Martingale-Based Computational Liquidity Risk Premium

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Abstract: In this paper, we consider the pricing of liquidity risk in normal market. By employing the no-arbitrage idea of financial calculus and finance engineering, we discuss the pricing of market risk and liquidity risk under martingale measure, and obtain two separate market prices of risk for all tradable assets via the change of equivalent measure to make discounted assets into martingales, and then provide the pricing formula of liquidity risk premium, in which the market price of risk in the same market for all tradable assets and for all the investors is the same, not varying with the levels of risk aversion of investors.

Key–Words: Liquidity Risk, Market Risk, Risk Premium, Market Price, Martingale, No-Arbitrage

1 Introduction

In the past decades, several large cyclical fluctuations of the global economy have demonstrated the following characteristics: liquidity expansion and contraction, asset price boom and crash. In efforts to cope with the financial crises, the governments have made little progress. Pro-cyclical monetary policy showed its limitations and lag in dealing with liquidity risk. In fact, recent financial crises (such as in Asia or in Russia or in US) suggest that illiquidity of assets can lead to liquidity risk (Brunnermeier and Pedersen [1]; Boyson et al [2]), largely associated with the unreasonable pricing of the assets (i.e., deviation of price from value). Liquidity risk thus refers to the potential loss due to illiquidity or change of liquidity of assets. Traditional asset pricing theories consider market risk and ignore liquidity risk. In recent years, many empirical studies show that liquidity risk is also a systemic risk and affects asset prices (Amihud [3]; Chordia et al [4]; Holmstrom and Tirole [5]; Pastor and Stambaugh [6]; Gibson and Mougeot [7]; Bekaert et al [8]; Wu Wenfeng et al [9]; Su Dongwei and Mai Yuanxun[10]; Luo Dengyue et al [11]; Liu Yang and Liu Shancun [12]; Zhou Fang and Zhang Wei [13]). The studies also suggest that we should not only take into account market risk and also need to pay attention to liquidity risk in asset pricing. Therefore, the question of how to price liquidity risk or how to compute liquidity risk premium has become one of the most important and central issues in the asset pricing theories.

In the study of Longstaff [14], he suggested that the discount for nonmarketable security can be valued by applying option pricing theory. He define a security to be nonmarketable if it cannot be traded at all for some fixed period of time, or if it can only be traded after a delay. In this sense if an investor is restricted from selling the security at any point in time, the cost of being forced to postpone trading (usually called opportunity cost) or the possible loss can be regarded as liquidity risk. Longstaff [14] identified a contingent claim (put option) that would compensate for the largest possible loss an investor could incur by foregoing the right to sell the security for a fixed period of time, and derived a no-arbitrage upper bound on the value of the discount for illiquidity depending on the volatility of the security. Nevertheless, this analysis presents the upper bound on the discounts of illiquid securities instead of modeling a rational or equilibrium value of the discount for illiquidity, so these results are difficult to reconcile within the context of the traditional assert pricing models.

Longstaff [15] proposed a new idea of estimating the price discount of illiquid asset from the perspective of optimal portfolio. When an investor faces liquidity constraints, he may take riskier positions than one in a perfectly liquid market. Since liquidity affects investor's decisions, the investor facing illiquidity would take trading strategy that maximizes the utility of the portfolio. Thus, the price discount or compensation for holding illiquid assets can be computed through comparing the maximum utility of the optimal portfolio with that of portfolio in a perfectly liquid market in a continuous time case. However, his study focuses on a two-asset portfolio with one risky asset and one risk-free asset rather than a portfolio with many assets.

Wu Weixing and Wang Yongxiang [16] considered the optimal search behaviors of rational investors, and presented a model for premium of liquidity risk caused by the sudden announcement of circulation of state-owned shares. Because the presence of illiquid shares deters some investors from entering the market, with the result of an endogenously thin market, any rational investor will ask for compensation for this thinness. Though considering the impact of event risk on market liquidity, Wu Weixing and Wang Yongxiang [16] likewise only discussed a simple case of a two-security portfolio.

Following the methodology of Longstaff [14], Liang Zhaohui and Zhang Wei [17] introduced an option-theoretical approach to value the discount for illiquidity securities, and provided a benchmark for assessing the potential costs from non-marketability and thinly-traded market. However, their findings are similar to the results of Longstaff [14], they offered an upper bound on the discount for illiquidity rather than an equilibrium price.

Liang Zhaohui et al [18] further proposed a practical framework for the quantification of liquidity risk premium by designing an optimal liquidation strategy. That is, in a thin market, as investors' dealings impact the market price, a rational investor will adopt such optimal trading strategy as liquidating his position gradually to maximize his utility, which is different from the strategy under a perfectly liquid market. According to the no-arbitrage principle, the liquidity risk premium can be calculated as investors obtain the same utility as in a perfectly liquid market. Nevertheless, this liquidity risk premium may vary with the optimal trading strategy for different investors (Almgren and Chriss [19] proved that for each level of risk aversion there is a uniquely determined optimal trading strategy).

Acharya and Pedersen [20], Liu weimin[21], Zou Xiaopeng et al [22], Chen Qing and Li Zibai [23], Zhou Fang and Zhang Wei [13] etc., introduced a liquidity factor into the traditional capital asset pricing model, and established respective expanded CAPM models with the liquidity factor , thus liquidity risk premium can be calculated. Acharya and Pedersen [20] derived a liquidity-adjusted CAPM by decomposing a single-factor CAPM model (in which the net return of a security or market portfolio is defined as the difference between return and illiquidity costs). Although considering liquidity costs, the model has a potential problem. That is, market risk and liquidity risk is priced the same as the risk premium on market portfolio, this means that the model, in essence, is still a single-factor model. Zou Xiaopeng et al [22] further took into account liquidity demand and elasticity value of price impact (the impact of trading volume on transaction costs), and constructed a pricing model on Acharya and Pedersen's [20] work, whereby there is a similar problem aforementioned in their model. Chen Qing and Li Zibai [23], Zhou Fang and Zhang Wei [13], proposed a two-factor model incorporating the market and liquidity, in which there implies an assumption, namely that "market portfolio only has market risk but no liquidity risk", so the risk premium on market portfolio shall be the market risk premium (this assumption is reasonable and also very useful, in our study we adopt this assumption). According to their respective liquidity measures, the stocks are grouped, and then the compensation for liquidity risk can be calculated as the difference between the return on a portfolio of low liquidity stocks and the return on a portfolio of high liquidity stocks. Nevertheless, such quantifying of liquidity risk premium or pricing of liquidity risk crucially depends on liquidity indicator, so liquidity risk premium may vary with different liquidity indicators. In fact, the liquidity is not observed directly but rather has a number of aspects that cannot be captured in a single measure (Amihud and Mendelson [24]), even there exist many proxies for liquidity that measure different aspects of liquidity, such as trading costs (Brennan and Subrahmanyam [25]), trading volume (Brennan et al.[26]), turnover rate (Datar et al.[27]), bid-ask spread (Amihud and Mendelson [28]), illiquidity ratio (Amihud [3]), etc. However, the problem is that there is no single liquidity measure that reflects all its aspects (Amihud [3]; Pastor and Stambaugh [6]; Zhou Fang and Zhang Wei [13]). Moreover, the approach to computational liquidity risk premium does not take into account the effect of market risk on the liquidity risk premium, in the sense that their results may not be accurate.

As discussed above, if the premium of liquidity risk is simply priced, there may be some problems such as non-unique, non-equilibrium, etc. However, from the previous literature, we can get a new idea: in a perfect market, all the systemic risk (including liquidity risk) should have its own "price", and the "price" of each risk for all the financial products should be the same (otherwise there will be arbitrage opportunities in the market). Thus, once the "price" of a risk is determined, we can directly obtain the risk premium if the risk is measured (namely, the"size"of risk multiplied by the "price" of risk). Therefore, in our study, we will employ the no-arbitrage idea of financial mathematics and financial engineering to price the liquidity risk under the martingale measure. Using martingale methods, via the change of equivalent measure to make the discounted prices of tradable assets martingales, we can obtain the market price of liquidity risk, and then construct a new and simple formula for calculating the liquidity risk premium.

The remainder of the paper is organized as follows: section 2 introduces the basic concepts and theories related to Martingale used in this study; section 3 derives the market price of liquidity risk under the martingale measure; section 4 discusses the liquidity risk premium; section 5 presents an example of computational liquidity risk premium; section 6 summarizes the results and makes concluding remarks.

2 Martingale and related theories

Martingale has been applied to the asset pricing analysis since the fundamental theorem of asset pricing was proved by Harrison and Kreps [29], and Harrison and Pliska [30]. And then, martingale has become an important and necessary mathematical tool for the pricing of modern financial products.

The following concepts (definitions) and theorems and corollaries can be the basic theories used in our study, given by Martin Baxter and Andrew Rennie [31].

Definition 1 *A stochastic process $X_t(t \in [0,\infty))$ is a martingale with respect to a measure P if and only if*

$$(1)\ \ E_p(|X_t|) < \infty,\ for\ all\ \ t \in [0,\infty),$$

$$(2)\ \ E_p(X_t|F_s) = X_s,\ for\ all\ \ s \le t$$

where F_s is the history of the process X_t up to time s.

A martingale measure is one which makes the expected future value conditional on its present value and past history merely its present value. It is not expected to drift upwards or downwards.

Theorem 2 *If $X_t(t \in [0,\infty))$ is a stochastic process with volatility σ_t which satisfies the technical condition $E[(\int_0^T \sigma_s^2 ds)^{\frac{1}{2}}] < \infty$, where T is some time horizon, then X_t is a martingale if and only if X_t is driftless.*

Definition 3 *Suppose we have a market of securities and a numeraire cash bond B_t(a basic security relative to which the value of other securities can be judged) under a measure P. An equivalent martingale measure(EMM) is a measure Q equivalent toP, under which the bond-discounted securities are all Q-martingales.*

Theorem 4 *(Cameron-Martin-Girsanov Theorem) If $W(t)$ is a P-Brownian motion and $\gamma(t)$ is a F-previsible process satisfying the boundedness condition*

$$E_p \exp(\frac{1}{2}\int_0^T |\gamma(t)|^2 dt) < \infty, \tag{1}$$

then there exists a measure Q equivalent to P such that

$$\widetilde{W}(t) = W(t) + \int_0^t \gamma(s)ds \tag{2}$$

is a Q -Brownian motion up to time T .

We can use Theorem 4 to make the discounted price processes into martingales under a new measure. Therefore, Theorem 2 and Theorem 4 present an efficient method for us to price the liquidity risk under martingale measure.

Definition 5 *Given a numeraire B_t and an asset S_t , a process S_t represents a tradable asset if and only if its discounted value $B_t^{-1}S_t$ is actually a Q- martingale, where Q is the measure under which the discounted asset $B_t^{-1}S_t$ is a martingale.*

Definition 6 *A market is arbitrage-free if there is no way of making riskless profits. An arbitrage opportunity would be a trading strategy which started with zero value and terminated at some definite date T with a positive value. A market is arbitrage-free if there are absolutely no such arbitrage opportunities.*

Theorem 7 *(Fundamental Theorem of Asset Pricing) There are no free lunches if and only if all the bond-discounted securities are Q- martingales.*

According to Theorem 7, any asset in the market can be reasonably priced. Then, from Theorem 7, we have the following corollaries.

Corollary 8 *Martingales mean no arbitrage.*

Corollary 9 *Non-martingales are non-tradable.*

Theorem 7 provides a theoretical support and practical approach for us to price the liquidity risk.

3 The market price of liquidity risk

Although a number of empirical studies have shown that liquidity risk is an undiversifiable risk affecting asset prices, unfortunately, there is no theoretical model that is directly applicable to pricing liquidity risk and calculating liquidity risk premium. Notice that in recent studies on asset pricing models, Gibson and Mougeot [7], Acharya and Pedersen [20], Zou Xiaopeng et al [22], Liu [21], Chen Qing and Li Zibai [23], Zhou Fang and Zhang Wei [13] considered market risk and liquidity risk as two systematic risks, which implies that asset prices are affected by both the market risk and the liquidity risk.

Even though some scholars such as Chan and Faff [32], and Pastor and Stambaugh [6] established the four-factor model with liquidity risk compensation by adding liquidity factor to the Fama and French [33] three-factor model, Hearn et al.[34], on the other hand, built a three-factor model containing market, company size and liquidity factor through modifying the Fama-French three-factor model. Whether it is the four-factor model of Chan and Faff, or the three-factor model of Hearn et al., there might be a problem where one risk factor is related to or dependent upon other risk factors. For instance, companies with low liquidity often have small size and high book-to-market ratio. On the contrary, small companies providing high return have correspondingly illiquidity (Amihud and Mendelson [28]; Liu Weimin[21]). Firm size (the market value of stock)is related to liquidity since a larger stock issue has smaller price impact for a given order flow and a smaller bid-ask spread (Amihud [3]).

Several scholars have examined the relationship among some specific priced factors such as firm size and book-to-market ratio and liquidity. Liao Shiguang [35] employed trading volume, stock prices and stock price volatility as indicators to measure the liquidity of the stock (on the grounds that these indicators are related to liquidity) and found that there exists a significant positive correlation between firm size and stock liquidity. Zhou Fang and Zhang Wei [36] applied generalized impulse-response function in Chinese stock market and found that liquidity affects firm size and book-to-market ratio with a longer continuity than the effect of firm size and book-to-market ratio on liquidity, which indicates that liquidity can be a surrogate for firm size and book-to-market ratio in explaining asset returns. Zhou Fang et al. [37] further discussed the relationship among the risk factors such as firm size, book-to-market ratio and liquidity by using dynamic regression model and quantile regression model. Their results provide a significant positive correlation between firm size and liquidity and a significant negative correlation between book-to-market ratio and liquidity after taking into account the lagged effect of liquidity on firm size and book-to-market ratio, which reveals the reason of that liquidity premium theory can explain size effect and value effect (Chen Qing and Li Zibai [23]; Zhou Fang and Zhang Wei [13]). Thus the studies imply that the risk factors related to firm size and book-to-market can be subsumed by the liquidity factor.

Therefore, we assume that market factor and illiquidity factor are two systematic factors causing risk premium on securities in our study.

So suppose that there is a market, in which there are two risk sources $W_1(t)$ and $W_2(t)$ from market and illiquidity respectively. In order to facilitate the measurement of liquidity risk, and also in order to eliminate the impact of market risk on liquidity risk premium, we suppose that $W_1(t)$ and $W_2(t)$ are two independent P-Brownian motions, namely $cov(W_1(t), W_2(t)) = 0$ (this assumption is tenable and reasonable under the measure of liquidity risk below given in this paper). Suppose further that, there are no arbitrage opportunities (that is, all the bond-discounted securities are Q- martingales).

Given two securities S_t and X_t. Let S_t be a risky security only with market risk but no liquidity risk (S_t can be viewed as the market portfolio), and X_t be another risky security with both market risk and liquidity risk. In addition suppose that there is a cash bond B_t. According to Martin Baxter and Andrew Rennie's work [31] and risk composition theory, the price processes B_t, S_t and X_t can be defined by means of their stochastic differential equations (SDEs), typically

$$dB_t = r\, B_t\, dt, \tag{3}$$

$$dS_t = S_t(\sigma_m\, dW_1(t) + udt), \tag{4}$$

$$dX_t = X_t(\sigma_1\, dW_1(t) + \sigma_2\, dW_2(t) + vdt). \tag{5}$$

Where r is the risk-free interest rate, u and v are the respective drifts of S_t and X_t, namely, the rates of expected return (continuous compounding). σ_m can be the volatility of S_t, σ_1 and σ_2 are respectively from Brownian motion components $W_1(t)$ and $W_2(t)$, and contribute to the volatility of X_t. If let σ be the aggregate volatility of X_t, then $\sigma = \sqrt{\sigma_1^2 + \sigma_2^2}$ (the non-systematic risk is ignored).

These have solutions

$$B_t = \exp(rt), \tag{6}$$

$$S_t = S_0 \exp(\sigma_m\, W_1(t) + (u - \frac{1}{2}\sigma_m^2)t), \tag{7}$$

$$\begin{aligned} X_t = \;& X_0 \exp(\sigma_1\, W_1(t) + \sigma_2\, W_2(t) \\ & +(v - \tfrac{1}{2}\sigma_1^2 - \tfrac{1}{2}\sigma_2^2)t). \end{aligned} \tag{8}$$

Let Y_t and Z_t denote, respectively, the discounted prices of two securities S_t and X_t. Then, $Y_t = B_t^{-1} S_t$, $Z_t = B_t^{-1} X_t$. We have

$$Y_t = S_0 \exp(\sigma_m\, W_1(t) + (u - \frac{1}{2}\sigma_m^2 - r)t), \tag{9}$$

$$\begin{aligned} Z_t = \;& X_0 \exp(\sigma_1\, W_1(t) + \sigma_2\, W_2(t) \\ & +(v - \tfrac{1}{2}\sigma_1^2 - \tfrac{1}{2}\sigma_2^2 - r)t). \end{aligned} \tag{10}$$

By applying Ito's formula, the SDEs of Y_t and Z_t can be written as

$$dY_t = Y_t(\sigma_m\, dW_1(t) + (u - r)dt), \tag{11}$$

$$\begin{aligned} dZ_t = \quad & Z_t(\sigma_1\, dW_1(t) + \sigma_2\, dW_2(t) \\ & +(v-r)dt). \end{aligned} \tag{12}$$

If we find another measure Q equivalent to the original measure P, under which the discounted prices $Y_t = B_t^{-1}S_t$ and $Z_t = B_t^{-1}X_t$ are Q- martingales simultaneously. Then, we will obtain the market prices of risk which reflect the drift change to the underlying Brownian motion given by Theorem 4.

Since there exist two sources of risk, as previously noted, $W_1(t)$ from market and $W_2(t)$ from illiquidity, there will be two separate prices of risk. Respectively, $\gamma_1(t)$ will be the price of $W_1(t)$ -risk and $\gamma_2(t)$ will be the price of $W_2(t)$-risk. In other words, the market price of risk will be a vector $\gamma(t) = (\gamma_1(t), \gamma_2(t))$.

We want to choose $\gamma_1(t)$ and $\gamma_2(t)$, such that the drift terms in dY_t and dZ_t vanish simultaneously. Then theorem 4 says that there is a measure Q such that $\widetilde{W}(t) = W(t) + \int_0^t \gamma(s)ds$ is Q-Brownian motion. This means that the discounted prices Y_t and Z_t of S_t and X_t will be Q- martingales.

Therefore, let

$$\widetilde{W_i}(t) = W_i(t) + \int_0^t \gamma_i(s)ds, \quad i = 1,\ 2, \tag{13}$$

such that $\widetilde{W}(t) = (\widetilde{W_1}(t), \widetilde{W_2}(t))$ is 2-dimensional Q -Brownian motion.

Then under the measure Q , $Y_t = B_t^{-1}S_t$ and $Z_t = B_t^{-1}X_t$ have the following SDEs

$$\begin{aligned} dY_t = \quad & Y_t(\sigma_m\, d\widetilde{W_1}(t) \\ & +(u - r - \sigma_m\gamma_1(t))dt), \end{aligned} \tag{14}$$

$$\begin{aligned} dZ_t = \quad & Z_t(\sigma_1\, d\widetilde{W_1}(t) + \sigma_2\, d\widetilde{W_2}(t) \\ & +(v - r - \sigma_1\gamma_1(t) - \sigma_2\gamma_2(t))dt. \end{aligned} \tag{15}$$

To make the drift terms of dY_t and dZ_t vanish, we must have that

$$u - r - \sigma_m\gamma_1(t) = 0, \tag{16}$$

$$v - r - \sigma_1\gamma_1(t) - \sigma_2\gamma_2(t) = 0. \tag{17}$$

Because $\sigma_2 \neq 0$, $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ must exist and be equal to

$$\gamma_1(t) = \frac{u-r}{\sigma_m}, \tag{18}$$

$$\gamma_2(t) = \frac{v - r - \sigma_1\gamma_1(t)}{\sigma_2}. \tag{19}$$

According to the no-arbitrage assumption and theorem 7, we now see that this is to make sure that there exists a measure Q which makes the discounted stock prices Y_t and Z_t of S_t and X_t into Q- martingales. On the other hand, S_t and X_t are tradable, under the martingale measure Q, satisfying

$$dY_t = Y_t\sigma_m\, d\widetilde{W_1}(t), \tag{20}$$

$$dZ_t = Z_t(\sigma_1\, d\widetilde{W_1}(t) + \sigma_2\, d\widetilde{W_2}(t)). \tag{21}$$

Thus under Q , S_t and X_t can be written as

$$S_t = S_0\exp(\sigma_m\widetilde{W_1}(t) + (r - \frac{1}{2}\sigma_m^2)t), \tag{22}$$

$$\begin{aligned} X_t = \quad & X_0\exp(\sigma_1\widetilde{W_1}(t) + \sigma_2\widetilde{W_2}(t) \\ & +(r - \tfrac{1}{2}\sigma_1^2 - \tfrac{1}{2}\sigma_2^2)t). \end{aligned} \tag{23}$$

We can see that, $\gamma_1(t)$ and $\gamma_2(t)$ must be the desired market prices of market risk $W_1(t)$ and liquidity risk $W_2(t)$.

Note that, in the market price formula of the liquidity risk, both the market risk measure σ_1 and the liquidity risk measure σ_2 of X_t are unknown, thus the market price for liquidity risk of $\gamma_2(t)$ is not determined yet. However, consider that S_t is a security only with the market risk but no liquidity risk, as defined before. Thus if we establish the link between X_t and S_t , then using S_t as a benchmark to measure the market risk σ_1 of X_t , we can determine liquidity risk measure σ_2 of X_t (because $\sigma = \sqrt{\sigma_1^2 + \sigma_2^2}$).

Here, we draw inspiration from the quanto model of Martin Baxter and Andrew Rennie [31]. Consider, using

$$dX_t = X_t(\sigma_1\, dW_1(t) + \sigma_2\, dW_2(t) + vdt), \tag{24}$$

with

$$\sigma_1 = \rho\sigma,\ \sigma_2 = \bar{\rho}\sigma. \tag{25}$$

Where $\bar{\rho}$ is the orthogonal complement of ρ, namely $\bar{\rho} = \sqrt{1-\rho^2}$.

We can write the original processes S_t and X_t as

$$S_t = S_0\exp(\sigma_m\, W_1(t) + (u - \frac{1}{2}\sigma_m^2)t), \tag{26}$$

$$\begin{aligned} X_t = \quad & X_0\exp(\rho\sigma\, W_1(t) + \bar{\rho}\sigma\, W_2(t) \\ & +(v - \tfrac{1}{2}\sigma^2)t). \end{aligned} \tag{27}$$

Consider the covariance of S_t and X_t . If we write the model in vector form, the vector random variable $(\log S_t, \log X_t)$ is jointly-normally distributed with mean vector $(\log S_0 + (u - \frac{1}{2}\sigma_m^2)t, \log X_0 + (v - \frac{1}{2}\sigma^2)t)$, and variance-covariance matrix

$$\begin{aligned} & \begin{bmatrix} \sigma_m & o \\ \rho\sigma & \bar{\rho}\sigma \end{bmatrix} \begin{bmatrix} t & o \\ o & t \end{bmatrix} \begin{bmatrix} \sigma_m & o \\ \rho\sigma & \bar{\rho}\sigma \end{bmatrix}^T \\ & = \begin{bmatrix} \sigma_m^2 & \rho\sigma\sigma_m \\ \rho\sigma\sigma_m & \sigma^2 \end{bmatrix} t. \end{aligned} \tag{28}$$

This ensures that, a volatility of S_t is σ_m, a volatility of X_t is σ, that is, the standard deviations of return or aggregate risk measures of S_t and X_t are, respectively, σ_m and σ, and a correlation coefficient between them is ρ . Thus, we have now established the link between X_t and S_t by the correlation coefficient ρ.

Using S_t as a benchmark, we can determine both the market risk measure $\rho\sigma$ and liquidity risk measure $\bar{\rho}\sigma$ of X_t through the correlation coefficient ρ between X_t and S_t.

Here then, we have the market prices for market risk $W_1(t)$ and liquidity risk $W_2(t)$ of $\gamma(t) = (\gamma_1(t), \gamma_2(t))$, given by

$$\gamma_1(t) = \frac{u - r}{\sigma_m}. \tag{29}$$

$$\gamma_2(t) = \frac{v - r - \rho\sigma\gamma_1(t)}{\bar{\rho}\sigma}. \tag{30}$$

We should point out that, whether there is one risk source or two risk sources, the market price of each risk for all the tradable securities in a market should be the same, otherwise, there can be arbitrage.

To see why, suppose that we have a couple of tradable risky securities X_t^1 and X_t^2 , both in the same market, and both are defined by their SDEs

$$\begin{aligned} dX_t^1 = \; & X_t^1(\rho_1\sigma_1 \, dW_1(t) + \bar{\rho}_1\sigma_1 \, dW_2(t) \\ & + v_1 dt), \end{aligned} \tag{31}$$

$$\begin{aligned} dX_t^2 = \; & X_t^2(\rho_2\sigma_2 \, dW_1(t) + \bar{\rho}_2\sigma_2 \, dW_2(t) \\ & + v_2 dt). \end{aligned} \tag{32}$$

We want the discounted prices of X_t^1 and X_t^2 to be martingales under the same measure Q. So the vector

$$\widetilde{W}(t) = W(t) + \int_0^t \gamma^i(s) ds,$$

must be 2-dimensional Q-Brownian motion for $i = 1, 2$. But this can happen if and only if the two changes of drift ($\gamma_j^i(t), i = 1, 2$) are the same. That is

$$\gamma_1^1(t) = \gamma_1^2(t), \; \gamma_2^1(t) = \gamma_2^2(t).$$

In addition, we should note that, almost all of the securities in a market will have more or less market risk and liquidity risk. To determine a security S_t with only market risk and no liquidity risk, there are two ways. One is, by employing the idea of the combination of financial engineering, to copy such a required security S_t through combining two securities (or portfolios) with different levels of market risk but the same levels of liquidity risk. The other is to choose a particular tradable security S_t defined its liquidity risk as 0, then S_t can be viewed as a benchmark to determine liquidity risk of other securities by comparing with it.

4 Liquidity risk premium

In the derivation of the pricing formulas of market risk $W_1(t)$ and liquidity risk $W_2(t)$ by using the above-mentioned no-arbitrage pricing method, we assume that the aggregate volatilities of two securities do not contain non-systemic risk. In fact, the non-systemic risk can be eliminated by portfolio diversification.

Therefore, we can consider two portfolios, the market portfolio M and a fully diversified portfolio P (such as a fully diversified fund), as two new "securities", which have only systemic risk because all non-systemic risk have been diversified away, viewing the market portfolio M as the representative of the security S_t with only liquidity risk but no market risk and the portfolio P as a representative of the security X_t with both market risk and liquidity risk.

Assume that $E(r_m)$ is rate of expected return of the market portfolio M , σ_m is the aggregate risk measure of M. $E(r_p)$ is rate of expected return of a completely diversified portfolio P, σ_p is the aggregate risk measure of P. Thus, the market prices for market risk $W_1(t)$ and liquidity risk $W_2(t)$ of $\gamma_1(t)$ and $\gamma_2(t)$ can be written as follows

$$\gamma_1(t) = \frac{E(r_m) - r_f}{\sigma_m}. \tag{33}$$

$$\gamma_2(t) = \frac{E(r_p) - r_f - \rho_{mp}\sigma_p\gamma_1(t)}{\bar{\rho}_{mp}\sigma_p}. \tag{34}$$

where r_f represents the risk-free rate of returnthat is $r = r_f$. ρ_{mp} represents the correlation between the returns of M and P, $\bar{\rho}_{mp}$ represents the orthogonal complement of ρ_{mp}, namely $\bar{\rho}_{mp} = \sqrt{1 - \rho_{mp}^2}$.

The above shows that, if the non-systematic risk is ignored, two completely diversified security portfolios M and P (their non-systematic risks are zero) can be regarded as benchmark securities S_t with the market risk measure σ_m but the liquidity risk measure zero, and X_t with both the market risk measure $\rho_{mp}\sigma_p$ and the liquidity risk measure $\bar{\rho}_{mp}\sigma_p$ (σ_p represents the aggregate systematic risk measure of portfolio P). Because $E(r_m) - r_f$ is the market risk premium on the market portfolio M, and $E(r_p) - r_f$ is the aggregate risk premium on portfolio P, therefore, $\gamma_1(t)$ is the premium per market risk. Thus, $\rho_{mp}\sigma_p\gamma_1(t)$ is the aggregate market risk premium on portfolio P, and $E(r_p) - r_f - \rho_{mp}\sigma_p\gamma_1(t)$ is the aggregate liquidity risk premium on portfolio P. Then, $\gamma_2(t)$ is the premium per liquidity risk.

For any security $X_i(t)$ (individual security or portfolio) in a market, if its market risk and liquidity risk are denoted by $\sigma_{i(1)}$ and $\sigma_{i(2)}$, the aggregate systematic risk of $X_i(t)$ can be written by

$$\sigma_{i(0)} = \sqrt{\sigma_{i(1)}^2 + \sigma_{i(2)}^2}. \quad (35)$$

Thus the market risk premium and the liquidity risk premium on $X_i(t)$ are $\gamma_1(t)\sigma_{i(1)}$ and $\gamma_2(t)\sigma_{i(2)}$, and the aggregate risk premium is

$$E(r_i) - r_f = \gamma_1(t)\sigma_{i(1)} + \gamma_2(t)\sigma_{i(2)}. \quad (36)$$

For any security $X_i(t)$ in a market, because its aggregate risk σ_i includes the non-systematic risk, which is not related to risk compensation. Thus, the liquidity risk premium on $X_i(t)$ is $\gamma_2\sigma_{i(2)}$, then

$$\gamma_2\sigma_{i(2)} \le \gamma_2\bar{\rho}_{mi}\sigma_i. \quad (37)$$

The aggregate risk premium on $X_i(t)$ is $E(r_i) - r_f$, thus

$$E(r_i) - r_f \le \gamma_1\rho_{mi}\sigma_i + \gamma_2\bar{\rho}_{mi}\sigma_i. \quad (38)$$

However, for any completely diversified portfolio P in a market, such as a completely diversified fund, if its expected return rate and aggregate risk measure are denoted by $E(r_p)$ and σ_p, due to the diversification of portfolio eliminates the non-systematic risk, thus the aggregate risk can be only systematic risk, including market risk and liquidity risk. Then, the aggregate risk premium on a completely diversified portfolio P is

$$E(r_p) - r_f = \gamma_1\rho_{mp}\sigma_p + \gamma_2\bar{\rho}_{mp}\sigma_p. \quad (39)$$

Risk premium and frontier of risk premium on security $X_i(t)$ are shown in Figure 1 and Figure 2.

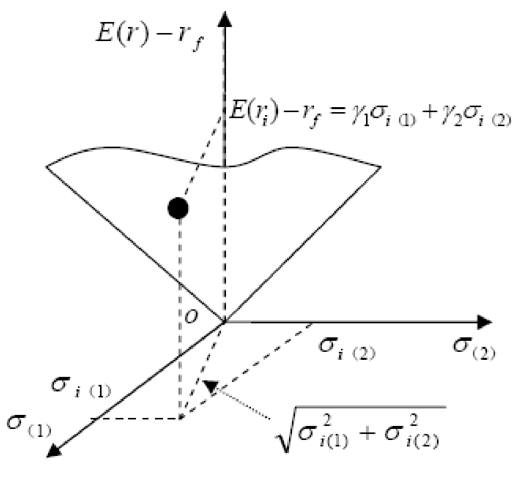

Figure 1 Risk Premium

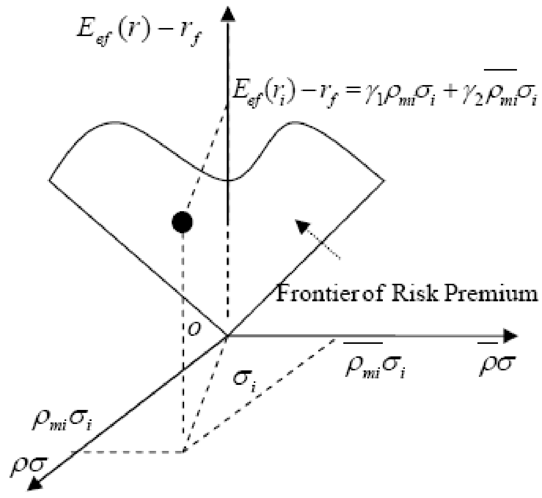

Figure 2 Frontier of Risk Premium

5 An example of liquidity risk premium

To illustrate our results, we suppose that M is the market portfolio with rate of expected return $E(r_m) = 16\%$ per period and volatility rate $\sigma_m = 30\%$, and P is a completely diversified fund with rate of expected return $E(r_p) = 20\%$ per period and volatility rate $\sigma_p = 50\%$. The correlation between the rates of return of M and P is $\rho_{mp} = 0.6$. The risk-free interest rate is $r_f = 4\%$.

According to the previous formula (33) and (34), we can calculate the market prices of the market risk $W_1(t)$ and liquidity risk $W_2(t)$, respectively

$$\gamma_1(t) = \frac{16\% - 4\%}{30\%} = 0.4,$$

$$\gamma_2(t) = \frac{20\% - 4\% - 0.6 \times 50\% \times 0.4}{0.8 \times 50\%} = 0.1.$$

This tells us that, the market price of market risk is 0.4, the market price of liquidity risk is 0.1. In other words, the premium per liquidity risk is 0.1 while the premium per market risk is 0.4.

Thus for any given individual security $X_i(t)$ with the aggregate risk σ_i, if the correlation between the rates of return of $X_i(t)$ and market portfolio M is ρ_{im}, then, while $X_i(t)$'s market risk is $\rho_{im}\sigma_i$, its liquidity risk is less than

$\bar{\rho}_{im}\sigma_i = \sqrt{1 - \rho_{im}^2}\sigma_i$.

Therefore, while the market risk premium on $X_i(t)$ is $0.4\rho_{im}\sigma_i$, the liquidity risk premium is less than $0.1\sqrt{1 - \rho_{im}^2}\sigma_i$. Thus the aggregate risk premium on $X_i(t)$ is less than

$0.4\rho_{im}\sigma_i + 0.1\sqrt{1-\rho_{im}^2}\sigma_i$.

If given a completely diversified portfolio P with the aggregate risk σ_p, then its market risk and liquidity risk is $\rho_{mp}\sigma_p$ and $\sqrt{1-\rho_{mp}^2}\sigma_p$, respectively. And thus market risk premium and liquidity risk premium on P are $0.4\rho_{mp}\sigma_p$ and $0.1\sqrt{1-\rho_{mp}^2}\sigma_p$. So the aggregate risk premium is

$0.4\rho_{mp}\sigma_p + 0.1\sqrt{1-\rho_{mp}^2}\sigma_p$.

In this case, risk premium on an individual security $X_i(t)$ and risk premium on a fully diversified portfolio P are shown in Figure 3 and Figure 4.

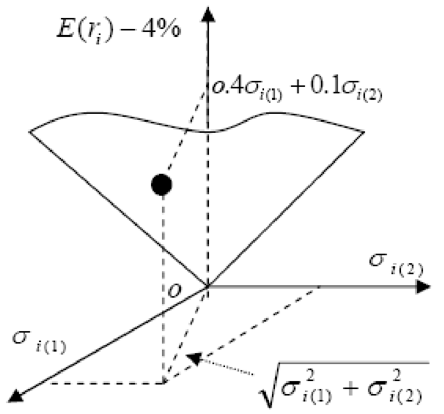

Figure 3 Risk Premium on an Individual Security

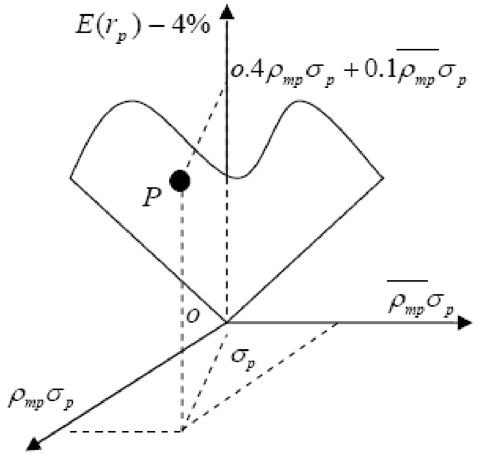

Figure 4 Risk Premium on a Fully Diversified Portfolio

6 Conclusion

Bid-ask spread, short selling restrictions, etc., which are often regarded as liquidity costs or transaction costs in some literatures (e.g., Amihud and Mendelson [28]; Brennan and Subrahmanyam [25]), are considered as risk factors that cause the illiquidity of assets in this study. In this paper, we study the pricing of liquidity risk in more general, derive the market prices of market risk $W_1(t)$ and liquidity risk $W_2(t)$ (ie, the premium per liquidity risk and the premium per market risk):

$$\gamma_1(t) = \frac{u-r}{\sigma_m}, \tag{40}$$

$$\gamma_2(t) = \frac{v-r-\rho\sigma\gamma_1(t)}{\bar{\rho}\sigma}. \tag{41}$$

In a market, all tradable securities have the same market price of risk, namely, the same premium per market risk and the same premium per liquidity risk; and the market price of risk for all investors is the same, not varying with the levels of risk aversion of investors. However, in different markets, the premium per liquidity risk and the premium per market risk may vary with the selected benchmark securities. Moreover, for different securities in the same market, their market risk premium and liquidity risk premium are often different.

In general, if the respective drifts for S_t and X_t of $u(t)$ and $v(t)$, the respective volatilities for S_t and X_t of $\sigma_m(t)$ and $\sigma(t)$, the correlation between them of $\rho(t)$, and the risk-free interest rate $r(t)$ are previsible processes, the market prices of the market risk $W_1(t)$ and liquidity risk $W_2(t)$ (ie, the premium per market risk and the premium per liquidity risk) can be expressed as

$$\gamma_1(t) = \frac{u(t)-r(t)}{\sigma_m(t)}, \tag{42}$$

$$\gamma_2(t) = \frac{v(t)-r(t)-\rho(t)\sigma(t)\gamma_1(t)}{\overline{\rho(t)}\sigma(t)}. \tag{43}$$

where $\gamma_1(t)$ and $\gamma_2(t)$ are market prices of risk depending on the time and the state. Then, in a market, all tradable securities have instantaneously the same premium per market risk and the same premium per liquidity risk, and the risk price for all investors is the same.

In this paper, we present a pricing formula of liquidity risk as well as a liquidity risk measure for completely diversified portfolio, and then offer a new approach for calculating the liquidity risk premium on completely diversified portfolio (such as a diversified fund). Compared with the results of other studies, such computational liquidity risk premium does not depend on any liquidity indicators, that makes the pricing of liquidity risk and calculating of liquidity risk premium easier and more efficient. Nevertheless, our work leaves some questions. Is it possible to measure liquidity risk for an individual security? If possible, then how do we get the liquidity risk measure for an individual security? Furthermore, how do we price liquidity risk premium on a single security? These and other interesting questions are left to future work.

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