1(\Omega),
\]
with \( c_1 = (\nu C + 1)^{-\frac{1}{2}} \) and \( C = \max \{ \alpha_1; 1 \} \). □

This result allows us to prove that \( (H^1(\Omega), \| . \|_{J,\Omega}) \) is a Hilbert space which is obliged condition for obtain the existence and uniqueness of the solution.

**Theorem 2** \( (H^1(\Omega), \| . \|_{J,\Omega}) \) is a real Hilbert space.

**Proof:** \( (H^1(\Omega), \| . \|_{J,\Omega}) \) is a real Hilbert space and \( \| . \|_{1,\Omega} \) and \( \| . \|_{J,\Omega} \) are equivalent norms, then \( (H^1(\Omega), \| . \|_{J,\Omega}) \) is a real Hilbert space. □

**Theorem 3** \( B \) satisfies the inf-sup condition: there exists a constant \( \beta > 0 \) such that:
\[
\sup_{\overline{\mathbf{w}} \in H^1(\Omega)} \frac{B(\overline{\mathbf{w}}, q)}{\| \overline{\mathbf{w}} \|_{J,\Omega}} \geq \beta \| q \|_{0,\Omega}.
\]
for all \( q \in L^2_0(\Omega) \).

**Proof.** The same proof of [2] it suffices to see that \( H^1_0(\Omega) \subset H^1(\Omega) \) and \( \| \overline{\mathbf{v}} \|_{J,\Omega} = \| \overline{\mathbf{v}} \|_{1,\Omega} \) in \( H^1_0(\Omega) \).

We define the “big” symmetric bilinear form
\[
C[\langle \overline{\mathbf{w}}, p \rangle; \langle \overline{\mathbf{v}}, q \rangle] = A(\overline{\mathbf{w}}, \overline{\mathbf{v}}) + B(\overline{\mathbf{w}}, q) + B(\overline{\mathbf{v}}, q)
\]
(17)

And the corresponding function \( F(\overline{\mathbf{v}}, q) = L(\overline{\mathbf{v}}) \)
choosing the successive test vectors \( \overline{\mathbf{w}}, 0 \) and \( \overline{\mathbf{v}}, q \) shows that the Stokes problem (12) can be rewritten in the form:

Find \( \langle \overline{\mathbf{w}}, p \rangle \in H^1(\Omega) \times L^2_0(\Omega) \) such that
\[
C[\langle \overline{\mathbf{w}}, p \rangle; \langle \overline{\mathbf{v}}, q \rangle] = F(\overline{\mathbf{v}}, q)
\]
(18)

for all \( \langle \overline{\mathbf{w}}, q \rangle \in H^1(\Omega) \times L^2_0(\Omega) \). □

The bilinear form \( A \) is positive continuous and \( H^1(\Omega) \)-elliptic and the bilinear form \( b \) is continuous and satisfies the inf-sup condition. Then the problem (12) is well-posed, and the forms bilinear \( C \) and \( A \) satisfies the following propositions.

**Proposition 4** for all \( \langle \overline{\mathbf{w}}, s \rangle \in H^1(\Omega) \times L^2_0(\Omega) \), we have
\[
\sup_{\langle \overline{\mathbf{w}}, q \rangle \in H^1 \times L^2_0} \frac{C[\langle \overline{\mathbf{w}}, p \rangle; \langle \overline{\mathbf{v}}, q \rangle]}{\| \overline{\mathbf{v}} \|_{J,\Omega} + \| q \|_{0,\Omega}} \geq \delta(\| \overline{\mathbf{w}} \|_{J,\Omega} + \| s \|_{0,\Omega}).
\]
(19)

**Proof.** See [1].

**Proposition 5** For all \( \langle \overline{\mathbf{w}}, s \rangle \in H^1 \times L^2_0(\Omega) \), we have
\[
\sup_{\langle \overline{\mathbf{w}}, q \rangle \in H^1 \times L^2_0} \frac{A(\overline{\mathbf{w}}, \overline{\mathbf{v}}) + d(s, q)}{\| \overline{\mathbf{v}} \|_{J,\Omega} + \| q \|_{0,\Omega}} \geq \frac{1}{2}(\| \overline{\mathbf{w}} \|_{J,\Omega} + \| s \|_{0,\Omega}),
\]
(20)

where \( d(s, q) = \int_\Omega s q \).

**Proof.** Let \( \langle \overline{\mathbf{w}}, q \rangle \in H^1(\Omega) \times L^2_0(\Omega) \). We will take \( q = 0 \) and \( \overline{\mathbf{v}} = \overline{\mathbf{w}} \) in the first and \( \overline{\mathbf{v}} = \overline{\mathbf{v}} \) and \( q = s \) in the second, we obtain
\[
\sup_{\langle \overline{\mathbf{w}}, q \rangle \in H^1 \times L^2_0} \frac{A(\overline{\mathbf{w}}, \overline{\mathbf{v}}) + d(s, q)}{\| \overline{\mathbf{v}} \|_{J,\Omega} + \| q \|_{0,\Omega}} \geq \| \overline{\mathbf{v}} \|_{J},
\]
(21)

we gather (21) and (22) to get (20). □

The bilinear form \( A \) is symmetric and continuous and semi positive definite on \( H^1(\Omega) \), in this case we say the problem (12) is a type of saddle-point problem. The results (13)-(16) ensure the existence and uniqueness of the solution of the problem (12) (see Theorem 6.2 in [1]). In the following section we will solve this problem by mixed finite element method.

**5 Mixed finite element approximation**

Let \( T_h; h \geq 0 \), be a family of rectangles sharing at least one edge with element \( T \), \( \hat{T} \) is the set of rectangles sharing at least one vertex with \( T \). Also, for an element edge \( E \), \( \hat{E} \) denotes the union of rectangles sharing \( E \)., while \( \hat{E} \) is the set of rectangles sharing at least one vertex whit \( E \). Next, \( \partial T \) is the set of the four edges of \( T \) we denote by \( \varepsilon(T) \) and \( N_T \) the set of its edges and vertices, respectively. We let \( \varepsilon_h = \bigcup_{T \in T_h} \varepsilon(T) \) denotes the set of all edges split into interior and boundary edges. \( \varepsilon_h = \varepsilon_{h,\Omega} \bigcup \varepsilon_{h,\Gamma} \) where
\[
\varepsilon_{h,\Omega} = \{ E \in \varepsilon_h : E \subset \Omega \}
\]
\[
\varepsilon_{h,\Gamma} = \{ E \in \varepsilon_h : E \subset \partial \Omega \}
\]
We denote by \( h_T \) the diameter of a simplex, by \( h_E \) the diameter of a face \( E \) of \( T \), and we set \( h = \max_{T \in T_h} \{ h_T \} \). A discrete weak formulation is defined using finite dimensional spaces \( X^1_h \subset H^1(\Omega) \) and \( M^h \subset L^2_0(\Omega) \).
The discrete version of (12) is:

\[
\begin{align*}
\text{find } & \quad \mathbf{v}_h \in X_h^1 \text{ and } p_h \in M_h \text{ such that: } \\
A(\mathbf{v}_h, \mathbf{v}_h) + B(\mathbf{v}_h, p_h) &= L(\mathbf{v}_h), \\
B(\mathbf{v}_h, p_h) &= 0
\end{align*}
\]

for all \( \mathbf{v}_h \in X_h^1 \) and \( q_h \in M_h \).

We use a set of vector-valued basis functions \{\mathbf{v}_n\}_{n=1}^N\) and set

\[
p_h = \sum_{k=1}^{n_p} p_k \phi_k
\]

where \( n_u \) and \( n_p \) are the numbers of velocity and pressure basis functions, respectively.

We find that the discrete formulation (23) can be expressed as a system of linear equations

\[
\begin{pmatrix}
A_0 & B_0^T \\
B_0 & 0
\end{pmatrix}
\begin{pmatrix}
U \\
P
\end{pmatrix} =
\begin{pmatrix}
f \\
0
\end{pmatrix}.
\]

The system is referred to as the discrete Newton problem. The matrix \( A_0 \) is the vector Laplacian matrix and \( B_0 \) is the divergence matrix

\[
A_0 = [a_{i,j}];
\]

\[
a_{i,j} = \int_\Omega \nabla \mathbf{v}_i : \nabla \mathbf{v}_j + \int_{\partial \Omega} \frac{a}{b} \mathbf{v}_i \cdot \mathbf{v}_j.
\]

\[
B_0 = [b_{k,j}]; b_{k,j} = -\int_\Omega \psi_k \nabla \mathbf{v}_j.
\]

\[
f = [f_i]; f_i = \nu \int_{\partial \Omega} \mathbf{v}_i \cdot \mathbf{n} + \int_{\partial \Omega} \frac{1}{b} \mathbf{v}_i
\]

for \( i, j = 1, \ldots, n_u \), \( k = 1, \ldots, n_p \), and the function pair \((\mathbf{v}_h, p_h)\) obtained by substituting the solution vectors \( U \in \mathbb{R}^{n_u} \) and \( P \in \mathbb{R}^{n_p} \) into (24) and (25) is the mixed finite element solution. The system (26)-(29) is henceforth referred to as the discrete stokes problem. We use the iterative methods Minimum Residual Method (MINRES) for solving the symmetric system.

6 A Posteriori error estimator

In this section, we propose two types of a posteriori error indicator; The first one is the residual error estimator and the second one is the local Poisson problem estimator. Which are shown to give global error estimates that are equivalent to the true error.

6.1 A Residual error estimator

The bubble functions on the reference element \( \tilde{T} = (0, 1) \times (0, 1) \) are defined as follows

\[
\begin{align*}
b_{\tilde{T}} &= 2^1 x(1-x)y(1-y), \\
b_{\tilde{E}_{11}} &= 2^2 x(1-x)(1-y), \\
b_{\tilde{E}_{12}} &= 2^2 y(1-y)x, \\
b_{\tilde{E}_{21}} &= 2^2 y(1-y)(1-x).
\end{align*}
\]

Here; \( b_{\tilde{T}} \) is the reference element bubble function, and \( b_{\tilde{E}_{1i}}, i=1:4 \) are reference edge bubble functions. For any \( T \in T_h \), the element bubble functions is \( b_T = b_{\tilde{T}} \circ F_T \) and the element edge bubble functions is \( b_{E_{iT}} = b_{\tilde{E}_{1i}} \circ F_T \) where \( F_T \) the affine map form \( \tilde{T} \) to \( T \).

For an interior edge \( E \in \tilde{E}_{1i} \), \( b_E \) is defined piecewise, so that \( b_{E_{iT}} = b_{\tilde{E}_{1i}} \circ F_T \) where \( E = \partial \Omega_1 \cap \partial \Omega_2 \). For a boundary edge \( E \in \tilde{E}_{1i}, b_E = b_{E_{iT}} \), where \( T \) is the rectangle such that \( E \in \partial T \).

With these bubble functions, ceruse et al ([19], lemma 4.1) established the following lemma.

Lemma 6 Let \( T \) be an arbitrary rectangle in \( \tau_h \) and \( E \in \partial T \). For any \( \mathbf{v}_T \in P_{k_0}(T) \) and \( \mathbf{v}_E \in P_{k_1}(E) \), the following inequalities hold.

\[
c_k \| \mathbf{v}_T \|_{0,T} \leq \| \mathbf{v}_T b_T \|_{0,T} \leq C_k \| \mathbf{v}_T \|_{0,T}
\]

\[
| \mathbf{v}_T b_T |_{1,T} \leq C_k h_T^{-1} \| \mathbf{v}_T \|_{0,T}
\]

\[
c_k \| \mathbf{v}_E \|_{0,E} \leq \| \mathbf{v}_E b_E \|_{0,E} \leq C_k \| \mathbf{v}_E \|_{0,E}
\]

\[
| \mathbf{v}_E b_E |_{1,E} \leq C_k h_E^{-1} \| \mathbf{v}_E \|_{0,E},
\]

where \( c_k \) and \( C_k \) are tow constants which only depend on the element aspect ratio and the polynomial degrees \( k_0 \) and \( k_1 \).

Here, \( k_0 \) and \( k_1 \) are fixed and \( c_k \) and \( C_k \) can be associated with generic constants \( c \) and \( C \). In addition, \( \mathbf{v}_E \) which is only defined on the edge \( E \) also denotes its natural extension to the element \( T \). From the inequalities (33) and (34) we will establish the following lemma:
Lemma 7 Let $T$ be an rectangle and $E \in \partial T \cap \varepsilon_{h,\Gamma}$. For any $\overrightarrow{v}_E \in P_{k_1}(E)$, the following inequalities hold.

$$\|\overrightarrow{v}_E b_E\|_{1,T} \leq Ch_E^{-\frac{1}{2}}\|\overrightarrow{v}_E\|_{0,E}. \quad (35)$$

Proof. Since $\overrightarrow{v}_E b_E = \overrightarrow{0}$ in the other three edges of rectangle $T$, it can be extended to the whole of $\Omega$ by setting $\overrightarrow{v}_E b_E = \overrightarrow{0}$ in $\Omega \setminus T$, then

$$\|\overrightarrow{v}_E b_E\|_{1,T} = \|\overrightarrow{v}_E b_E\|_{1,\Omega}$$

and

$$\|\overrightarrow{v}_E b_E\|_{1,T} = \|\overrightarrow{v}_E b_E\|_{1,\Omega}.$$

Using the inequalities (13), (33) and (34), gives

$$\|\overrightarrow{v}_E b_E\|_{1,T} = \|\overrightarrow{v}_E b_E\|_{1,\Omega} \leq c_2\|\overrightarrow{v}_E b_E\|_{1,\Omega} = c_2\|\overrightarrow{v}_E b_E\|_{1,\Omega} \leq c_2(\|\overrightarrow{v}_E b_E\|_{0,T}^2 + \|\overrightarrow{v}_E b_E\|_{1,T}^2)^{1/2} \leq c_2C_k(h_E + h_E^{-1})^{1/2}\|\overrightarrow{v}_E\|_{0,E} \leq c_2C_k(D^2 + 1)^{1/2}h_E^{-2}\|\overrightarrow{v}_E\|_{0,E} \leq Ch_E^{-\frac{1}{2}}\|\overrightarrow{v}_E\|_{0,E}$$

with $D$ is the diameter of $\Omega$ and $C = c_2C_k(D^2 + 1)^{1/2}$.

□

Lemma 8 Clement interpolation estimate: Given $\overrightarrow{v}_E \in H^1(\Omega)$, let $\overrightarrow{v}_h \in X_h^1$ be the quasi-interpolant of $\overrightarrow{v}_E$ defined by averaging as in [20]. For any $T \in T_h$ and for all $E \in \partial T$, we have

$$\|\overrightarrow{v}_T - \overrightarrow{v}_h\|_{0,T} \leq Ch_T\|\overrightarrow{v}_T\|_{1,\partial T} \quad (36)$$

and

$$\|\overrightarrow{v}_T - \overrightarrow{v}_h\|_{0,E} \leq Ch_E^{-\frac{1}{2}}\|\overrightarrow{v}_T\|_{1,\partial E} \quad (37)$$

We let $(\overrightarrow{v}_T, p)$ denote the solution of (12) and let $(\overrightarrow{u}_h, p_h)$ denote the solution of (23) with an approximation on a rectangular subdivision $T_h$.

Our aim is to estimate the velocity and the pressure errors $\overrightarrow{v}_T - \overrightarrow{v}_h \in H^1(\Omega)$ and $p - p_h \in L_0^2(\Omega)$.

The element contribution $\eta_{R,T}$ of the residual error estimator $\eta_R$ is given by

$$\eta_{R,T}^2 = h_T^2\|\overrightarrow{R}_T\|_{2,T}^2 + \|R_T\|_{0,T}^2 + \sum_{E \in \partial T} h_E\|\overrightarrow{R}_E\|_{0,E}^2 \quad (38)$$

and the components in (38) are given by

$$\overrightarrow{R}_T = \{\overrightarrow{f} + \nabla \overrightarrow{u}_h - \nabla p_h\}|_T,$$

$$R_T = \{\nabla \cdot \overrightarrow{u}_h\}|_T,$$

With the key contribution coming from the stress jump associated with an edge $E$ adjoining elements $T$ and $S$:

$$\begin{align*}
[[\nabla \overrightarrow{u}_h - p_h I]] &= ((\nabla \overrightarrow{u}_h - p_h I)|_T - (\nabla \overrightarrow{u}_h - p_h I)|_S)\overrightarrow{r}_{E,T}.
\end{align*}$$

The global residual error estimator is given by:

$$\eta_R = \left(\sum_{T \in \mathcal{T}_h} \eta_{R,T}^2\right)^{1/2}.$$

Our aim is to bound $\|\overrightarrow{v}_T - \overrightarrow{v}_h\|_X$ and $\|p - p_h\|_M$ with respect to the norm $\|\cdot\|$ for velocity $\|\overrightarrow{v}\|_X = \|\overrightarrow{v}\|_{1,\Omega}$ and the quotient norm for the pressure $\|p\|_X = \|p\|_{0,\Omega}.

For any $T \in T_h$, and $E \in \partial T$, we define the following two functions:

$$\overrightarrow{w}_T = \overrightarrow{R}_T b_T; \overrightarrow{w}_E = \overrightarrow{R}_E b_E$$

- $\overrightarrow{w}_T = \overrightarrow{0}$ on $\partial T$.
- If $E \in \partial T \cap \varepsilon_{h,\Omega}$ then $\overrightarrow{w}_E = \overrightarrow{0}$ on $\partial E$.
- If $E \in \partial T \cap \varepsilon_{h,\Omega}$ then $\overrightarrow{w}_E = \overrightarrow{0}$ in the other three edges of rectangle $T$.

- $\overrightarrow{w}_T$ and $\overrightarrow{w}_E$ can be extended to whole of $\Omega$ by setting:

$$\overrightarrow{w}_T = \overrightarrow{0} \text{ in } \Omega - T,$$

$$\overrightarrow{w}_E = \overrightarrow{0} \text{ in } \Omega - \overrightarrow{w}_E \text{ if } E \in \partial T \cap \varepsilon_{h,\Omega}.$$

$$\overrightarrow{w}_T = \overrightarrow{0} \text{ in } \Omega - T \text{ if } E \in \partial T \cap \varepsilon_{h,\Omega}.$$

With these two functions we have the following lemmas:

Lemma 9 For any $T \in T_h$ we have:

$$\int_T \overrightarrow{f} \cdot \overrightarrow{w}_T = \int_T (\nabla \overrightarrow{v}_T - p I) : \nabla \overrightarrow{w}_T \quad (39)$$

Proof. By applying the Green formula and $\overrightarrow{w}_T = \overrightarrow{0}$ on $\partial T$, and using (1) we obtain

$$\int_T (\nabla \overrightarrow{v} - p I) : \nabla \overrightarrow{w}_T = \int_T (-\nabla^2 \overrightarrow{u} + \nabla p) \cdot \overrightarrow{w}_T + \int_{\partial T} (\nabla \overrightarrow{v} - p I) \cdot \overrightarrow{n} \overrightarrow{w}_T = \int_T \overrightarrow{f} \cdot \overrightarrow{w}_T.$$

Lemma 10 i) if $E \in \partial T \cap \varepsilon_{h,\Omega}$, we have:

$$\int_{\partial E} \overrightarrow{f} \overrightarrow{w}_E = \int_{\partial E} (\nabla \overrightarrow{v} - p I) : \nabla \overrightarrow{w}_E. \quad (40)$$
ii) if $E \in \partial T \cap \varepsilon_{h,T}$, we have:
\[
\int_{T} \vec{f} \cdot \vec{w}_E = \int_{T} (\nabla \vec{v} - pI) : \nabla \vec{w}_E + \int_{\partial T} \left( \frac{a}{b} \vec{u} - \frac{1}{b} \vec{t} \right) \cdot \vec{w}_E.
\]

**Proof.** i) The same proof of (39).

ii) if $E \in \partial T \cap \varepsilon_{h,T}$, we have: Using (1) gives
\[
\int_{T} \vec{f} \cdot \vec{w}_E = \int_{T} (-\nabla^2 \vec{v} + \nabla p) \cdot \vec{w}_E.
\]

By applying the Green formula, we obtain
\[
\int_{T} \vec{f} \cdot \vec{w}_E = \int_{T} (\nabla \vec{v} - pI) : \nabla \vec{w}_E + \int_{\partial T} \left( \frac{a}{b} \vec{u} - \frac{1}{b} \vec{t} \right) \cdot \vec{w}_E.
\]

Since $\vec{w}_E = \vec{v}$ in the other three edges of rectangle $T$ and we have by (3),
\[
a \vec{v} + b(\frac{\vec{u}}{b} - \vec{v} \cdot \vec{p}) = \vec{r} \quad \text{in} \quad E \subset \partial T \cap \Omega,
\]
then
\[
\int_{T} \vec{f} \cdot \vec{w}_E = \int_{T} (\nabla \vec{v} - pI) : \nabla \vec{w}_E + \int_{\partial T} \left( \frac{a}{b} \vec{u} - \frac{1}{b} \vec{t} \right) \cdot \vec{w}_E.
\]

**Theorem 11** For any mixed finite element approximation (not necessarily inf-sup stable) defined on rectangular grids $T_h$, the residual estimator $\eta_R$ satisfies:
\[
\|\vec{v}\|_{J,\Omega} + \|\varepsilon\|_{0,\Omega} \leq C \eta_R
\]

\[
\eta_{R,T} \leq C' \left( \sum_{T' \in \omega_T} \{ \|\vec{v}\|^2_{I,T'} + \|\varepsilon\|^2_{0,T'} \} \right)^{\frac{1}{2}}.
\]

Note that the constant $C$ in the local lower bound is independent of the domain, and
\[
\|\vec{v}\|^2_{I,T'} = \int_{T'} \nabla \vec{v} : \nabla \vec{v} + \int_{\partial T'} \frac{a}{b} \vec{v} \cdot \vec{v} + \int_{\partial T'} \varepsilon^2.
\]

**Proof:** To establish the upper bound we let $(\vec{v}, q) \in H^1(\Omega) \times L^2(\Omega)$ and $\vec{v} \in X_h$ be the element interpolant of $\vec{v}$, then
\[
C'[(\vec{v}, \varepsilon); (\vec{v}, q)] = C'[(\vec{v}, \varepsilon); (\vec{v} - \vec{v}_h, q)] = C'[(\vec{v}, \varepsilon); (\vec{v} - \vec{v}_h, q)]
\]
\[
- C[(\vec{v}_h, p); (\vec{v} - \vec{v}_h, q)] = L(\vec{v} - \vec{v}_h) - A(\vec{v}_h, \vec{v} - \vec{v}_h) - B(\vec{v} - \vec{v}_h, p) - B(\vec{v}_h, q).
\]

Finally, using (19), gives:
\[
\|\vec{v}\|_{J,\Omega} + \|\varepsilon\|_{0,\Omega} \leq C' \left( \sum_{T' \in \omega_T} \{ \|\vec{v}\|^2_{I,T'} + \|\varepsilon\|^2_{0,T'} \} \right)^{\frac{1}{2}}
\]

This establishes the upper bound. Turning to the local lower bound. First, for the element residual part, we have
\[
\int_{T} \vec{R}_T \cdot \vec{w}_T = \int_{T} (\vec{f} + \nabla^2 \vec{v}_h - \nabla p_h) \cdot \vec{w}_T
\]
\[
= \int_{T} (\vec{f} \cdot \vec{w}_T) - \int_{T} \left( \nabla \vec{v}_h - p_h I \right) : \nabla \vec{w}_T + \int_{\partial T} \left( \nabla \vec{v}_h - p_h I \right) \vec{w}_T.
\]

Using (39), (31) and $\vec{w}_T = \vec{v}$ in $\partial T$, give:
\[
\int_{T} \vec{R}_T \cdot \vec{w}_T = \int_{T} (\nabla \vec{v} - \varepsilon I) : \nabla \vec{w}_T
\]
\[
\leq \left( \|\vec{v}\|_{1,\Omega} + \|\varepsilon\|_{0,\Omega} \right) \|\vec{w}\|_{1,\Omega}
\]
\[
\leq C(\|\vec{v}\|^2_{I,\Omega} + \|\varepsilon\|^2_{0,\Omega}) h^{-1} \|\vec{R}_T\|_{0,\Omega}
\]
In addition, from the inverse inequality (30)
\[ \int_T R_T \cdot \tilde{\omega}_T = \| R_T b_T \|^2_{0,T} + c E \| R_T \|^2_{0,T}. \]

Thus,
\[ h_E^2 \| R_T \|^2_{0,T} \leq C(\| \tilde{\omega} \|^2_{J,T} + \| \epsilon \|^2_{0,T}). \]

(42)

Next comes the divergence part,
\[ \| R_T \|_{0,T} = \| \nabla (\tilde{\omega} - \bar{\omega}_h) \|_{0,T} \]
\[ \leq \sqrt{2} \| \tilde{\omega} - \bar{\omega}_h \|_{1,T} \]
\[ \leq \sqrt{\frac{\nu}{\mu}} \| \tilde{\omega} \|^2_{J,T}. \]

(43)

Finally, we need to estimate the jump term. For
an edge \( E \in \partial T \cap \epsilon h, \Omega \). We have
\[ 2 \int_E \vec{R}_E \cdot \vec{\omega}_E = \sum_{i=1,2} \int_{\partial T_i} (\nabla \vec{\omega}_h - p_h I) \vec{\omega}_E \cdot \vec{\omega}_E \]
\[ = \int_{\omega_E} (\nabla \vec{\omega}_h - p_h I) : \nabla \vec{\omega}_E \]
\[ + \sum_{i=1,2} \int_{T_i} (\nabla^2 \vec{\omega}_h - \nabla p_h) \vec{\omega}_E \]

Using (40) and \( \vec{\omega}_E = \vec{\omega} \) in \( \partial \omega_E \), gives:
\[ 2 \int_E \vec{R}_E \cdot \vec{\omega}_E = - \int_{\omega_E} (\nabla \vec{\omega} - \epsilon I) : \nabla \vec{\omega}_E \]
\[ + \sum_{i=1,2} \int_{T_i} \vec{R}_T \cdot \vec{\omega}_E \]
\[ \leq (\| \tilde{\omega} \|_{1,T} + \| \epsilon \|_{0,T}) \| \vec{\omega}_E \|_{1,T} \]
\[ + \sum_{i=1,2} \| \vec{R}_T \|_{0,T} \| \vec{\omega}_E \|_{0,T}. \]

Using (33) and (35), give
\[ \int_E \vec{R}_E \cdot \vec{\omega}_E \leq C(\| \tilde{\omega} \|^2_{1,T} + \| \epsilon \|^2_{0,T}) \frac{1}{2} h_E \| \vec{R}_E \|_{0,E}. \]
\[ + \sum_{i=1,2} \| \vec{R}_T \|_{0,T} h_E \| \vec{R}_E \|_{0,E}. \]

Using (32), gives
\[ \int_E \vec{R}_E \cdot \vec{\omega}_E \leq C(\| \tilde{\omega} \|^2_{2,E} + \| \epsilon \|^2_{0,E}) \frac{1}{2} h_E \| \vec{R}_E \|_{0,E}. \]

(44)

Using (42), gives
\[ \int_E \vec{R}_E \cdot \vec{\omega}_E \leq C(\| \tilde{\omega} \|^2_{2,E} + \| \epsilon \|^2_{0,E}) \frac{1}{2} h_E \| \vec{R}_E \|_{0,E}. \]

(44)

Using (32), gives
\[ \int_E \vec{R}_E \cdot \vec{\omega}_E = \| \vec{R}_E \|_{0,E} \geq c \| \vec{R}_E \|_{0,E}. \]

and thus using (44) gives
\[ h_E \| \vec{R}_E \|^2_{0,E} \leq C(\| \tilde{\omega} \|^2_{2,E} + \| \epsilon \|^2_{0,E}). \]

(45)

We also need to show that (45) holds for boundary edges. For an \( E \in \partial T \cap \epsilon h, \Gamma \), we have
\[ \int_E \vec{R}_E \cdot \vec{\omega}_E \]
\[ = \int_{\partial T} \left[ \frac{\alpha}{\beta} \vec{\omega}_h - \frac{1}{\beta} \vec{\omega} \right] \cdot \vec{\omega}_E \]
\[ + \int_T (\nabla \vec{\omega}_h - p_h I) : \nabla \vec{\omega}_E \]
\[ + \int_T (\nabla^2 \vec{\omega}_h - \nabla p_h) \vec{\omega}_E \]

Using (41) and (44), gives
\[ \int_E \vec{R}_E \cdot \vec{\omega}_E = - \int_T (\nabla \vec{\omega} - \epsilon I) : \nabla \vec{\omega}_E \]
\[ - \int_{\partial T} \left[ \frac{\alpha}{\beta} \vec{\omega}_h - \frac{1}{\beta} \vec{\omega} \right] \cdot \vec{\omega}_E \]
\[ \leq C(\| \tilde{\omega} \|^2_{1,T} + \| \epsilon \|_{0,T}) \| \vec{\omega}_E \|_{1,T} \]
\[ + \| \vec{R}_T \|_{0,T} \| \vec{\omega}_E \|_{0,T} \]
\[ + C(\| \tilde{\omega} \|^2_{2,T} + \| \epsilon \|^2_{0,T}) \| \vec{\omega}_E \|_{2,T} \]
\[ + \| \vec{R}_T \|_{0,T} \| \vec{\omega}_E \|_{0,T}. \]

Using (33) and (35), give
\[ \int_E \vec{R}_E \cdot \vec{\omega}_E \leq C(\| \tilde{\omega} \|^2_{1,E} + \| \epsilon \|_{0,E}) \| \vec{\omega}_E \|^2_{0,E} \]
\[ + \| \vec{R}_T \|_{0,T} \| \vec{\omega}_E \|^2_{0,E}. \]

(44)

Using (42), gives
\[ \int_E \vec{R}_E \cdot \vec{\omega}_E \leq C(\| \tilde{\omega} \|^2_{2,E} + \| \epsilon \|^2_{0,E}) \frac{1}{2} h_E \| \vec{R}_E \|^2_{0,E}. \]

(46)

Using (32), gives
\[ \int_E \vec{R}_E \cdot \vec{\omega}_E = \| \vec{R}_E \|_{0,E} \geq c \| \vec{R}_E \|_{0,E}. \]

and thus using (46) gives
\[ h_E \| \vec{R}_E \|^2_{0,E} \leq C(\| \tilde{\omega} \|^2_{2,E} + \| \epsilon \|^2_{0,E}). \]

(47)

Finally, combining (42), (43), (45) and (47) establishes the local lower bound. 

Remark. Theorem 11 also holds for stable (and unstable) mixed approximations defined on a triangular subdivision if we take the obvious interpretation of The Proof is identical except for the need to define appropriate element and edge bubble functions.
6.2 A Local Poisson Problem Estimator

The local Poisson problem estimator:

\[ \eta_P = \sqrt{\sum_{T \in T_h} \eta_{P,T}^2} \]

as follows

\[ \eta_{P,T}^2 = \| \bar{e}_{P,T}\|_{L^2(T)}^2 + \| e_{P,T}\|_{L^2(T)}^2 \quad (48) \]

Let

- \( V_T = H^1(T) \)
- \( A_T(\bar{\varphi}_{P,T}, \varphi) = \int_T \nabla \bar{\varphi}_{P,T} : \nabla \varphi + \int_{\partial T} b \bar{\varphi}_{P,T} \varphi \)

\( \bar{e}_{P,T} \in V_T \) satisfies the uncoupled Poisson problems

\[ A_T(\bar{\varphi}_{P,T}, \varphi) = (\bar{R}_T, \varphi) - \sum_{E \in e(T)} (\bar{R}_E, \bar{\varphi}_E) \quad (49) \]

\[ e_{P,T} = \nabla \bar{u}_h / T \quad (50) \]

for any \( \bar{\varphi} \in V_T \).

**Theorem 12** The estimator \( \eta_{P,T} \) is equivalent to the \( \eta_{R,T} \) estimator:

\[ c \eta_{P,T} \leq \eta_{R,T} \leq C \eta_{P,T} \]

**Proof.** For the upper bound, we first let \( \bar{w}_T = \bar{R}_T b_T \) (\( b_T \) is an element interior bubble function). From (49),

\[ (\bar{R}_T, \bar{w}_T)_T = (\bar{e}_{P,T}, \bar{w}_T)_T \leq \| \bar{e}_{P,T}\|_{L^2(T)} \| \bar{w}_T\|_{L^2(T)} \]

Using (31) we get

\[ (\bar{R}_T, \bar{w}_T)_T \leq C \frac{1}{h_T} \| \bar{R} \|_{L^2(T)} \| \bar{e}_{P,T}\|_{L^2(T)}^2 + \| e_{P,T}\|_{L^2(T)}^2 \|^2 \quad (51) \]

In addition, from the inverse inequalities (30), \( \| \bar{R} \|_{L^2(T)}^2 \leq C. (\bar{R}_T, \bar{w}_T)_T \) and using (51), to get

\[ h_T^2 \| \bar{R}_T \|_{L^2(T)}^2 \leq C (\| \bar{e}_{P,T}\|_{L^2(T)}^2 + \| e_{P,T}\|_{L^2(T)}^2) \quad (52) \]

Next, we let \( \bar{w}_E = \bar{R}_E b_E \) is an edge bubble function.

If \( E \in \partial T \cap \varepsilon_{h,T} \) using (49), (33) and (35) give

\[ (\bar{R}_E, \bar{w}_E)_E = -A_T(\bar{\varphi}_{P,T}, \bar{w}_E) + (\bar{R}_T, \bar{w}_E)_T \]

\[ \leq \| \bar{e}_{P,T}\|_{L^2(T)} \| \bar{w}_E\|_{L^2(T)} + \| \bar{R}_E\|_{L^2(T)} \| \bar{w}_E\|_{L^2(T)} \]

\[ \leq C h_T^{-\frac{1}{2}} \| \bar{R}_E\|_{L^2(T)} \| \bar{e}_{P,T}\|_{L^2(T)} + \| e_{P,T}\|_{L^2(T)}^2 \]
Now, since \( \vec{v} \) is zero at the four vertices of \( T \), a scaling argument and the usual trace theorem, see e.g. [14, Lemma 1.5], shows that \( \vec{v} \) satisfies

\[
\|\vec{v}\|_{0,E} \leq Ch_E^{\frac{1}{2}}|\vec{v}|_{1,T} \tag{57}
\]

\[
\|\vec{v}\|_{0,E} \leq Ch_T|\vec{v}|_{1,T} \tag{58}
\]

Combining these two inequalities with (56) immediately gives the lower bound in the equivalence relation.

**Theorem 13** For any mixed finite element approximation (not necessarily inf-sup stable) defined on rectangular grids \( T_h \) the estimator \( \eta_P \) satisfies:

\[
\|\vec{e}\|_{J,\Omega} + \|\varepsilon\|_{0,\Omega} \leq C\eta_P
\]

and

\[
\eta_{P,T} \leq C\left( \sum_{T \in T_h} \left( \|\vec{e}\|_{0,T'}^2 + \|\varepsilon\|_{0,T'}^2 \right) \right)^{\frac{1}{2}}
\]

Note that the constant \( C \) in the local lower bound is independent of the domain.

### 7 Numerical simulation

In this section some numerical results of calculations with mixed finite element Method will presented in two examples [14, 16].

**Example 14** Square domain, enclosed flow boundary condition.

This is a classic test problem used in fluid dynamics, known as driven-cavity flow. It is a model of the flow in a square cavity with the lid moving from left to right. Let the computational model:

\[
\{ y = 1; -1 \leq x \leq 1/u_x = 1 - x^2 \}, \text{ a regularized cavity.}
\]

The \( C_{a,b} \) condition is satisfied, just take \( a \) and \( b \) two real number strictly positive such that \( a \gg b \),

\[
T^* = (a(1 - x^2); 0) \text{ on } \Gamma_1 = (y = 1; -1 \leq x \leq 1) \text{ and } T^* = (0; 0) \text{ on the other three boundary of the square domain.}
\]

The streamlines are computed from the velocity solution by solving the Poisson equation numerically subject to a zero Dirichlet boundary condition.

![Fig. 1. Uniform streamline plot by MFE (left) associated with a 64 \times 64 square grid, \( P_1 - P_0 \) approximation, and uniform streamline plot (right) computed with ADINA system.](image1)

![Fig. 2. Velocity vectors solution by MFE (left) associated with a 64 \times 64 square grid, \( P_1 - P_0 \) approximation, and uniform streamline plot (right) computed with ADINA system.](image2)
approximation and velocity vectors solution (right) computed with ADINA system.

Fig. 3. Pressure plot for the flow with a $64 \times 64$ square grid.

<table>
<thead>
<tr>
<th>Grid</th>
<th>$\eta_R$</th>
<th>$\eta_P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$8 \times 8$</td>
<td>2.616759 e-001</td>
<td>8.722532 e-001</td>
</tr>
<tr>
<td>$16 \times 16$</td>
<td>1.695069 e-001</td>
<td>6.053819 e-001</td>
</tr>
<tr>
<td>$32 \times 32$</td>
<td>1.020879 e-001</td>
<td>3.781035 e-001</td>
</tr>
<tr>
<td>$64 \times 64$</td>
<td>6.063286 e-002</td>
<td>2.220984 e-001</td>
</tr>
</tbody>
</table>

Table 1. $\eta_R$ is the residual error estimator and $\eta_P$ is the local Poisson problem error estimator for a colliding flow.

Example 15 Square domain $\Omega$, analytic solution. This analytic test problem is associated with the following solution of the Stokes equation system:

$$u_x = 20xy^3; \quad u_y = 20x^4 - 5y^4; \quad (59)$$

$$p = 60x^2y - 20y^3 + \text{constant.}$$

It is a simple model of colliding flow, and a typical solution of streamline is illustrated in Figure 4. To solve this problem numerically, the finite element interpolant of the velocity in (59) is specified everywhere on $\partial \Omega$. The Dirichlet boundary condition for the stream function calculation is the interpolant of the exact stream function: $\psi(x, y) = 5xy^4 - x^5$. See that the $C_{a,b}$ condition is satisfied with $a \gg b$ a real number very large and $\mathbf{t} = (20axy^3; 20x^4 - 5y^4)a$ on $\Gamma$.

Fig. 4. Uniform streamline plot by MFE associated with a $64 \times 64$ square grid.

Fig. 5. Pressure plot for the flow with a $64 \times 64$ square grid.

Fig. 6. Estimated error $\eta_{P,T}$ associated with $64 \times 64$ square grid.

Figure 5 shows the pressure plot and figure 6 shows the estimated error associated with $64 \times 64$ square grid.
Table 2. $\eta_R$ is the residual error estimator and $\eta_P$ is the local Poisson problem error estimator for a colliding flow.

<table>
<thead>
<tr>
<th>grid</th>
<th>$\eta_R$</th>
<th>$\eta_P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>8 × 8</td>
<td>2.371793e+000</td>
<td>8.147694e+000</td>
</tr>
<tr>
<td>16 × 16</td>
<td>1.303621e+000</td>
<td>4.599936e+000</td>
</tr>
<tr>
<td>32 × 32</td>
<td>7.205132e-001</td>
<td>2.401784e+000</td>
</tr>
<tr>
<td>64 × 64</td>
<td>3.407043e-001</td>
<td>1.218978e+000</td>
</tr>
</tbody>
</table>

8 Conclusion

In this work, we were interested in the numerical solution of the partial differential equations by simulating the flow of an incompressible fluid. We introduced the Stokes equations with a new boundary condition noted $C_{a,b}$.

The weak formulation obtained is a problem of saddle point type. We have shown the existence and uniqueness of the solution of this problem. We used the discretization by mixed finite element method with two type of the posteriori error estimation of the computed solutions. Both types proved that the larger the grid is, the better the approximation is.

Numerical experiments were carried out and compared with satisfaction with other numerical results, either resulting from the literature, or resulting from calculation with commercial software like Adina system.

Acknowledgments

The authors would like to appreciate the referees for giving us the several corrections.

References:


[19] E. Creuse, G. Kunert, S. Nicaise, A posteriori error estimation for the Stokes problem: Anisotropic and isotropic discretizations,