Hedging strategy for unit-linked life insurance contracts in stochastic volatility models

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Abstract: A general class of stochastic volatility model is considered for modeling risky asset. This class of stochastic volatility model contains most of those without jump component which are commonly used in research. We obtain the minimal martingale measure and locally risk minimizing hedging strategy in these models, and employ the results to the unit-linked life insurance contracts. Moreover, we also investigate the locally risk minimizing hedging strategy for unit-linked life insurance contracts in a Barndorff-Nielsen and Shephard stochastic volatility model.

Key–Words: Locally risk minimizing; Stochastic volatility; Unit-linked life insurance contracts

1 Introduction

In the well known Black-Scholes model, volatility is assumed to be constant, but this hypothesis is far from being realistic, it does have known biases. Two empirical phenomenons have received much attention recently: the asymmetric leptokurtic features and the volatility smile. We know, if the Black-Scholes model is correct, then the implied volatility should be constant. In reality, it is widely known that the implied volatility curve resembles a "smile". Over the past decades, some different models are also provided to incorporate the "volatility smile" in option pricing. For example, Scott [6], Hull and White [3], Wiggins [2], Heston [12] and so on. In this paper, we consider a finance market with a risky asset and a risk-free asset. The price process of the risky asset follows a general class of stochastic volatility model, since there are more than one source of randomness, the finance market is incomplete. In a complete market, a contingent claim can be replicated perfectly by a portfolio of risk free bonds and the underlying asset. In an incomplete market, however, such a replication is not possible, we have to choose some approaches to hedge derivatives . In this paper, we shall use the criterion of risk minimization. The risk minimization concept first discussed in Föllmer and Sondermann [1] when the asset price process is a martingale under the empirical measure. This local risk minimization concept was introduced in Schweizer [8]. Møller [13] considered a model describing the uncertainty of the financial market and a portfolio of insured individuals simultaneously, the risk-minimizing trading strategies and the associated intrinsic risk processes are determined for different types of unit-linked life insurance contracts. Chan [14] found a locally risk minimizing strategy when the price process was driven by a general Lévy process. Riesner [7] extended the Møller 's model in [13], supposed that the risky asset price process was discontinuous as it followed a geometric lévy process, and obtained the risk minimizing hedging strategy of life insurance contracts in a lévy process financial market. However, Vandaele and Vanmaele [10] pointed out that the result of Riesner [7] was not correct, and found that the risk minimizing hedging strategy was not the locally risk minimizing hedging strategy under the original measure. Bi and Guo [5] also considered the risk minimizing hedging problem for unit-linked life insurance contracts in a financial market driven by a shot-noise process. However, the above research papers haven't involved stochastic volatility models. In this paper, we suppose that the risky asset follows a general class of stochastic volatility model, and obtain a locally risk minimizing hedging strategy for unit-linked life insurance contracts.

The outline of this paper is as follows: the model is developed in Section 2. Then the third section states a review of risk minimizing. The main theorem is derived in Section 4. Employing the results of Section 4, the locally risk minimizing hedging strategy of unit-linked life contracts is presented in Section 5. In Section 6, we derive a locally risk minimizing hedging strategy for unit-linked life insurance contracts in a Barndorff-Nielsen and Shephard stochastic volatility model. Conclusions are stated in Section 7.

2 The model

In this section, the two basis elements of the model, the financial market and a portfolio of individuals to be insured, are introduced. Let the probability space $\big(\Omega_1, \mathcal{G}, (\mathcal{G}_t)_{(0\le t\le T)}, P_1\big)$ denote the financial market and $\big(\Omega_2, \mathcal{H}, (\mathcal{H}_t)_{(0\le t\le T)}, P_2\big)$ be used to describe the insurance portfolio. Define $\big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{(0\le t\le T)}, P\big)$ as a product space of the above two independent probability space. Moreover, the probability space is supposed to satisfy the usual conditions of right continuity and completeness.

2.1 The financial market

We consider a continuous time model with two primary traded assets, namely a stock with price process $S = (S_t)_{0\le t\le T}$ and a bank account with price process $B = (B_t)_{0\le t\le T}$. The asset price processes are defined on a complete probability space $\big(\Omega, (\mathcal{G}_t)_{(0\le t\le T)}, P_1\big)$ and are given by as following

$$
\begin{aligned}
dB_t &= r_t B_t dt, \\
dS_t &= \mu_t S_t dt + S_t f(Y_t) d\widetilde{W}_t^1, \quad (1)\\
dY_t &= g(Y_t)dt + h(Y_t) d\widetilde{W}_t^2, \quad (2)
\end{aligned}
$$

for $0 \le t \le T$, where f, g and h are measurable Lipschitz functions, μ_t and r_t are time-dependent, strictly positive and deterministic, $\widetilde{W}_t^1$,$\widetilde{W}_t^2$ are two standard Brownian motions and $Cov(d\widetilde{W}_t^1, d\widetilde{W}_t^2) = \rho dt$, here, we assume that $\rho \in (-1, 1)$. In this paper, we consider a general class of stochastic volatility model, which contains most of stochastic volatility models without jump component. For example, if $f(y) = y, g(y) = 0, h(y) = \sigma, \rho = 0$, then this stochastic volatility model reduces to Hull/White [3] volatility model; if $f(y) = \sqrt{y}, g(y) = k(\theta - y), h(y) = \sigma\sqrt{y}$, this stochastic volatility model becomes Heston [12] volatility model. Let $\widetilde{S}_t$ denote the discount risky asset price process, it means $\widetilde{S}_t = S_t e^{-\int_0^t r_u du}$. Moreover, we define $M_t = \int_0^t \widetilde{S}_u f(Y_u) d\widetilde{W}_u^1$ and $A_t = \int_0^t (\mu_u - r_u)\widetilde{S}_u du$. Notice that M is the local martingale part of S and A is its predictable part. We assume that $E_{P_1}\left[\int_0^t \widetilde{S}_u^2 f^2(Y_u) du\right] < \infty$ and $E_{P_1}\left[\int_0^t \widetilde{S}_u^2 (\mu_u - r_u)^2 du\right] < \infty$ for $0 \le t \le T$, where $E_{P_1}[\cdot]$ denotes the expectation under the statistic probability measure P_1. Hence, $\widetilde{S}$ becomes a square integrable semi-martingale with decomposition $\widetilde{S}_t = M_t + A_t$. Since $E_{P_1}\left[\int_0^t \widetilde{S}_u^2 f^2(Y_u) du\right] < \infty$ for $0 \le t \le T$, then M_t is not only a local martingale but also a square integrable martingale and A_t is an increasing process.

2.2 The insurance portfolio

The insurance market is described on the probability space $\big(\Omega, \mathcal{H}, (\mathcal{H}_t)_{(0\le t\le T)}, P_2\big)$, where the filtration $(\mathcal{H}_t)_{(0\le t\le T)}$ is the natural filtration generated by $I_{\{T_i\le t\}}$ with $i = 1, 2, ..., N$. The number N denotes the number of individuals all of equal age x and with i.i.d nonnegative lifetimes $T_1, T_2, ..., T_N$. Their hazard rate $\mu_{x+\tau}$ is given by

$$
\begin{aligned}
{}_tP_x &= P_2(T_1 > x + t | T_1 > x) \\
&= \exp\left\{-\int_0^t \mu_{x+\tau} d\tau\right\}. \quad (3)
\end{aligned}
$$

The number of deaths until time t is denoted by $N_t^I = \sum_{i=1}^N I_{\{T_i\le t\}}$ for $0 \le t \le T$. In addition, we assume that the P_2 martingale $M^I = (M_t^I)_{0\le t\le T}$ is defined as

$$
M_t^I = N_t^I - \int_0^t \lambda_u du, \quad (4)
$$

where $\lambda_t = (N - N_t^I)\mu_{x+t}$.

3 A review of risk minimization

In this section we will introduce the definitions and notations of risk minimization, for all unexplained notations we refer the reader to Schweizer [9] and Jacod and Shiryaev [4].

Definition 1 *A couple $\varphi = (\xi, \eta)$ is called a strategy if ξ is a predictable process and $\|\xi\|_{L^2(S)} = \left(E_P\left[\int_0^T \xi_u^2 d[\widetilde{S}]_u\right]\right)^{\frac{1}{2}} < \infty$, η is an adapted process and $V = \xi\widetilde{S} + \eta$ has right continuous paths and $E_P[V_t^2] < \infty$ for every $t \in [0, T]$, where P is the statistic probability measure.*

Definition 2 *A martingale measure $\hat{Q}$ which is equivalent to the statistical measure P will be called minimal if*

$$
\hat{Q} = P \qquad on \qquad \mathcal{F}_0
$$

and if any square-integrable P martingale that is orthogonal to the martingale part M of the semimartingale S under P remains a martingale under $\hat{Q}$.

We define the cost process of a trading strategy via

$$C_t(\varphi) = V_t - \int_0^t \xi_s d\widetilde{S}_s.$$

Note that a strategy is self-financing if and only if the cost process is constant.

Definition 3 *A strategy φ is called pseudo locally risk-minimizing if the associated cost process $C(\varphi)$ is a martingale under P, and orthogonal to the martingale part M of the semi-martingale S.*

Lemma 4 *Assume that the semi-martingale S satisfy the following conditions, then the pseudo locally risk-minimizing strategy φ is a locally risk-minimizing strategy.*

(A1): $< M >$ should be P-almost surely strictly increasing on the whole interval $[0, T]$;

(A2): A is P-almost surely continuous;

(A3): A is absolutely continuous with respect to $< M >$ with a density α satisfying $E_P\left[|\alpha| \log^+ |\alpha|\right] < \infty$, where M is the martingale part of the discount asset price process $\widetilde{S}$ in the canonical decomposition under P.

For (A3), a sufficient condition is that $E_P\left[< \alpha dM >\right] < \infty$. The interesting readers can see it in Schweizer [9] or Vandaele and Vanmaele [10].

Definition 5 The residual process of φ is defined by $R_t(\varphi) = E_P\left[(C_T(\varphi) - C_t(\varphi))^2 \Big| \mathcal{F}_t\right]$.

4 Minimal martingale measure and locally risk minimizing hedging strategy

We shall consider the problem of hedging a contingent claim. Since S is the only traded asset in the model, and Y is not traded, then our market is incomplete, we have to add some criterion to determine hedging strategies. In this paper, we will use the criterion of risk minimization. Suppose that $\Psi(S_T)$ is a $\mathcal{F}_T$ measure claim for which $\sup_{t\in[0,T]} E_Q\left[\left(B_t^{-1}\Psi(S_t)\right)^2\right] < \infty$, where Q is a risk-neutral martingale measure. According to risk-neutral valuation, the arbitrage-free price $\Phi(t, S_t, Y_t)$ of the claim $\Psi(S_T)$ is given by $E_Q\left[B_t B_T^{-1}\Psi(S_T)|\mathcal{F}_t\right]$ for $0 \le t \le T$.

Since the financial market is incomplete, there are infinite equivalent martingale measures, we first provide a general equivalent probability measure which is described by the following Girsanov density:

$$\begin{aligned}\frac{dQ}{dP_1}\Big|_{\mathcal{F}_t} &= D_t = 1 + \int_0^t D_u G(u, S_u, Y_u) d\widetilde{W}_u^1 \\ &+ \int_0^t D_u H(u, S_u, Y_u) d\widetilde{W}_u^2. \qquad (5)\end{aligned}$$

Using Girsanov's theorem, we have that under the risk-neutral measure Q

$$\begin{aligned}W_t^1 &= \widetilde{W}_t^1 - \int_0^t G(u, S_u, Y_u) du \\ &- \rho \int_0^t H(u, S_u, Y_u) du, \qquad (6)\\ W_t^2 &= \widetilde{W}_t^2 - \int_0^t H(u, S_u, Y_u) du \\ &- \rho \int_0^t G(u, S_u, Y_u) du, \qquad (7)\end{aligned}$$

are two standard Brownian motions, in addition $Cov(dW_t^1, dW_t^2) = \rho dt$. Therefore, $\widetilde{S}_t$ can be expressed as

$$\begin{aligned}d\widetilde{S}_t &= (\mu_t - r_t)\widetilde{S}_t dt + \widetilde{S}_t f(Y_t) dW_t^1 + \widetilde{S}_t f(Y_t) \\ \times\ & G(t, S_t, Y_t) dt + \rho \widetilde{S}_t f(Y_t) H(t, S_t, Y_t) dt. \qquad (8)\end{aligned}$$

Then, for all $0 \le t \le T$, $\widetilde{S}_t$ is a Q martingale if and only if the following condition is satisfied

$$\mu_t + f(Y_t) G(t, S_t, Y_t) + \rho f(Y_t) H(t, S_t, Y_t) = r_t. \quad (9)$$

Hence, under the measure Q, risky asset price process S_t satisfies the following

$$d\widetilde{S}_t = \widetilde{S}_t f(Y_t) dW_t^1.$$

From the above Eq.(9), we know that $G(t, S_t, Y_t)$ and $H(t, S_t, Y_t)$ are not unique. In the following, we will obtain an especially equivalent martingale measure, the minimal martingale measure. Before the problem is discussed, we first consider an optimal hedge risk strategy. In what follows, we provide the locally risk minimizing strategy, first show the following theorem.

Theorem 6 *Let* $V_t = E_Q\left[B_T^{-1}\Psi(S_T)\Big|\mathcal{F}_t\right] = B_t^{-1}\Phi(t, S_t, Y_t)$, *then*

$$\begin{aligned}V_t &= V_0 + \int_0^t \Phi_x(u, S_u, Y_u) \widetilde{S}_u f(Y_u) dW_u^1 \\ &+ \int_0^t B_u^{-1} \Phi_y(u, S_u, Y_u) h(Y_u) dW_u^2, \quad (10)\end{aligned}$$

where $\Phi_t(t,S_t,Y_t)$, $\Phi_x(t,S_t,Y_t)$, $\Phi_y(t,S_t,Y_t)$ denote the first derivative of $\Phi(t,S_t,Y_t)$ with respect to variable t, S_t and Y_t respectively, $\Phi_{xx}(t,S_t,Y_t)$ denotes the second derivative of $\Phi(t,S_t,Y_t)$ with respect to variable S_t, $\Phi_{xy}(t,S_t,Y_t)$ denotes the second mixed derivative of $\Phi(t,S_t,Y_t)$ with respect to variable S_t and Y_t.

Proof: Using the Itô's formula, we get

$$\begin{aligned}
dV_t &= -r_tB_t^{-1}\Phi(t,S_t,Y_t)dt + B_t^{-1}\Phi_t(t,S_t,Y_t)dt \\
&+ B_t^{-1}\Phi_x(t,S_t,Y_t)dS_t + B_t^{-1}\Phi_y(t,S_t,Y_t)dY_t \\
&+ \frac{1}{2}B_t^{-1}\Phi_{xx}(t,S_t,Y_t)d<S>_t \\
&+ \frac{1}{2}B_t^{-1}\Phi_{yy}(t,S_t,Y_t)d<Y>_t \\
&+ B_t^{-1}\Phi_{xy}(t,S_t,Y_t)d<S,Y>_t \\
&= -r_tB_t^{-1}\Phi(t,S_t,Y_t)dt + B_t^{-1}\Phi_t(t,S_t,Y_t)dt \\
&+ B_t^{-1}\Phi_x(t,S_t,Y_t)\left(S_tf(Y_t)d\widetilde{W}_t^1 + \mu_tS_tdt\right) \\
&+ B_t^{-1}\Phi_y(t,S_t,Y_t)\left(h(Y_t)d\widetilde{W}_t^2 + g(Y_t)dt\right) \\
&+ \frac{1}{2}B_t^{-1}\Phi_{xx}(t,S_t,Y_t)S_t^2f^2(Y_t)dt \\
&+ \frac{1}{2}B_t^{-1}\Phi_{yy}(t,S_t,Y_t)h^2(Y_t)dt \\
&+ B_t^{-1}\Phi_{xy}(t,S_t,Y_t)\rho S_tf(Y_t)h(Y_t)dt,
\end{aligned}$$

together with equations (6) and (7), we may write

$$\begin{aligned}
&dV_t = -r_tB_t^{-1}\Phi(t,S_t,Y_t)dt + B_t^{-1}\Phi_t(t,S_t,Y_t)dt \\
&+ B_t^{-1}\Phi_x(t,S_t,Y_t)S_tf(Y_t)dW_t^1 \\
&+ B_t^{-1}\Phi_x(t,S_t,Y_t)S_tf(Y_t)(G(t,S_t,Y_t) \\
&+ \rho H(t,S_t,Y_t))dt + B_t^{-1}\Phi_y(t,S_t,Y_t)h(Y_t)dW_t^2 \\
&+ B_t^{-1}\Phi_y(t,S_t,Y_t)h(Y_t)(H(t,S_t,Y_t) \\
&+ \rho G(t,S_t,Y_t))dt + B_t^{-1}\Phi_x(t,S_t,Y_t)\mu_tS_tdt \\
&+ B_t^{-1}\Phi_y(t,S_t,Y_t)g(Y_t)dt \\
&+ \frac{1}{2}B_t^{-1}\Phi_{xx}(t,S_t,Y_t)S_t^2f^2(Y_t)dt \\
&+ \frac{1}{2}B_t^{-1}\Phi_{yy}(t,S_t,Y_t)h^2(Y_t)dt \\
&+ B_t^{-1}\Phi_{xy}(t,S_t,Y_t)\rho S_tf(Y_t)h(Y_t)dt.
\end{aligned}$$

Again, since V_t is a martingale, finally we obtain

$$\begin{aligned}
&\Phi_t(t,S_t,Y_t) + \Phi_x(t,S_t,Y_t)\mu_tS_t + \Phi_y(t,S_t,Y_t)g(Y_t) \\
&+ \Phi_x(t,S_t,Y_t)S_tf(Y_t)(G(t,S_t,Y_t) + \rho H(t,S_t,Y_t)) \\
&+ \Phi_y(t,S_t,Y_t)h(Y_t)(H(t,S_t,Y_t) + \rho G(t,S_t,Y_t)) \\
&+ \frac{1}{2}\Phi_{xx}(t,S_t,Y_t)S_t^2f^2(Y_t) + \frac{1}{2}\Phi_{yy}(t,S_t,Y_t)h^2(Y_t) \\
&+ \Phi_{xy}(t,S_t,Y_t)\rho S_tf(Y_t)h(Y_t) = r_t\Phi(t,S_t,Y_t). \qquad (11)
\end{aligned}$$

Therefore, by the above equation, the proof is completed. □

From the Definition 3, and suppose that $\varphi_u = (\xi_u, \eta_u)$ is pseudo-locally risk minimizing strategy. Let

$$L_t = V_t - V_0 - \int_0^t \xi_u d\widetilde{S}_u = C_t(\varphi) - V_0, \qquad (12)$$

then L_t must satisfy the following two conditions

- ◦ L_t is a P_1 martingale;
- ◦ L_t is orthogonal to M_t which is a local martingale part of semimartingale decomposition $\widetilde{S}_t$.

Theorem 7 *Suppose that $E_{P_1}\left[\int_0^t \frac{1}{f^2(Y_u)}du\right] < \infty$ for all $0 \le t \le T$, then the locally risk minimizing hedging strategy $\varphi_t = (\xi_t, V_t - \xi_t\widetilde{S}_t)$, where*

$$\begin{aligned}
\xi_t &= \frac{\widetilde{S}_t\Phi_x(t,S_t,Y_t)f(Y_t)}{\widetilde{S}_tf(Y_t)} \\
&+ \frac{\rho h(Y_t)B_t^{-1}\Phi_y(t,S_t,Y_t)}{\widetilde{S}_tf(Y_t)}. \qquad (13)
\end{aligned}$$

The residual risk $R_t(\varphi)$ is

$$\begin{aligned}
&R_t(\varphi) \\
&= E_{P_1}\Big[\int_t^T \Big(\widetilde{S}_u^2f^2(Y_u)(\Phi_x(u,S_u,Y_u) - \xi_u)^2 \\
&+ B_u^{-2}h^2(Y_u)\Phi_y^2(u,S_u,Y_u) \\
&+ 2\rho B_u^{-1}\widetilde{S}_uf(Y_u)h(Y_u)(\Phi_x(u,S_u,Y_u) - \xi_u) \\
&\times \Phi_y(u,S_u,Y_u)\Big)du\Big|\mathcal{F}_t\Big]. \qquad (14)
\end{aligned}$$

Proof: By equations (6), (7), (10) and (12), leads to

$$\begin{aligned}
L_t &= V_t - V_0 - \int_0^t \xi_u d\widetilde{S}_u \\
&= \int_0^t \Phi_x(u,S_u,Y_u)\widetilde{S}_uf(Y_u)dW_u^1 \\
&+ \int_0^t \Phi_y(u,S_u,Y_u)B_u^{-1}h(Y_u)dW_u^2 \\
&- \int_0^t \xi_u\widetilde{S}_uf(Y_u)dW_u^1 \\
&= \int_0^t \widetilde{S}_uf(Y_u)(\Phi_x(u,S_u,Y_u) - \xi_u) \\
&\times \Big(d\widetilde{W}_u^1 - (G(u,S_u,Y_u) + \rho H(u,S_u,Y_u))du\Big) \\
&+ \int_0^t B_u^{-1}h(Y_u)\Phi_y(u,S_u,Y_u)\Big(d\widetilde{W}_u^2 \\
&- (H(u,S_u,Y_u) + \rho G(u,S_u,Y_u))du\Big). \qquad (15)
\end{aligned}$$

Due to $\varphi_u = (\xi_u, \eta_u)$ is supposed as pseudo locally risk-minimizing hedging strategy, L_t is P_1 martingale, that is, the drift coefficient of the above equation is 0 at any time $t \in [0, T]$. It implies that

$$\begin{aligned} & \widetilde{S}_t f(Y_t)\left(\Phi_x(t, S_t, Y_t) - \xi_t\right)\Big(\rho H(t, S_t, Y_t) \\ + \;& G(t, S_t, Y_t)\Big) + B_t^{-1} h(Y_t)\Phi_y(t, S_t, Y_t) \\ \times \;& (\rho G(t, S_t, Y_t) + H(t, S_t, Y_t)) = 0. \end{aligned} \tag{16}$$

Hence

$$\begin{aligned} L_t &= \int_0^t \widetilde{S}_u f(Y_u)(\Phi_x(u, S_u, Y_u) - \xi_u) d\widetilde{W}_u^1 \\ &+ \int_0^t B_u^{-1} h(Y_u)\Phi_y(u, S_u, Y_u) d\widetilde{W}_u^2. \end{aligned} \tag{17}$$

In addition,

$$M_t = \int_0^t \widetilde{S}_u f(Y_u) d\widetilde{W}_u^1,$$

combining the Eq.(17) and the above equation, we obtain

$$\begin{aligned} [L, M]_t = \int_0^t \widetilde{S}_u^2 f^2(Y_u)(\Phi_x(u, S_u, Y_u) - \xi_u) du \\ + \int_0^t \rho \widetilde{S}_u B_u^{-1} f(Y_u) h(Y_u)\Phi_y(u, S_u, Y_u) du. \end{aligned} \tag{18}$$

Moreover, using Itô's formula of integration by parts, we have

$$\begin{aligned} L_t M_t &= L_0 M_0 + \int_0^t L_{s-} dM_s \\ &+ \int_0^t M_{s-} dL_s + [L, M]_t, \end{aligned} \tag{19}$$

then $L_t M_t$ is a P martingale if and only if at any time $u \in [0, T]$,

$$\begin{aligned} & \widetilde{S}_u^2 f^2(Y_u)\left(\Phi_x(u, S_u, Y_u) - \xi_u\right) + \rho \widetilde{S}_u B_u^{-1} \\ \times \;& \Phi_y(u, S_u, Y_u) f(Y_u) h(Y_u) = 0, \quad P_1 - a.s. \end{aligned} \tag{20}$$

Therefore, we have

$$\xi_u = \frac{\widetilde{S}_u f(Y_u)\Phi_x(u, S_u, Y_u) + \rho B_u^{-1} h(Y_u)\Phi_y(u, S_u, Y_u)}{\widetilde{S}_u f(Y_u)}.$$

Now, we prove that for the conditions (A1)-(A3) of Lemma 4 are satisfied.

- A1.

$$\begin{aligned} < M >_t &= < \int_0^t \widetilde{S}_u f(Y_u) d\widetilde{W}_u^1 > \\ &= \int_0^t \widetilde{S}_u^2 f^2(Y_u) du. \end{aligned} \tag{21}$$

Hence $< M >_t$ is P_1 almost surely strictly increasing on the whole $[0, T]$ if and only if $\widetilde{S}_u^2 Y_u > 0$ for every $u \in [0, T]$, and this condition is satisfied.

- A2. The finite variation part

$$A_t = \int_0^t \widetilde{S}_u(\mu_u - r_u) du \tag{22}$$

is continuous.

- A3. Combining the Eqs.(21) and (22), we can get

$$\begin{aligned} \lambda_t = \frac{dA_t}{d < M >_t} &= \frac{\widetilde{S}_t(\mu_t - r_t) dt}{\widetilde{S}_t^2 f^2(Y_t) dt} \\ &= \frac{\mu_t - r_t}{\widetilde{S}_t f^2(Y_t)}. \end{aligned}$$

Moreover, since $E_{P_1}\left[\int_0^t \frac{1}{f^2(Y_u)} du\right] < \infty$, then

$$\begin{aligned} & E_{P_1}\left[< \int_0^t \lambda_u dM_u >\right] = E_{P_1}\left[\int_0^t \lambda_u^2 d < M >_u\right] \\ & = E_{P_1}\left[\int_0^t \frac{(\mu_u - r_u)^2}{f^2(Y_u)} du\right] < \infty. \end{aligned}$$

Now, we will calculate the residual risk process $R_t(\varphi)$. From the Definition 3 and equation (12), we obtain

$$\begin{aligned} & R_t(\varphi) = E_{P_1}\left[(C_T(\varphi) - C_t(\varphi))^2 \Big| \mathcal{F}_t\right] \\ = \;& E_{P_1}\left[(L_T - L_t)^2 \Big| \mathcal{F}_t\right] \\ = \;& E_{P_1}\Big[\Big(\int_t^T \widetilde{S}_u f(Y_u)(\Phi_x(u, S_u, Y_u) - \xi_u) d\widetilde{W}_u^1 \\ + \;& \int_t^T B_u^{-1} h(Y_u)\Phi_y(u, S_u, Y_u) d\widetilde{W}_u^2\Big)^2 \Big| \mathcal{F}_t\Big] \\ = \;& E_{P_1}\Big[\int_t^T \Big(\widetilde{S}_u^2 f^2(Y_u)(\Phi_x(u, S_u, Y_u) - \xi_u)^2 \\ + \;& B_u^{-2} h^2(Y_u)\Phi_y^2(u, S_u, Y_u) + 2\rho B_u^{-1} \widetilde{S}_u f(Y_u) \\ \times \;& h(Y_u)(\Phi_x(u, S_u, Y_u) - \xi_u)\Phi_y(u, S_u, Y_u)\Big) du \Big| \mathcal{F}_t\Big]. \end{aligned}$$

Thus, we complete the proof. □

We substitute the Eq.(13) into the Eq.(16), then leads to the following condition, for all $u \in [0, T]$,

$$(1 - \rho^2) B_u^{-1} h(Y_u)\Phi_y(u, S_u, Y_u) H(u, S_u, Y_u) = 0,$$

thus $H(u, S_u, Y_u) = 0$, for $\forall u \in [0, T]$. We now recall the condition (9) and get $G(u, S_u, Y_u) = \frac{r_u - \mu_u}{f(Y_u)}$.

Theorem 8 *Suppose that the condition $E_{P_1}\left[\int_0^t \frac{1}{f^2(Y_u)}du\right] < \infty$ for all $0 \le t \le T$ is satisfied. Let*

$$\frac{d\hat{Q}}{dP_1}\Big|_{\mathcal{F}_t} = D_t = \exp\Big\{\int_0^t G(u,S_u,Y_u)d\widetilde{W}_u^1 - \frac{1}{2}\int_0^t G(u,S_u,Y_u)^2du\Big\},$$

where

$$G(u,S_u,Y_u) = \frac{r_u - \mu_u}{f(Y_u)}.$$

Then, the probability measure $\hat{Q}$ is the minimal martingale measure.

Proof: Assuming that Γ is a P_1-martingale orthogonal to the martingale part M of the semimartingale $\widetilde{S}$. Since

$$D_t = 1 + \int_0^t D_u G(u,S_u,Y_u)d\widetilde{W}_u^1,$$

we recall that under the measure P, M_t is given by:

$$M_t = \int_0^t \widetilde{S}_u f(Y_u)d\widetilde{W}_u^1,$$

then

$$D_t = 1 + \int_0^t D_u \frac{r_u - \mu_u}{\widetilde{S}_u f^2(Y_u)}dM_u.$$

We can now easily obtain

$$[\Gamma, D]_t = \int_0^t D_u \frac{r_u - \mu_u}{\widetilde{S}_u f^2(Y_u)} d\,[\Gamma, M]_u\,.$$

Suppose that the condition $E_{P_1}\left[\int_0^t \frac{1}{f^2(Y_u)}du\right] < \infty$ for all $0 \le t \le T$ is satisfied, and note that Γ is a P_1-martingale orthogonal to the martingale M, thus Γ is also orthogonal to the martingale D. It means that Γ is still a martingale under the measure $\hat{Q}$. Then, according to the Definition 2, the measure $\hat{Q}$ is the minimal martingale measure. Hence, we complete the proof. □

5 Locally risk minimizing hedging strategy for unit-linked contracts

In this section, we employ the results derived in Section 4 to the unit-linked life insurance contracts.

5.1 The pure endowment

The total claim for N pure endowment contracts is

$$H = B_T^{-1}\Psi(S_T)\sum_{i=1}^N I_{\{T_i>T\}} = B_T^{-1}\Psi(S_T)(N - N_T^I).$$

Let $V_t^* = E_{Q^*}[H|\mathcal{F}_t]$, where $Q^* = \hat{Q}\times P_2$, since the independence of the financial market and the insurance portfolio, then

$$\begin{aligned}
V_t^* &= E_{Q^*}\left[B_T^{-1}\Psi(S_T)(N - N_T^I)|\mathcal{F}_t\right]\\
&= E_{P_2}\left[(N - N_T^I)|\mathcal{F}_t\right]E_{\hat{Q}}\left[B_T^{-1}\Psi(S_T)|\mathcal{F}_t\right]\\
&= (N - N_t^I)_{T-t}P_{x+t}V_t.
\end{aligned}$$

By product rule yields

$$d\Big((N - N_s^I)_{u-s}P_{x+s}\Big) = -_{u-s}P_{x+s}dM_s^I.$$

Thus we have

$$\begin{aligned}
V_T^* &= V_0^* + \int_0^T (N - N_{s-}^I)_{T-s}P_{x+s}dV_s\\
&- \int_0^T {}_{T-s}P_{x+s}V_s dM_s^I.
\end{aligned}$$

Due to

$$V_t = V_0 + L_t + \int_0^t \xi_u d\tilde{S}_u$$

and Eq.(17), we can obtain

$$\begin{aligned}
V_T^* &= V_0^* + \int_0^T (N - N_{s-}^I)_{T-s}P_{x+s}\xi_s d\widetilde{S}_s\\
&+ \int_0^T (N - N_{s-}^I)_{T-s}P_{x+s}\widetilde{S}_s f(Y_s)\\
&\times (\Phi_x(s,S_s,Y_s) - \xi_s)d\widetilde{W}_s^1\\
&+ \int_0^T (N - N_{s-}^I)_{T-s}P_{x+s}h(Y_s)B_s^{-1}\\
&\times \Phi_y(s,S_s,Y_s)d\widetilde{W}_s^2 - \int_0^T {}_{T-s}P_{x+s}V_s dM_s^I,
\end{aligned}$$

where M_s^I is defined in Section 2.

Therefore, the optimal portfolio invests $\xi_t^* = (N - N_{t-}^I)_{T-t}P_{x+t}\xi_t$ in the risky asset and $\eta_t^* = (N - N_t^I)_{T-t}P_{x+t}V_t - \xi_t^*\widetilde{S}_t$ in the riskless asset for $0 \le t \le T$, the cost process is

$$\begin{aligned}
C_t(\varphi^*) &= V_0^* + \int_0^t (N - N_{s-}^I)_{T-s}P_{x+s}\widetilde{S}_s f(Y_s)\\
&\times (\Phi_x(s,S_s,Y_s) - \xi_s)d\widetilde{W}_s^1\\
&+ \int_0^t (N - N_{s-}^I)_{T-s}P_{x+s}h(Y_s)B_s^{-1}\\
&\times \Phi_y(s,S_s,Y_s)d\widetilde{W}_s^2 - \int_0^t {}_{T-s}P_{x+s}V_s dM_s^I.
\end{aligned}$$

The residual risk process is given by

$$
\begin{aligned}
& R_t(\varphi^*) = E_P\Big[\int_t^T (N-N^I_{s-})^2_{T-s}P^2_{x+s} \\
\times \;& \Big(f(Y_s^2)\widetilde{S}_s^2(\Phi_x(s,S_s,Y_s)-\xi_s)^2 + h(Y_s)^2B_s^{-2} \\
\times \;& \Phi_y^2(s,S_s,Y_s) + 2\rho\widetilde{S}_s f(Y_s)h(Y_s)B_s^{-1}\Phi_y(s,S_s,V_s) \\
\times \;& (\Phi_x(s,S_s,V_s)-\xi_s)\Big)ds + \int_t^T {}_{T-s}P^2_{x+s}V_s^2 \\
\times \;& (N-N^I_{s-})\mu_{x+s}ds\Big|\mathcal{F}_t\Big].
\end{aligned}
$$

5.2 The term insurance

The payment $\Psi(u,S_u)$ is time-dependent but we assume that the insurance company only pays out at time T. Thus the claim for a portfolio of N term insurance contract is

$$
\begin{aligned}
H_T &= B_T^{-1}\sum_{i=1}^N B_T B_{T_i}^{-1}\Psi(T_i,S_{T_i})I_{\{T_i\le T\}} \\
&= \int_0^T B_u^{-1}\Psi(u,S_u)dN_u^I.
\end{aligned}
$$

Again, we will apply the results of Section 4 to the term insurance. For the term insurance, $V_t = V(t,u)$, with $V(t,u) = E_{Q^*}\left[B_u^{-1}\Psi(u,S_u)|\mathcal{F}_t\right]$ for all $t \le u \le T$. Therefore, ξ_t becomes $\xi(t,u)$ and L_t becomes $L(t,u)$. Hence

$$
\begin{aligned}
V^*_{t,T} &= E_{Q^*}\left[H_T|\mathcal{F}_t\right] = \int_0^t B_u^{-1}\Psi(u,S_u)dN_u^I \\
&+ E_{Q^*}\left[\int_t^T B_u^{-1}\Psi(u,S_u)dN_u^I\Big|\mathcal{F}_t\right] \\
&= \int_0^t B_u^{-1}\Psi(u,S_u)dN_u^I + \int_t^T V(t,u) \\
&\times (N-N_t^I)_{u-t}P_{x+t}\mu_{x+u}du.
\end{aligned}
$$

Using the Itô's formula, $V^*_{s,T}$ can be rewritten as

$$
\begin{aligned}
& dV^*_{s,T} = B_s^{-1}\Psi(s,S_s)dN_s^I - B_s^{-1}\Psi(s,S_s) \\
\times \;& (N-N_s^I)\mu_{x+s}ds \\
+ \;& \int_s^T \Big((N-N^I_{s-})_{u-s}P_{x+s}\mu_{x+u}du\Big)dV(s,u) \\
+ \;& \int_s^T V(s,u)d\Big((N-N_s^I)_{u-s}P_{x+s}\Big)\mu_{x+u}du.
\end{aligned}
$$

Since we have

$$
\begin{aligned}
dV(s,u) &= \xi(s,u)d\widetilde{S}_s + dL(s,u), \\
d\Big((N-N_s^I)_{u-s}P_{x+s}\Big) &= -{}_{u-s}P_{x+s}dM_s^I,
\end{aligned}
$$

then

$$
\begin{aligned}
V^*_{t,T} &= V^*_{0,T} + \int_0^t B_s^{-1}\Psi(s,S_s)dM_s^I \\
&+ \int_0^t\int_s^T (N-N_s^I)_{u-s}P_{x+s}\mu_{x+u}\xi(s,u)dud\widetilde{S}_s \\
&+ \int_0^t\int_s^T (N-N^I_{s-})_{u-s}P_{x+s}\mu_{x+u}dudL(s,u) \\
&- \int_0^t\int_s^T V(s,u)_{u-s}P_{x+s}\mu_{x+u}dudM_s^I.
\end{aligned}
$$

Define

$$
\begin{aligned}
\xi^*(s,T) &= \int_s^T (N-N^I_{s-})_{u-s}P_{x+s}\mu_{x+u}\xi(s,u)du, \\
K(t,T) &= \int_0^t B_s^{-1}\Psi(s,S_s)dM_s^I - \int_0^t\int_s^T V(s,u) \\
&\times {}_{u-s}P_{x+s}\mu_{x+u}dudM_s^I + \int_0^t\int_s^T (N-N_s^I) \\
&\times {}_{u-s}P_{x+s}\mu_{x+u}dudL(s,u).
\end{aligned}
$$

Therefore

$$
V^*_{t,T} = V^*_{0,T} + \int_0^t \xi^*(s,T)d\widetilde{S}_s + K(t,T). \tag{23}
$$

For $0 \le t \le T$, the unique admissible locally risk minimizing hedging strategy $\varphi^*(\xi^*,\eta^*)$ for the term insurance is given by

$$
\begin{aligned}
\xi^*(t,T) &= \int_t^T (N-N^I_{t-})_{u-t}P_{x+t}\mu_{x+u}\xi(t,u)du, \\
\eta^*(t,T) &= \int_0^t B_u^{-1}\Psi(u,S_u)dN_u^I + \int_t^T V(t,u) \\
&\times (N-N_t)_{u-t}P_{x+t}\mu_{x+u}du - \xi^*(t,T)\widetilde{S}_t.
\end{aligned}
$$

According to the Definitions 3 and 4, and Eq.(23), the residual risk process is

$$
\begin{aligned}
& R_{t,T}(\varphi^*) = E_P\left[\Big(K(T,T)-K(t,T)\Big)^2\Big|\mathcal{F}_t\right] \\
= \;& E_P\Big[\int_t^T \Big(B_s^{-1}\Phi(s,S_s) - \int_s^T V(s,u) \\
\times \;& {}_{u-s}P_{x+s}\mu_{x+u}du\Big)^2(N-N_s^I)\mu_{x+s}ds \\
+ \;& \int_t^T \Big(\int_s^T (N-N_s^I)_{u-s}P_{x+s}\mu_{x+u}du\Big)^2 \\
\times \;& \Big(\tilde{S}_s^2 f^2(Y_s)(\Phi_x(s,S_s,Y_s)-\xi_s)^2 \\
+ \;& B_s^{-2}h^2(Y_s)\Phi_y(s,S_s,Y_s)^2 + 2\rho B_s^{-1}f(Y_s) \\
\times \;& h(Y_s)S_s(\Phi_x(s,S_s,Y_s)-\xi_s)\Phi_y(s,S_s,Y_s)\Big)ds\Big|\mathcal{F}_t\Big].
\end{aligned}
$$

6 Barndorff-Nielsen and Shephard stochastic volatility model

In this section, we assume that the risky asset is evolving according to the stochastic volatility model proposed by Barndorff-Nielsen and Shephard [11], where the squared volatility is given by a non-Gaussian Ornstein-Uhlenbeck process:

$$\begin{aligned} dS_t &= (\mu_t + \beta Y_t) S_t dt + \sqrt{Y_t} S_t d\widetilde{W}_t \quad (24) \\ dY_t &= -\lambda Y_t dt + dL(\lambda t) \quad (25) \end{aligned}$$

where β, λ are constant, μ_t is time-dependent, strictly positive and deterministic, $\widetilde{W}_t$ is a standard Brownian motion and $L(t)$ is a pure-jump subordinator. We let $\{\mathcal{F}_t\}_{t\geq 0}$ be the completion of the filtration $\sigma(\widetilde{W}_s, L(\lambda s); s \leq t)$ generated by the Brownian motion and the subordinator such that $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ becomes a complete filtered probability space. The Lévy measure of the subordinator is denoted $\widetilde{\nu}(dy)$ and satisfies by definition $\int_0^\infty \min(1, y)\widetilde{\nu}(dy) < \infty$. $\mu(dy, dt)$ and $\widetilde{\nu}(dy, dt) = \widetilde{\nu}(dy)dt$ denote the jump measure and its compensator, respectively. We refer to Jacod and Shiryaev [4] with respect to the notation used in this paper. From the Eq.(24), we have the following semi-martingale decomposition

$$d\widetilde{S}_t = \bar{M}_t + \bar{A}_t, \quad (26)$$

where $\widetilde{S}$ is the discounted asset price process of S and

$$\begin{aligned} \bar{M}_t &= \int_0^t \sqrt{Y_u} \widetilde{S}_u d\widetilde{W}_u, \quad (27) \\ \bar{A}_t &= \int_0^t (\mu_u + \beta Y_u - r_u) \widetilde{S}_u du. \end{aligned}$$

Since the market is incomplete, there are infinite equivalent martingale measures, we first define a martingale measure by the following

$$\begin{aligned} \frac{d\bar{Q}}{dP_1} &= \bar{D}_t = 1 + \int_0^t G(u, S_u, Y_u) \bar{D}_u d\widetilde{W}_u \\ &+ \int_0^t (H(u, S_u, Y_u) - 1)\, \widetilde{\mu}(dy, du). \quad (28) \end{aligned}$$

By Girsanov's theorem, under the new measure $\bar{Q}$, we know that $W_t = \widetilde{W}_t - \int_0^t G(u, S_u, Y_u)du$ is a standard Brownian motion and the compensator $\nu(dy, dt) = H(t, S_t, Y_t)\widetilde{\nu}(dy)dt$. Under the equivalent martingale measure $\bar{Q}$, the discounted risky asset price process is a martingale, we see easier that the following martingale condition holds

$$\mu_t + \beta Y_t - r_t + \sqrt{Y_t} G(t, S_t, Y_t) = 0. \quad (29)$$

Theorem 9 *Let* $V_t = E_{\bar{Q}}\left[B_T^{-1}\Psi(S_T)\middle|\mathcal{F}_t\right] = B_t^{-1}\Phi(t, S_t, Y_t)$, *then*

$$\begin{aligned} V_t &= V_0 + \int_0^t B_t^{-1}\Phi_x(u, S_u, Y_u)\sqrt{Y_u} S_u dW_u \\ &+ \int_0^t \int_0^\infty B_u^{-1}\Big(\Phi(u, S_u, Y_{u-} + y) \\ &- \Phi(u-, S_{u-}, Y_{u-})\Big)\widetilde{\mu}(dy, du), \end{aligned}$$

where $\Phi_t(t, S_t, Y_t)$, $\Phi_x(t, S_t, Y_t)$, $\Phi_y(t, S_t, Y_t)$ *denote the first derivative of* $\Phi(t, S_t, Y_t)$ *with respect to variable t,* S_t *and* Y_t *respcetively,* $\Psi(S_T)$ *is a* $\mathcal{F}_T$ *measure claim.*

Proof: By the Itô's formula, we obtain

$$\begin{aligned} & dV_t = -r_t B_t^{-1}\Phi(t, S_t, Y_t)dt + B_t^{-1}\Phi_t(t, S_t, Y_t)dt \\ +\ & B_t^{-1}\Phi_x(t, S_t, Y_t)dS_t + B_t^{-1}\Phi_y(t, S_t, Y_t)dY_t^c \\ +\ & \frac{1}{2}B_t^{-1}\Phi_{xx}(t, S_t, Y_t)d < S^c >_t \\ +\ & B_t^{-1}\left(\Phi(t, S_t, Y_t) - \Phi(t-, S_{t-}, Y_{t-})\right) \\ =\ & -r_t B_t^{-1}\Phi(t, S_t, Y_t)dt + B_t^{-1}\Phi_t(t, S_t, Y_t)dt \\ +\ & B_t^{-1}\Phi_x(t, S_t, Y_t)(\mu + \beta Y_t)S_t dt \\ +\ & B_t^{-1}\Phi_x(t, S_t, Y_t)\sqrt{Y_t} S_t d\widetilde{W}_t - B_t^{-1}\Phi_y(t, S_t, Y_t) \\ \times\ & \lambda Y_t dt + \frac{1}{2}B_t^{-1}\Phi_{xx}(t, S_t, Y_t)Y_t S_t^2 dt \\ +\ & B_t^{-1}\left(\Phi(t, S_t, Y_t) - \Phi(t-, S_{t-}, Y_{t-})\right) \\ =\ & -r_t B_t^{-1}\Phi(t, S_t, Y_t)dt + B_t^{-1}\Phi_t(t, S_t, Y_t)dt \\ +\ & B_t^{-1}\Phi_x(t, S_t, Y_t)(\mu + \beta Y_t)S_t dt \\ +\ & B_t^{-1}\Phi_x(t, S_t, Y_t)\sqrt{Y_t} S_t \left(d\widetilde{W}_t + G(t, S_t, Y_t)dt\right) \\ -\ & B_t^{-1}\Phi_y(t, S_t, Y_t)\lambda Y_t dt + \frac{1}{2}B_t^{-1}\Phi_{xx}(t, S_t, Y_t)Y_t \\ \times\ & S_t^2 dt + \int_0^\infty B_t^{-1}\Big(\Phi(t, S_t, Y_{t-} + y) \\ -\ & \Phi(t-, S_{t-}, Y_{t-})\Big)\widetilde{\mu}(dy, dt) \\ +\ & \int_0^\infty B_t^{-1}\Big(\Phi(t, S_t, Y_{t-} + y) \\ -\ & \Phi(t-, S_{t-}, Y_{t-})\Big)\left(H(t, S_t, Y_t) - 1\right)\widetilde{\nu}(dy, dt), \end{aligned}$$

where S^c and Y^c denote continuous parts of S and Y respectively.

Then V_t is a martingale only and only if the following condition is satisfied

$$\begin{aligned} & \Phi_t(t, S_t, Y_t) + \Phi_x(t, S_t, Y_t)(\mu + \beta Y_t \\ +\ & \sqrt{Y_t} G(t, S_t, Y_t))S_t - \Phi_y(t, S_t, Y_t)\lambda Y_t \\ +\ & \int_0^\infty \left(\Phi(t, S_t, Y_{t-} + y) - \Phi(t-, S_{t-}, Y_{t-})\right) \end{aligned}$$

$$
\begin{aligned}
\times & \quad (H(u,S_u,Y_u)-1)\,\widetilde{\nu}(dy,du) \\
= & \quad r_t\Phi(t,S_t,Y_t)-\frac{1}{2}\Phi_{xx}(t,S_t,Y_t)Y_tS_t^2.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
V_t &= V_0+\int_0^t B_u^{-1}\Phi_x(u,S_u,Y_u)\sqrt{Y_u}S_u dW_u \\
&+ \int_0^t\int_0^\infty B_u^{-1}\Big(\Phi(u,S_u,Y_{u-}+y) \\
&- \Phi(u-,S_{u-},Y_{u-})\Big)\widetilde{\mu}(dy,du).
\end{aligned}
$$

□

Theorem 10 *The locally risk minimizing strategy* $\varphi_t=(\bar{\xi}_t,V_t-\bar{\xi}_t\widetilde{S}_t)$, *where*

$$\bar{\xi}_t=\Phi_x(t,S_t,Y_t). \tag{30}$$

The residual risk $R_t(\varphi)$ *is*

$$
\begin{aligned}
R_t(\varphi) &= E_{P_1}\Big[\int_t^T\int_0^\infty B_u^{-2}\Big(\Phi(u,S_u,Y_{u-}+y) \\
&- \Phi(u-,S_{u-},Y_{u-})\Big)^2\widetilde{\nu}(dy)du\Big|\mathcal{F}_t\Big]. \quad (31)
\end{aligned}
$$

Proof:

$$
\begin{aligned}
\bar{L}_t &= V_t-V_0-\int_0^t\bar{\xi}_u d\widetilde{S}_u \\
&= \int_0^t\Phi_x(u,S_u,Y_u)\sqrt{Y_u}\widetilde{S}_u dW_u+\int_0^t\int_0^\infty B_u^{-1} \\
&\times \Big(\Phi(u,S_u,Y_{u-}+y)-\Phi(u-,S_{u-},Y_{u-})\Big)\widetilde{\mu}(dy,du) \\
&- \int_0^t\bar{\xi}_u\Big[(\mu_u+\beta Y_u-r_u)\,\widetilde{S}_u du+\sqrt{Y_u}\widetilde{S}_u d\widetilde{W}_u\Big] \\
&= \int_0^t\left(\Phi_x(u,S_u,Y_u)-\bar{\xi}_u\right)\sqrt{Y_u}\widetilde{S}_u d\widetilde{W}_u \\
&+ \int_0^t\int_0^\infty B_u^{-1}\Big(\Phi(u,S_u,Y_{u-}+y) \\
&- \Phi(u-,S_{u-},Y_{u-})\Big)\left(\mu(dy,du)-\widetilde{\nu}(dy,du)\right) \\
&+ \int_0^t\int_0^\infty (H(u,S_u,Y_u)-1)\Big(\Phi(u,S_u,Y_{u-}+y) \\
&- \Phi(u-,S_{u-},Y_{u-})\Big)B_u^{-1}P_1(dy,du) \\
&- \int_0^t\Phi_x(u,S_u,Y_u)\sqrt{Y_u}\widetilde{S}_u G(u,S_u,Y_u)du \\
&- \int_0^t\bar{\xi}_u\left(\mu_u+\beta Y_u-r_u\right)\widetilde{S}_u du.
\end{aligned}
$$

We find that if we want $\bar{L}$ to be a martingale under P, the drift term of L should be zero:

$$
\begin{aligned}
& 0=\int_0^\infty\Big(\Phi(u,S_u,Y_{u-}+y)-\Phi(u-,S_{u-},Y_{u-})\Big) \\
\times & \quad B_u^{-1}\left(H(u,S_u,Y_u)-1\right)\widetilde{\nu}(dy)-\Phi_x(u,S_u,Y_u) \\
\times & \quad \sqrt{Y_u}\widetilde{S}_u G(u,S_u,Y_u)-\bar{\xi}_u\left(\mu_u+\beta Y_u-r_u\right)\widetilde{S}_u. \quad (32)
\end{aligned}
$$

Thus

$$
\begin{aligned}
\bar{L}_t &= \int_0^t\left(\Phi_x(u,S_u,Y_u)-\bar{\xi}_u\right)\sqrt{Y_u}\widetilde{S}_u d\widetilde{W}_u \\
&+ \int_0^t\int_0^\infty B_u^{-1}\Big(\Phi(u,S_u,Y_{u-}+y) \\
&- \Phi(u-,S_{u-},Y_{u-})\Big)\left(\mu(dy,du)-\widetilde{\nu}(dy,du)\right).
\end{aligned}
$$

Combined the above formula and the Eq.(26), we get

$$\left[\bar{L},\bar{M}\right]_t=\int_0^t\left(\Phi_x(u,S_u,Y_u)-\bar{\xi}_u\right)Y_u\widetilde{S}_u^2du,$$

Finally, we recall that Definition 3 and obtain $\bar{\xi}_u=\Phi_x(u,S_u,Y_u)$. Furthermore we obtain that the residual risk process is given by:

$$
\begin{aligned}
& R_t(\varphi)=E_{P_1}\left[\left(C_T(\varphi)-C_t(\varphi)\right)^2\Big|\mathcal{F}_t\right] \\
= & \quad E_{P_1}\left[\left(\bar{L}_T-\bar{L}_t\right)^2\Big|\mathcal{F}_t\right] \\
= & \quad E_{P_1}\Big[\Big\{\int_t^T\left(\Phi_x(u,S_u,Y_u)-\bar{\xi}_u\right)\sqrt{Y_u}\widetilde{S}_u d\widetilde{W}_u \\
+ & \quad \int_t^T\int_0^\infty B_u^{-1}\Big(\Phi(u,S_u,Y_{u-}+y) \\
- & \quad \Phi(u-,S_{u-},Y_{u-})\Big)\left(\mu(dy,du)-\widetilde{\nu}(dy,du)\right)\Big\}^2\Big|\mathcal{F}_t\Big] \\
= & \quad E_{P_1}\Big[\int_t^T\Big\{\widetilde{S}_u^2Y_u(\Phi_x(u,S_u,Y_u)-\bar{\xi}_u)^2 \\
+ & \quad \int_0^\infty B_u^{-2}\Big(\Phi(u,S_u,Y_{u-}+y) \\
- & \quad \Phi(u-,S_{u-},Y_{u-})\Big)^2\widetilde{\nu}(dy)\Big\}du\Big|\mathcal{F}_t\Big] \\
= & \quad E_{P_1}\Big[\int_t^T\int_0^\infty B_u^{-2}\Big(\Phi(u,S_u,Y_{u-}+y) \\
- & \quad \Phi(u-,S_{u-},Y_{u-})\Big)^2\widetilde{\nu}(dy)du\Big|\mathcal{F}_t\Big].
\end{aligned}
$$

Hence, we complete the proof of above theorem. □

Now recall the condition (29) and get $G(u,S_u,Y_u)=\frac{r_u-\beta Y_u-\mu_u}{\sqrt{Y_u\widetilde{S}_u}}$. We substitute it into the Eq.(32), then leads to the following condition

$$
\begin{aligned}
& \int_0^\infty B_u^{-1}\Big(\Phi(u,S_u,Y_{u-}+y)-\Phi(u-,S_{u-},Y_{u-})\Big) \\
\times & \quad \left(H(u,S_u,Y_u)-1\right)\widetilde{\nu}(dy)=0,\forall u\in[0,T],
\end{aligned}
$$

thus $H(u, S_u, Y_u) = 1, \forall u \in [0, T]$.

In the following, we consider the locally risk minimizing hedging strategy of unit-linked life contracts when volatility satisfies Barndorff-Nielsen and Shephard volatility model. We will adopt the similar procedure of Section 5.

For the pure endowment, we can get

$$\begin{aligned} V_T^* &= V_0^* + \int_0^T (N - N_{s-}^I)_{T-s}P_{x+s}dV_s \\ &- \int_0^T {}_{T-s}P_{x+s}V_s dM_s^I \\ &= V_0^* + \int_0^T (N - N_{s-}^I)_{T-s}P_{x+s}\bar{\xi}_s d\widetilde{S}_s \\ &+ \int_0^T (N - N_{s-}^I)_{T-s}P_{x+s}d\bar{L}_s \\ &- \int_0^T {}_{T-s}P_{x+s}V_s dM_s^I. \end{aligned}$$

Then the optimal portfolio invests $\bar{\xi}_t^* = (N - N_{t-}^I)_{T-t}P_{x+t}\bar{\xi}_t$ in the risky asset and $\bar{\eta}_t^* = (N - N_t^I)_{T-t}P_{x+t}V_t - \bar{\xi}_t^*\widetilde{S}_t$ in the riskless asset for $0 \le t \le T$, the cost process is

$$\begin{aligned} C_t(\bar{\varphi}^*) &= V_0^* + \int_0^t (N - N_{s-}^I)_{T-s}P_{x+s}d\bar{L}_s \\ &- \int_0^t {}_{T-s}P_{x+s}V_s dM_s^I. \end{aligned}$$

As to the term insurance, $V_t = V(t, u)$, with $V(t, u) = E_{Q^*}\left[B_u^{-1}\Psi(u, S_u)|\mathcal{F}_t\right]$ for all $t \le u \le T$. Therefore, ξ_t becomes $\xi(t, u)$ and L_t becomes $L(t, u)$. We can obtain, for $0 \le t \le T$, the unique admissible locally risk minimizing hedging strategy $\bar{\varphi}^*(\bar{\xi}^*, \bar{\eta}^*)$ for the term insurance is given by

$$\begin{aligned} \bar{\xi}^*(t, T) &= \int_t^T (N - N_{t-}^I)_{u-t}P_{x+t}\mu_{x+u}\bar{\xi}(t, u)du, \\ \bar{\eta}^*(t, T) &= \int_0^t B_u^{-1}\Psi(u, S_u)dN_u^I + \int_t^T V(t, u) \\ &\times (N - N_t)_{u-t}P_{x+t}\mu_{x+u}du - \bar{\xi}^*(t, T)\widetilde{S}_t. \end{aligned}$$

7 Conclusion

We have discussed a general class of stochastic volatility model which contains most of those without jump component. The market considered is incomplete, we studied a locally risk-minimization strategy of unit-linked life insurance contracts. Furthermore, we also investigate the locally risk minimizing hedging strategy for unit-linked life insurance contracts in a Barndorff-Nielsen and Shephard stochastic volatility model.

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