Algorithms for finding the minimum norm solution of hierarchical fixed point problems

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Abstract: Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$, $\{T_k\}_{k=1}^{\infty} : C \rightarrow C$ an infinite family of nonexpansive mappings with the nonempty set of common fixed points $\bigcap_{k=1}^{\infty} Fix(T_k)$ and $S : C \rightarrow C$ a nonexpansive mapping. In this paper, we introduce an explicit algorithm with strong convergence for finding the minimum norm solution of the following hierarchical fixed point problem

$$\text{Find } x^* \in \bigcap_{k=1}^{\infty} Fix(T_k) \text{ and } \langle (I - S)x^*, x^* - x \rangle \leq 0, \forall x \in \bigcap_{k=1}^{\infty} Fix(T_k).$$

Key Words: Hierarchical fixed point, Iterative algorithm, Variational inequality, Minimum norm, Strong convergence

1 Introduction

Let $H$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively. Let $C$ be a nonempty closed convex subset of $H$. Let $f : C \rightarrow H$ be a $\alpha$-contraction, where $\alpha \in (0,1)$; namely,

$$\|f(x) - f(y)\| \leq \alpha \|x - y\|, \forall x, y \in C.$$

A mapping $T : C \rightarrow C$ is said to be nonexpansive, if

$$\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in C.$$

We use $Fix(T)$ to denote the set of fixed points of $T$, namely, $Fix(T) = \{x \in C : Tx = x\}$. The metric projection from $H$ onto $C$ is the mapping $P_C : H \rightarrow C$ which assigns to each point $x \in H$ the unique point $P_Cx \in C$ satisfying the property

$$\|x - P_Cx\| = \inf_{y \in C} \|x - y\| =: d(x,C).$$

Let $S, T : C \rightarrow C$ be two nonexpansive mappings. Now, we consider the following problem of finding hierarchically a fixed point of a nonexpansive mapping $T$ with respect to another mapping $S$, namely finding a point $x^*$ with the property

$$x^* \in Fix(T) \text{ such that } \langle (I - S)x^*, x^* - x \rangle \leq 0, \forall x \in Fix(T). \quad (1)$$

Problem (1) is very important in the area of optimization and related fields, such as signal processing and image reconstruction (see [1-4]). Recently, for solving (1), Mainge and Moudafi [5] introduced a hybrid iterative method and Lu, Xu and Yin [6] considered a regularization method. Related work in the field can be found in [7-14] and the references therein. It is needed to find a minimum norm solution in many problems. A typical example is the least-squares solution to the constrained linear inverse problem (see [15]). Therefore, it is an interesting problem to find the minimum norm solution of (1). Yao et al. [13] introduced an implicit algorithm and an explicit algorithm as following:

$$x_{s,t} = P_C[s(1-t)Sx_{s,t} + (1-s)Tx_{s,t}],$$

where $s, t \in (0,1)$, $P_C$ is the metric projection from $H$ to $C$.

$$x_{n+1} = P_C[\lambda_n(1-\alpha_n)Sx_n + (1-\lambda_n)Tx_n], \quad n \geq 0,$$

where $\{\lambda_n\}$ and $\{\alpha_n\}$ are two sequences in $(0,1)$ and $P_C$ is the metric projection from $H$ onto $C$. Under some mild assumptions, Yao et al. [13] proved that $\{x_{s,t}\}$ and $\{x_n\}$ converge strongly to the minimum norm solution $x^*$ of (1).

In order to deal with some problems involving the common fixed points of infinite family of nonexpansive mappings, $W$-mapping is often used. Let
where $\omega$ we introduce two explicit algorithms with iterations, we prove that the two proposed algorithms have

tions, we prove that the two proposed algorithms have

point problem (2). Under certain appropriate condi-

tions, we prove that the two proposed algorithms have

\[
\begin{align*}
U_{n,n+1} &= I, \\
U_{n,n} &= \xi_n T_n U_{n,n+1} + (1 - \xi_n) I, \\
U_{n,n-1} &= \xi_{n-1} T_{n-1} U_{n,n} + (1 - \xi_{n-1}) I, \\
\vdots \\
U_{n,k} &= \xi_k T_k U_{n,k+1} + (1 - \xi_k) I, \\
U_{n,k-1} &= \xi_{k-1} T_{k-1} U_{n,k} + (1 - \xi_{k-1}) I, \\
\vdots \\
U_{n,2} &= \xi_2 T_2 U_{n,3} + (1 - \xi_2) I, \\
W_n &= U_{n,1} = \xi_1 T_1 U_{n,2} + (1 - \xi_1) I,
\end{align*}
\]

Such $W_n$ is called the $W$-mapping generated by \( \{T_k\}_{k=1}^\infty \) and \( \{\xi_k\}_{k=1}^\infty \), see \cite{14,16-18}.

Now we consider the following hierarchical fixed point problem which includes (1) as a special case.

Find $x^* \in \bigcap_{k=1}^\infty Fix(T_k)$ such that

\[
\langle (I - S)x^*, x^* - x \rangle \leq 0, \quad \forall x \in \bigcap_{k=1}^\infty Fix(T_k),
\]

where \( \{T_k\}_{k=1}^\infty \) be an infinite family of nonexpansive mappings with \( \bigcap_{k=1}^\infty Fix(T_k) \neq \emptyset \). In [14], Yao, et al. considered an explicit algorithm which generated a iterative sequence \( \{x_n\} \) by

\[
x_{n+1} = \alpha_n S x_n + (1 - \alpha_n) W_n P_C [(1 - \beta_n) x_n], \quad n \geq 0,
\]

where \( \{\alpha_n\} \) and \( \{\beta_n\} \) are two sequences in \( (0,1) \), \( W_n : C \to C \) is the $W$-mapping. Under some mild assumptions, they proved that \( \{x_n\} \) generated by (3) converges strongly to the minimum norm solution of hierarchical fixed point problem (2).

Since $W$-mapping contains many composite operations of \( \{T_k\} \), it is complicated and needs large computational work. In this paper, we will introduce a new mapping to take the place of $W$-mapping for solving hierarchical fixed point problem (2). Let \( \{T_k\}_{k=1}^\infty : C \to C \) be an infinite family of nonexpansive mappings. The new mapping is defined as follows:

\[
L_n = \sum_{k=1}^n \frac{\omega_k}{S_n} T_k \quad (n = 1, 2, \ldots),
\]

where $\omega_k > 0$ with \( \sum_{k=1}^\infty \omega_k = 1 \), \( S_n = \sum_{k=1}^n \omega_k \).

Inspired and motivated by the work in the field, we introduce two explicit algorithms with $L_n$ for finding the minimum norm solution of hierarchical fixed point problem (2). Under certain appropriate conditions, we prove that the two proposed algorithms have strong convergence. Because $L_n$ used in our algorithms doesn’t contain many composite operations of \( \{T_k\} \) which are included in $W$-mapping, our introduced algorithms are more brief and need less computational work.

We will use the notations:

- $\rightarrow$ for weak convergence and $\to$ for strong convergence.
- $\omega_w(x_n) = \{ x : \exists x_{n_j} \to x \}$ denotes the weak $\omega$-limit set of \( \{x_n\} \).

## 2 Preliminaries

In this section, some lemmas are given which are important to prove our main results.

**Lemma 1** \cite{19} Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $T : C \to C$ be a nonexpansive mapping with $Fix(T) \neq \emptyset$. If \( \{x_n\} \) is a sequence in $C$ weakly converging to $x$ and if \( \{(I - T)x_n\} \) converges strongly to $y$, then \( (I - T)x = y \); in particular, if $y = 0$, then $x \in Fix(T)$.

**Lemma 2** \cite{13} Given $x \in H$ and $z \in C$.

1. That $z = P_C x$ if and only if there holds the relation:

\[
\langle x - z, y - z \rangle \leq 0 \quad \text{for all } y \in C.
\]

2. That $z = P_C x$ if and only if there holds the relation:

\[
\|x - z\|^2 \leq \|x - y\|^2 - \|y - z\|^2 \quad \text{for all } y \in C.
\]

3. There holds the relation:

\[
\langle P_C x - P_C y, x - y \rangle \geq \|P_C x - P_C y\|^2 \quad \text{for all } y \in H.
\]

**Lemma 3** \cite{20} Assume \( \{\alpha_n\} \) is a sequence of nonnegative real numbers such that

\[
\alpha_{n+1} \leq (1 - \gamma_n) \alpha_n + \gamma_n \delta_n, \quad n \geq 0,
\]

where \( \{\gamma_n\} \) is a sequence in \( (0,1) \) and \( \{\delta_n\} \) is a sequence in $\mathbb{R}$ such that

(i) \( \sum_{n=0}^\infty \gamma_n = \infty \); 

(ii) $\lim_{n \to \infty} \delta_n \leq 0$ or \( \sum_{n=0}^\infty |\gamma_n \delta_n| < \infty \).

Then $\lim_{n \to \infty} \alpha_n = 0$. 

Lemma 4 [21] Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $\{T_k\}_{k=1}^\infty : C \to C$ be an infinite family of nonexpansive mappings. Suppose $\bigcap_{k=1}^\infty \text{Fix}(T_k)$ is nonempty. Let $\{\omega_k\}_{k=1}^\infty$ be a sequence in $(0,1)$ with $\sum_{k=1}^\infty \omega_k = 1$. Then a mapping $L$ on $C$ defined by $Lx = \sum_{k=1}^\infty \omega_k T_k x$ for $x \in C$ is well defined, nonexpansive and $\text{Fix}(L) = \bigcap_{k=1}^\infty \text{Fix}(T_k)$ holds.

Lemma 5 [22] Let $H$ be a real Hilbert space, $\{T_k : k \in \mathbb{N}\}$ be a sequence of nonexpansive mappings on $H$ with $\bigcap_{k=1}^\infty \text{Fix}(T_k) \neq \emptyset$, and $\{\omega_k\}$ be a sequence of positive numbers with $\sum_{k=1}^\infty \omega_k = 1$. Let $L = \sum_{k=1}^\infty \omega_k T_k$, $L_m = \sum_{k=1}^m \frac{\omega_k}{\sum_{k=1}^m} T_k$, and $S_m = \sum_{k=1}^m \omega_k$. Then $L_m$ uniformly converges to $L$ in each bounded subset $S$ of $H$.

3 Main result

In this section, we first introduce an explicit scheme for finding the minimum norm solution of hierarchical fixed point problem (2). More precisely, starting with an arbitrary initial guess $x_0 \in C$, we define a sequence $\{x_n\}$ recursively by

$$x_{n+1} = P_C[\beta_n(1-\alpha_n)Sx_n + (1-\beta_n)\text{Fix}(L_n)x_n], \quad n \geq 0,$$  

where $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences in $(0,1)$, $S : C \to C$ is a nonexpansive mapping, $L_n : C \to C$ is the nonexpansive mapping defined by (4), $P_C : H \to C$ is the metric projection.

Remark 6 We note that the well-known Mann algorithm $x_{n+1} = \beta_n x_n + (1-\beta_n)Tx_n$ has only weak convergence; please see [23-29] for the related works. This implies that the algorithm

$$x_{n+1} = \beta_n Sx_n + (1-\beta_n)L_n x_n, \quad n \geq 0 \tag{6}$$

has only weak convergence. In order to obtain strong convergence, some modifications are needed. We modify the algorithm (6) by adding the factor $1-\alpha_n$ (where $\alpha_n \to 0$). However, we note that $(1-\alpha_n)x_n$ may not be in $C$. Hence, the projection $P_C$ is used in order to guarantee that the sequence $\{x_n\}$ is well-defined.

Next, we will show the strong convergence of the algorithm (5). As a matter of fact, we introduce a general algorithm which includes the algorithm (5) as a special case. For any $x_0 \in C$, define the sequence $\{x_n\}$ iteratively by

$$x_{n+1} = P_C[\beta_n(\alpha_n f(x_n) + (1-\alpha_n)Sx_n) + (1-\beta_n)L_n x_n], \quad n \geq 0,$$  

where $f : C \to H$ is a $\alpha$-contraction. It is clear that if we take $f = 0$, then (7) reduces to (5). For the strong convergence of the algorithm (5) and (7), we have the following theorem. Throughout, we use $\Omega$ to denote the set of solution to (2) and assume that $\Omega$ is nonempty.

Theorem 7 Let $C$ be a nonempty bounded closed convex subset of a real Hilbert space $H$. Let $f : C \to H$ be a $\alpha$-contraction with $\alpha \in [0,1]$, $S : C \to C$ a nonexpansive mapping, and $\{T_k\}_{k=1}^\infty : C \to C$ an infinite family of nonexpansive mapping with $\bigcap_{k=1}^\infty \text{Fix}(T_k) \neq \emptyset$. Let $L_n = \sum_{k=1}^\infty \frac{\omega_k}{\sum_{k=1}^\infty} T_k$. $S_n = \sum_{k=1}^n \omega_k$, and $\omega_k > 0$ with $\sum_{k=1}^\infty \omega_k = 1$. Suppose the following conditions are satisfied

(C1) $\lim_{n \to \infty} \alpha_n = \lim_{n \to \infty} \frac{\alpha_n \beta_n - \alpha_n \beta_{n-1}}{\alpha_n \beta_n} = \lim_{n \to \infty} \frac{1}{n} = 0$,

(C2) $\sum_{n=0}^\infty \alpha_n \beta_n = \infty$;

(C3) There exists some constant $\gamma > 0$ such that $\|x - L_n x\| \geq \gamma \text{Dist}(x, \bigcap_{k=1}^\infty \text{Fix}(T_k))$, where

$$\text{Dist}(x, \bigcap_{n=1}^\infty \text{Fix}(T_k)) = \inf_{y \in \bigcap_{n=1}^\infty \text{Fix}(T_k)} \|x-y\|.$$

Then the sequence $\{x_n\}$ generated by (7) converges strongly to $x^* \in \bigcap_{k=1}^\infty \text{Fix}(T_k)$ which is the unique solution of the variational inequality:

$$x^* \in \Omega, \quad \langle (I-f)x^*, x-x^* \rangle \geq 0, \quad \forall x \in \Omega. \tag{8}$$

In particular, if we take $f = 0$, then the sequence $\{x_n\}$ generated by (5) converges strongly to $x^* \in \bigcap_{k=1}^\infty \text{Fix}(T_k)$ which is the minimum norm solution of hierarchical fixed point problem (2).

Proof: We will use six steps to prove the result.

Step 1. We show that $\lim_{n \to \infty} \|x_{n+1} - x_n\| \to 0$.

For $n \geq 0$, set

$$y_n = \beta_n (\alpha_n f(x_n) + (1-\alpha_n)Sx_n) + (1-\beta_n)L_n x_n.$$

Then it holds that

$$y_n - y_{n-1} = \beta_n (\alpha_n f(x_n) + (1-\alpha_n)Sx_n) + (1-\beta_n)L_n x_n - \beta_n (\alpha_n f(x_{n-1}) + (1-\alpha_n)Sx_{n-1}) - (1-\beta_n)L_{n-1} x_{n-1}$$

$$= \alpha_n \beta_n [f(x_n) - f(x_{n-1})] + \beta_n (1-\alpha_n) [Sx_n - Sx_{n-1}] + (\alpha_n \beta_n - \alpha_n \beta_{n-1}) [f(x_{n-1}) - Sx_{n-1}] + (1-\beta_n) (L_n x_n - L_{n-1} x_{n-1}) + (1-\beta_n) (L_n x_n - L_{n-1} x_{n-1}) + (\beta_n - \beta_{n-1}) (Sx_{n-1} - L_n x_{n-1})$$
It follows that
\[
\begin{align*}
\|x_{n+1} - x_n\| & = \|Pcyn - Pcy_{n-1}\| \leq \|y_n - y_{n-1}\| \\
& \leq \alpha_n \beta_n \|f(x_n) - f(x_{n-1})\| + \beta_n \|(1 - \alpha_n)\|Sx_n - Sx_{n-1}\| \\
& + \|\alpha_n \beta_n - \alpha_n - \beta_n - 1\|f(x_{n-1}) - Sx_{n-1}\| \\
& + |\beta_n - \beta_n - 1|\|Sx_{n-1} - Lx_{n-1}\| \\
& \leq \alpha_n \beta_n \|x_n - x_{n-1}\| + \beta_n \|(1 - \alpha_n)\|x_n - x_{n-1}\| \\
& + \|\alpha_n \beta_n - \alpha_n - \beta_n - 1\|f(x_{n-1}) - Sx_{n-1}\| \\
& + |\beta_n - \beta_n - 1|\|Sx_{n-1} - Lx_{n-1}\| \\
& \leq \alpha_n \beta_n \|x_n - x_{n-1}\| + \beta_n \|(1 - \alpha_n)\|x_n - x_{n-1}\| \\
& + \|\alpha_n \beta_n - \alpha_n - \beta_n - 1\|f(x_{n-1}) - Sx_{n-1}\| \\
& + |\beta_n - \beta_n - 1|\|Sx_{n-1} - Lx_{n-1}\|.
\end{align*}
\]

We observe that
\[
\begin{align*}
\|L_n x_{n-1} - L_n x_{n-1}\| & = \|\sum_{k=1}^{n} \frac{w_k}{S_n} T_k x_{n-1} - \sum_{k=1}^{n-1} \frac{w_k}{S_n} T_k x_{n-1}\| \\
& = \|\sum_{k=1}^{n} \frac{w_k}{S_n} T_k x_{n-1} - \sum_{k=1}^{n-1} \frac{w_k}{S_n} T_k x_{n-1}\| \\
& \leq \|\sum_{k=1}^{n} \frac{w_k}{S_n} T_k x_{n-1}\| + \|\sum_{k=1}^{n-1} \frac{w_k}{S_n} T_k x_{n-1}\| \\
& \leq \frac{\omega_n}{S_n} \|T_n x_{n-1}\| + \|\sum_{k=1}^{n-1} \frac{w_k}{S_n} T_k x_{n-1}\| \\
& \leq \frac{\omega_n}{S_n} \|T_n x_{n-1}\| + \|\sum_{k=1}^{n-1} \frac{w_k}{S_n} T_k x_{n-1}\| \\
& \leq M \omega_n,
\end{align*}
\]
where \(M\) is a constant such that
\[
M \geq \sup_{1 \leq k \leq n} \frac{\|f(x_n)\| + \|Sx_n\|}{\omega_1}, \quad \left(\|Sx_{n-1} - L_n x_{n-1}\|, \|x_n - x_{n-1}\|\right).
\]
Therefore,
\[
\begin{align*}
\|x_{n+1} - x_n\| & \leq \left[1 - (1 - \alpha_n)\alpha_n \beta_n\right]\|x_n - x_{n-1}\| \\
& + \left|\alpha_n \beta_n - \alpha_n - \beta_n - 1\right| + (1 - \beta_n - 1)\omega_n \\
& + |\beta_n - \beta_n - 1|\|M\|
\end{align*}
\]
\[
\leq \frac{[1 - (1 - \alpha_n)\alpha_n \beta_n]\|x_n - x_{n-1}\| + \alpha_n \beta_n\left(\frac{\|\alpha_n \beta_n - \alpha_n - \beta_n - 1\|}{\alpha_n \beta_n} + \frac{|\beta_n - \beta_n - 1|}{\beta_n}\right)}{\beta_n}.
\]
Thus, from (C1), we have \(\limsup_{n \to \infty} \left(\frac{\omega_n}{\alpha_n \beta_n} + \frac{|\alpha_n \beta_n - \alpha_n - \beta_n - 1|}{\alpha_n \beta_n} + \frac{|\beta_n - \beta_n - 1|}{\beta_n}\right) = 0\). Hence, applying Lemma 3 to (9), we conclude immediately that
\[
\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0.
\]

**Step 2.** We prove that \(\omega_w(x_n) \subset Fix(L) = \bigcap_{k=1}^{\infty} Fix(T_k)\), where \(L = \sum_{k=1}^{\infty} \omega_k T_k\).

By (7), we get immediately
\[
\begin{align*}
\|x_{n+1} - L_n x_n\| & = \|Pcyn - PcL_n x_n\| \leq \|y_n - L_n x_n\| \\
& = \|\beta_n (\alpha_n f(x_n) + (1 - \alpha_n) Sx_n)\| + \|\beta_n - L_n x_n - L_n x_n\| \\
& \leq \beta_n \|\alpha_n f(x_n) + (1 - \alpha_n) Sx_n - L_n x_n\| \\
& \to 0 (n \to \infty).
\end{align*}
\]
Notice that
\[
\|x_n - L_n x_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - L_n x_n\| \\
+ \|L_n x_n - L x_n\|.
\]
Thus, from (10), (11) and Lemma 5, we deduce
\[
\lim_{n \to \infty} \|x_n - L_n x_n\| = 0.
\]
Since the sequence \(\{x_n\}\) is bounded, there exists a subsequence \(\{x_{n_j}\}\) of \(\{x_n\}\) which converges weakly to some \(\bar{x} \in H\). Therefore, we have \(\bar{x} \in Fix(L) = \bigcap_{k=1}^{\infty} Fix(T_k)\) by (12) and Lemma 1. Hence, \(\omega_w(x_n) \subset Fix(L) = \bigcap_{k=1}^{\infty} Fix(T_k)\).

**Step 3.** We claim that \(\omega_w(x_n) \subset \Omega\).

By (9), we get
\[
\begin{align*}
\|x_{n+1} - x_n\| & \leq \left[1 - (1 - \alpha_n)\alpha_n \beta_n\right]\|x_n - x_{n-1}\| \\
& + \|\alpha_n \beta_n - \alpha_n - \beta_n - 1\| + \omega_n + |\beta_n - \beta_n - 1| \\
& = \left[1 - (1 - \alpha_n)\alpha_n \beta_n\right]\|x_n - x_{n-1}\| \\
& + \|\alpha_n \beta_n - \alpha_n - \beta_n - 1\| + \omega_n + |\beta_n - \beta_n - 1| \\
& + \|\alpha_n \beta_n - \alpha_n - \beta_n - 1\| + \omega_n + |\beta_n - \beta_n - 1|
\end{align*}
\]
Thus, by virtue of condition (C1), we have
\[
\lim_{n \to \infty} \left( \frac{|\alpha_n \beta_n - \alpha_{n-1} \beta_{n-1}| + \omega_n}{\alpha_n \beta_n^2} + \frac{|\beta_n - \beta_{n-1} - 1|}{\alpha_n \beta_n} + \frac{1}{\alpha_n \beta_n} \left( \frac{1}{\beta_n} - \frac{1}{\beta_{n-1}} \right) \right) = 0.
\]

Hence, applying Lemma 3 to above last inequality, we conclude immediately that
\[
\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0. \quad (13)
\]

Rewriting (7) as
\[
x_{n+1} = PCy_n - y_n + \beta_n(\alpha_n f(x_n) + (1 - \alpha_n)Sx_n) + (1 - \beta_n)L_n x_n.
\]

We obtain
\[
x_n - x_{n+1} = -y_n - PCy_n + \alpha_n \beta_n (I - f)x_n + \beta_n (1 - \alpha_n) (I - S)x_n + (1 - \beta_n) (I - L_n) x_n.
\]

Set \(z_n = \frac{x_n - x_{n+1}}{\beta_n}\) for all \(n \geq 0\). That is
\[
z_n = \frac{y_n - PCy_n + \alpha_n (I - f)x_n + (1 - \alpha_n) (I - S)x_n + (1 - \beta_n) (I - L_n) x_n}{\beta_n}.
\]

Pick up \(u \in \bigcap_{k=1}^{\infty} Fix(T_k)\), then we have
\[
\langle z_n, x_n - u \rangle = \frac{1}{\beta_n} \langle y_n - PCy_n, PCy_n - u \rangle + \alpha_n \langle (I - f)x_n, x_n - u \rangle + (1 - \alpha_n) \langle (I - S)x_n, x_n - u \rangle + \frac{1}{\beta_n} \langle (I - L_n)x_n, x_n - u \rangle
\]
\[
= \frac{1}{\beta_n} \langle y_n - PCy_n, PCy_n - u \rangle + \alpha_n (I - f)x_n, x_n - u \rangle + (1 - \alpha_n) \langle (I - S)x_n, x_n - u \rangle + \frac{1}{\beta_n} \langle (I - L_n)x_n, x_n - u \rangle
\]
\[
+ \frac{1}{\beta_n} \langle y_n - PCy_n, PCy_n - 1 - PCy_n \rangle + \alpha_n (I - f)x_n, x_n - u \rangle + (1 - \alpha_n) \langle (I - S)u, x_n - u \rangle + (1 - \alpha_n) \langle (I - S)x_n, (I - S)u, x_n - u \rangle + \frac{1}{\beta_n} \langle (I - L_n)x_n, (I - L_n)u, x_n - u \rangle.
\]

Using the property of the projection (Lemma 2), we have
\[
\langle y_n - PCy_n, PCy_n - u \rangle \geq 0.
\]

Using monotonicity of \(I - S\) and \(I - L_n\), we derive the that
\[
\langle (I - S)x_n, (I - S)u, x_n - u \rangle \geq 0, \quad \langle (I - L_n)x_n, (I - L_n)u, x_n - u \rangle \geq 0.
\]

Therefore, we conclude that (noticing \(x_n = PCy_{n-1}\))
\[
\langle z_n, x_n - u \rangle \geq \frac{1}{\beta_n} \langle y_n - PCy_n, PCy_{n-1} - PCy_n \rangle + \alpha_n \langle (I - f)x_n, x_n - u \rangle + (1 - \alpha_n) \langle (I - S)u, x_n - u \rangle + (1 - \alpha_n) \langle (I - S)x_n, (I - S)u, x_n - u \rangle + (1 - \alpha_n) \langle (I - L_n)u, x_n - u \rangle.
\]

Since \(z_n \to 0, \alpha_n \to 0\) and \(\{x_n\}\) is bounded by assumption which implies \(\{y_n\}\) is bounded, we obtain from the above inequality that
\[
\limsup_{n \to \infty} \langle (I - S)u, x_n - u \rangle \leq 0, \quad \forall u \in \bigcap_{k=1}^{\infty} Fix(T_k).
\]

Therefore, we have
\[
\limsup_{j \to \infty} \langle (I - S)u, x_{n_j} - u \rangle \leq 0, \quad \forall u \in \bigcap_{k=1}^{\infty} Fix(T_k).
\]

Since \(x_{n_j} \rightharpoonup \hat{x} \in \omega_w(x_n)\), we obtain
\[
\limsup_{j \to \infty} \langle (I - S)u, x_{n_j} - u \rangle = \langle (I - S)u, \hat{x} - u \rangle.
\]

This implies that every weak cluster point \(\hat{x} \in \bigcap_{k=1}^{\infty} Fix(T_k)\) of the sequence \(\{x_n\}\) solves the variational inequality
\[
\langle (I - S)u, \hat{x} - u \rangle \leq 0, \quad \forall u \in \bigcap_{k=1}^{\infty} Fix(T_k).
\]

This is equivalent to its dual variational inequality
\[
\langle (I - S)\hat{x}, \hat{x} - u \rangle \leq 0, \quad \forall u \in \bigcap_{k=1}^{\infty} Fix(T_k). \quad (14)
\]

Hence, we get \(\omega_w(x_n) \subset \Omega\).

**Step 4.** We show that \(\limsup_{n \to \infty} \langle (I - f)x^*, x_n - x^* \rangle \geq 0\).
Since $f$ is a contraction, the solution set of the variational inequality (8) is a singleton. Let $x^*$ is the unique solution of the variational inequality (8). Now we take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ satisfying
\[
\lim_{n \to \infty} \langle (I - f)x^*, x_{n_k} - x^* \rangle = \lim_{k \to \infty} \langle (I - f)x^*, x_{n_k} - x^* \rangle.
\]
Without loss of generality, we may further assume that $x_{n_k} \to \bar{x}$, then $\bar{x} \in \Omega$. Therefore, noticing that $x^*$ is the solution of the variational inequality (8), we have
\[
\limsup_{n \to \infty} \langle (I - f)x^*, x_{n-k} - x^* \rangle = \langle (I - f)x^*, \bar{x} - x^* \rangle \geq 0.
\]

**Step 5.** We show that
\[
\limsup_{n \to \infty} \frac{1}{\alpha_n} \langle Sx^* - x^*, x_{n+1} - x^* \rangle \leq 0.
\]
We note that
\[
\langle Sx^* - x^*, x_{n+1} - x^* \rangle = \langle Sx^* - x^*, x_{n+1} - P_{[k=1}^{\infty} Fix(T_k) x_{n+1} \rangle + \langle Sx^* - x^*, P_{[k=1}^{\infty} Fix(T_k) x_{n+1} - x^* \rangle.
\]
Since $P_{[k=1}^{\infty} Fix(T_k) x_{n+1} \in \bigcap_{k=1}^{\infty} Fix(T_k)$, by (2) we have
\[
\langle Sx^* - x^*, P_{[k=1}^{\infty} Fix(T_k) x_{n+1} - x^* \rangle \leq 0,
\]
and by assumption (C3), we have
\[
\langle Sx^* - x^*, x_{n+1} - x^* \rangle \leq \langle Sx^* - x^*, x_{n+1} - x^* \rangle \leq \|Sx^* - x^*\| \|x_{n+1} - x^*\| + \|x_{n+1} - x^*\| \|x_{n+1} - L_n x_{n+1}\| \leq \frac{1}{\gamma} \|Sx^* - x^*\| \|x_{n+1} - L_n x_{n+1}\|.
\]
We note that
\[
\|x_{n+1} - L_n x_{n+1}\| \leq \|x_{n+1} - L_n x_{n+1}\| + \|L_n x_{n+1} - L_n x_{n+1}\| \leq \|P_C y_n - P_C L_n x_{n+1}\| \|L_n x_{n+1} - L_n x_{n+1}\| \leq \beta_n \|\alpha_n f(x_n) + (1 - \alpha_n) Sx_n - L_n x_{n+1}\| + \|x_n - x_{n+1}\| \|x_{n+1} - x^*\|/2.
\]
Therefore, we get
\[
\frac{1}{\alpha_n} \|x_{n+1} - L_n x_{n+1}\| \leq \frac{2}{\alpha_n} \|\alpha_n f(x_n) + (1 - \alpha_n) Sx_n - L_n x_{n+1}\| + \|x_n - x_{n+1}\| \|x_{n+1} - x^*\| / \beta_n \to 0.
\]
It follows that
\[
\limsup_{n \to \infty} \frac{1}{\alpha_n} \langle Sx^* - x^*, x_{n+1} - x^* \rangle \leq 0. \quad (16)
\]

**Step 6.** Finally, we prove that $x_n \to x^*$ as $n \to \infty$. From (7), we deduce that (noticing $x_{n+1} = P_C y_n$
\[
\|x_{n+1} - x^*\|^2 = \langle y_n - x^*, x_{n+1} - x^* \rangle + \langle P_C y_n - y_n, P_C y_n - x^* \rangle \leq \langle y_n - x^*, x_{n+1} - x^* \rangle + \|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2 + \|x_{n+1} - x^*\|^2.
\]
It turns out that
\[
\|x_{n+1} - x^*\|^2 \leq 1 - (1 - \alpha_n) \|x_n - x^*\|^2 + \frac{2 \beta_n}{1 - (1 - \alpha_n) \|x_n - x^*\|^2} \langle x_n, f(x_n) - x^* \rangle \langle x_n, x_{n+1} - x^* \rangle + \|x_{n+1} - x^*\|^2 + \|x_{n+1} - x^*\|^2 \to 0.
\]

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Thus, from (15) and (16), we have \( \lim_{n \to \infty} \alpha_n = 0 \) and point out that this assumption is weaker than \( \alpha_n \). Let \( \alpha_n \) be a sequence generated by \( x_{n+1} = \alpha_n S x_n + (1 - \alpha_n) L_n P C \) for all \( n \geq 0 \). If \( f = 0 \), then (17) is reduced to the iterative scheme:

\[
x_{n+1} = \alpha_n S x_n + (1 - \alpha_n) L_n P C \]

\[
(1 - \beta_n) x_n, n \geq 0.
\]

Suppose the following conditions are satisfied

\((A1)\) \( \lim_{n \to \infty} \alpha_n = \lim_{n \to \infty} \frac{\beta_n}{\alpha_n} = \lim_{n \to \infty} \frac{\alpha_n}{\beta_n} = 0 \),

\[(A2)\) \( \sum_{n=0}^{\infty} \beta_n = \infty; \)

\((A3)\) There exists some constant \( \gamma > 0 \) such that

\[ ||x - L_n x|| \geq \gamma \text{Dist}(x, \bigcap_{k=1}^{\infty} \text{Fix}(T_k)), \]

where \( \text{Dist}(x, \bigcap_{k=1}^{\infty} \text{Fix}(T_k)) = \inf_{y \in \bigcap_{k=1}^{\infty} \text{Fix}(T_k)} ||x - y|| \).

Then the sequence \( \{x_n\} \) generated by (17) converges strongly to \( x^* \in \bigcap_{k=1}^{\infty} \text{Fix}(T_k) \) which is the unique solution of the variational inequality:

\[ x^* \in \Omega, \quad \langle (I - f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in \Omega. \]

In particular, if we take \( f = 0 \), then the sequence \( \{x_n\} \) generated by (18) converges strongly to \( x^* \in \bigcap_{k=1}^{\infty} \text{Fix}(T_k) \) which is the minimum norm solution of hierarchical fixed point problem (2).

**Proof:** We will use six steps to complete the proof.

**Step 1.** We show that \( \lim_{n \to \infty} ||x_{n+1} - x_n|| \to 0 \).

Setting \( y_n = \beta_n f(x_n) + (1 - \beta_n) x_n \) for all \( n \geq 0 \), that is

\[ y_n - y_{n-1} = \beta_n (f(x_n) - f(x_{n-1})) + (1 - \beta_n) (x_n - x_{n-1}) + (1 - \beta_n) x_n - x_{n-1}. \]

It follows that

\[ ||y_n - y_{n-1}|| \leq [1 - (1 - \alpha) \beta_n] ||x_n - x_{n-1}|| + |\beta_n - \beta_{n-1}| (||f(x_{n-1})|| + ||x_{n-1}||). \]
From (17), we have
\[
x_{n+1} - x_n = \alpha_n Sx_n + (1 - \alpha_n)L_n PC y_n - \alpha_{n-1} Sx_{n-1} - (1 - \alpha_{n-1}) L_n PC y_{n-1}
\]
\[= \alpha_n (Sx_n - Sx_{n-1}) + (1 - \alpha_n) (L_n PC y_n - L_n PC y_{n-1}) + (\alpha_n - \alpha_{n-1}) Sx_{n-1} + (\alpha_{n-1} - \alpha_n) L_n PC y_{n-1}.
\]
Then, we obtain
\[
\|x_{n+1} - x_n\|
\leq \alpha_n \|x_n - x_{n-1}\| + (1 - \alpha_n) \|y_n - y_{n-1}\|
+ (1 - \alpha_n) \|L_n PC y_{n-1} - L_{n-1} PC y_{n-1}\|
+ |\alpha_n - \alpha_{n-1}| (\|Sx_{n-1}\| + \|L_{n-1} PC y_{n-1}\|).
\]
From theorem 7's proof, we can get
\[
\|L_n PC y_{n-1} - L_{n-1} PC y_{n-1}\| \leq M \omega_n,
\]
where $M$ is some constant such that
\[
M \geq \sup_{1 \leq k \leq n} \{(\|f(x_n)\| + \|Sx_n\|) \|T_k x_{n-1}\| / \omega_1, \|Sx_{n-1}\| + \|L_n x_{n-1}\|, \|x_n - x_{n-1}\|\}.
\]
Hence, we have
\[
\|x_{n+1} - x_n\|
\leq [1 - (1 - \alpha)n] \|x_n - x_{n-1}\| / \alpha_n
+ |\beta_n - \beta_{n-1}| (\|f(x_{n-1})\| + \|x_{n-1}\|)
+ |\alpha_n - \alpha_{n-1}| (\|Sx_{n-1}\| + \|L_{n-1} PC y_{n-1}\|)
+ M \omega_n.
\]
Therefore,
\[
\|x_{n+1} - x_n\| / \alpha_n
\leq [1 - (1 - \alpha)n] \|x_n - x_{n-1}\| / \alpha_n
+ |\beta_n - \beta_{n-1}| (\|f(x_{n-1})\| + \|x_{n-1}\|)
+ |\alpha_n - \alpha_{n-1}| (\|Sx_{n-1}\| + \|L_{n-1} PC y_{n-1}\|)
+ M \omega_n / \alpha_n
\leq [1 - (1 - \alpha)n] \|x_n - x_{n-1}\| / \alpha_n - 1
+ (1 - \alpha)n \|x_{n-1}\| / \alpha_{n-1}
+ M \omega_n / \alpha_n
\leq [1 - (1 - \alpha)n] \|x_n - x_{n-1}\| / \alpha_n
+ (1 - \alpha)n \|x_{n-1}\| / \alpha_{n-1}
+ M \omega_n / \alpha_n
\leq (1 - \alpha)n \|x_{n-1}\| / \alpha_{n-1}
+ M \omega_n / \alpha_n
\]
Thus, from (A1), we have
\[
\lim_{n \to \infty} \|x_{n+1} - x_n\| / \alpha_n = 0.
\]
This implies that
\[
\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0.
\]
Step 2. We prove that $\omega_w(x_n) \subset Fix(L) = \bigcap_{k=1}^{\infty} Fix(T_k)$.
From (17) and (20), we have
\[
\lim_{n \to \infty} \|x_n - L_n PC y_n\| = 0.
\]
By (17), we get
\[
\|PC y_n - x_n\| \leq \|y_n - x_n\| = \|\beta_n (f(x_n) - x_n)\|
\to 0.
\]
Notice that
\[
\|x_n - Lx_n\|
\leq \|x_n - L_n PC y_n\| + \|L_n PC y_n - Lx_n\|
\leq \|x_n - L_n PC y_n\| + \|PC y_n - x_n\|
+ \|L_n x_n - Lx_n\|.
\]
By (21)-(23) and Lemma 5, we get
\[
\lim_{n \to \infty} \|x_n - Lx_n\| = 0.
\]
Since the sequence $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges weakly to some $\tilde{x} \in H$. Therefore, we have $\tilde{x} \in Fix(L) = \bigcap_{k=1}^{\infty} Fix(T_k)$ by (24) and Lemma 1. Hence, $\omega_w(x_n) \subset Fix(L) = \bigcap_{k=1}^{\infty} Fix(T_k)$.
Step 3. We claim that $\omega_w(x_n) \subset \Omega$.
Rewriting (17) as
\[
x_n - x_{n+1} = \alpha_n (x_n - Sx_n) + (1 - \alpha_n) (PC y_n - L_n PC y_n) + (1 - \alpha_n) (y_n - PC y_n) + (1 - \alpha_n) (x_n - y_n),
\]
that is
\[
x_n - x_{n+1} = (I - S)x_n + (1 - \alpha_n) (PC y_n - L_n PC y_n) + (1 - \alpha_n) (y_n - PC y_n) + (1 - \alpha_n) (x_n - y_n),
\]
rewriting (17) as
\[
x_n - x_{n+1} = (I - S)x_n + (1 - \alpha_n) (PC y_n - L_n PC y_n) + (1 - \alpha_n) (y_n - PC y_n) + (1 - \alpha_n) (x_n - y_n),
\]
Set \( z_n = \frac{z_n - x_n - x_{n-1}}{\alpha_n} \) and pick up \( u \in \bigcap_{k=1}^{\infty} \). Then, we have

\[
\langle z_n, x_n - u \rangle = \langle (I - S)x_n, x_n - u \rangle + \frac{1 - \alpha_n}{\alpha_n} \langle P_C y_n - L_n P_C y_n, x_n - u \rangle + \frac{1 - \alpha_n}{\alpha_n} \langle (I - P_C)y_n, x_n - u \rangle + \beta_n(1 - \alpha_n) \langle (I - f)x_n, x_n - u \rangle.
\]

Using monotonicity of \( I - W_n \) and \( I - S \), we derive that

\[
\langle (I - S)x_n - (I - S)u, x_n - u \rangle \geq 0 \quad \text{and} \quad \langle (I - L_n)P_C y_n - (I - L_n)u, P_C y_n - u \rangle \geq 0.
\]

At the same time, we observe that

\[
y_n - L_n P_C y_n = \beta_n f(x_n) + (1 - \beta_n)x_n - L_n P_C y_n = \beta_n[f(x_n) - x_{n+1}] + (1 - \beta_n)(x_n - x_{n-1}) + x_{n-1} - L_n P_C y_n = \beta_n[f(x_n) - x_{n+1}] + (1 - \beta_n)(x_n - x_{n-1}) + \alpha_n(S x_n - L_n P_C y_n).
\]

Therefore,

\[
\langle z_n, x_n - u \rangle \geq \langle (I - S)u, x_n - u \rangle + \frac{1 - \alpha_n}{\alpha_n} \langle (I - L_n)P_C y_n, x_n - P_C y_n \rangle + \beta_n(1 - \alpha_n) \langle (I - f)x_n, x_n - u \rangle.
\]

Using the property of the projection (Lemma 2.2), we have

\[
\langle (I - P_C)y_n, P_C y_n - u \rangle \geq 0.
\]

But, since \( z_n \to 0 \), \( \beta_n \to 0 \), \( x_n - x_{n+1} \to 0 \) and \( (x_n - P_C y_n) \to 0 \), we obtain from the above inequality that

\[
\limsup_{n \to \infty} \langle (I - S)u, x_n - u \rangle \leq 0, \quad u \in \bigcap_{k=1}^{\infty} \text{Fix}(T_k).
\]

Therefore,

\[
\limsup_{j \to \infty} \langle (I - S)u, x_{n_j} - u \rangle \leq 0, \quad u \in \bigcap_{k=1}^{\infty} \text{Fix}(T_k).
\]

Since \( x_{n_j} \to \bar{x} \), we have

\[
\limsup_{j \to \infty} \langle (I - S)u, x_{n_j} - u \rangle = \langle (I - S)u, \bar{x} - u \rangle.
\]

This implies that every weak cluster point \( \bar{x} \in \bigcap_{k=1}^{\infty} \text{Fix}(T_k) \) of the sequence \( \{x_n\} \) solves the variational inequality

\[
\langle (I - S)u, \bar{x} - u \rangle \leq 0, \quad \forall u \in \bigcap_{k=1}^{\infty} \text{Fix}(T_k).
\]

This is equivalent to its dual variational inequality

\[
\langle (I - S)\bar{x}, \bar{x} - u \rangle \leq 0, \quad \forall u \in \bigcap_{k=1}^{\infty} \text{Fix}(T_k).
\]

Hence, we get \( \omega_w(x_n) \subset \Omega \).

**Step 4.** We show that \( \limsup_{n \to \infty} \langle (I - f)x^*, x_n - x^* \rangle \geq 0 \).

Since \( f \) is a contraction, the solution set of the variational inequality (19) is a singleton. Let \( x^* \) is the unique solution of the variational inequality (19). Now we take a subsequence \( \{x_{n_k}\} \) of \( \{x_n\} \) satisfying

\[
\limsup_{n \to \infty} \langle (I - f)x^*, x_n - x^* \rangle = \lim_{k \to \infty} \langle (I - f)x^*, x_{n_k} - x^* \rangle.
\]
Without loss of generality, we may further assume that $x_{n_k} \to \bar{x}$, then $\bar{x} \in \Omega$. Therefore, noticing that $x^*$ is the solution of the variational inequality (19), we have
\[
\limsup_{n \to \infty} \langle (I - f)x^*, x_n - x^* \rangle = \langle (I - f)x^*, \bar{x} - x^* \rangle \geq 0.
\tag{25}
\]

**Step 5.** We show that $\limsup_{n \to \infty} \frac{\alpha_n}{\beta_n} \langle Sx^* - x^*, x_{n+1} - x^* \rangle \leq 0$.

We note that
\[
\langle Sx^* - x^*, x_{n+1} - x^* \rangle = \langle Sx^* - x^*, x_{n+1} - P_{\bigcap_{k=1}^{\infty} Fix(T_k)}x_{n+1} \rangle + \langle Sx^* - x^*, P_{\bigcap_{k=1}^{\infty} Fix(T_k)}x_{n+1} - x^* \rangle.
\]
Since $P_{\bigcap_{k=1}^{\infty} Fix(T_k)}x_{n+1} \in \bigcap_{k=1}^{\infty} Fix(T_k)$, by (2) we have
\[
\langle Sx^* - x^*, P_{\bigcap_{k=1}^{\infty} Fix(T_k)}x_{n+1} - x^* \rangle \leq 0,
\]
and by assumption (A3), we have
\[
\langle Sx^* - x^*, x_{n+1} - x^* \rangle \leq \langle Sx^* - x^*, x_{n+1} - P_{\bigcap_{k=1}^{\infty} Fix(T_k)}x_{n+1} \rangle \leq \|Sx^* - x^*\| \|x_{n+1} - P_{\bigcap_{k=1}^{\infty} Fix(T_k)}x_{n+1}\| = \|Sx^* - x^*\| \text{Dist}(x_{n+1}, \bigcap_{k=1}^{\infty} Fix(T_k)) \leq \frac{1}{\gamma} \|Sx^* - x^*\| \|x_{n+1} - L_n x_{n+1}\|.
\]
We note that
\[
\|x_{n+1} - L_n x_{n+1}\| \leq \|x_{n+1} - L_n P_C y_n\| + \|L_n P_C y_n - L_n x\| + \|L_n x - L_n x_{n+1}\| \leq \alpha_n \|Sx_n - L_n P_C y_n\| + \|y_n - x\| + \|x_{n+1} - x\| \leq \alpha_n \|Sx_n - L_n P_C y_n\| + \beta_n \|f(x_n) - x_n\| + \|x_{n+1} - x\|.
\]
Therefore, we have
\[
\frac{\alpha_n}{\beta_n} \|x_{n+1} - L_n x_{n+1}\| \leq \frac{\alpha^2_n}{\beta_n} \|Sx_n - L_n P_C y_n\| + \alpha_n \|f(x_n) - x_n\| + \frac{\alpha_n^2}{\beta_n} \|x_n - x_{n+1}\| \alpha_n \to 0.
\]
It follows that
\[
\limsup_{n \to \infty} \frac{\alpha_n}{\beta_n} \langle Sx^* - x^*, x_{n+1} - x^* \rangle \leq 0. \tag{26}
\]

**Step 6.** We prove that $x_n \to x^*$ as $n \to \infty$. From (17), we have
\[
x_{n+1} - x^* = \alpha_n \langle Sx_n - Sx^*, x_{n+1} - x^* \rangle + (1 - \alpha_n)(L_n P_C y_n - x^*) + \alpha_n \langle Sx^* - x^*, x_{n+1} - x^* \rangle.
\]
Thus, we have
\[
\begin{align*}
\|x_{n+1} - x^*\|^2 &\leq \|\alpha_n \langle Sx_n - Sx^*, x_{n+1} - x^* \rangle + (1 - \alpha_n)(L_n P_C y_n - x^*)\|^2 + 2\alpha_n \langle Sx^* - x^*, x_{n+1} - x^* \rangle \|Sx_n - Sx^*\|^2 + 2\alpha_n \langle Sx^* - x^*, x_{n+1} - x^* \rangle \|x_n - x^*\|^2 + 2\alpha_n \langle Sx^* - x^*, x_{n+1} - x^* \rangle \|y_n - x^*\|^2 + 2\alpha_n \langle Sx^* - x^*, x_{n+1} - x^* \rangle \|x_{n+1} - x^*\|^2 + 2\alpha_n \langle Sx^* - x^*, x_{n+1} - x^* \rangle \|y_{n+1} - x^*\|^2.
\end{align*}
\tag{27}
\]
At the same time, we observe that
\[
\begin{align*}
\|y_n - x^*\|^2 &\leq \|(1 - \beta_n)(x_n - x^*) + \beta_n \langle f(x_n) - f(x^*) \rangle + \beta_n \|f(x_n) - f(x^*)\|^2 \leq \|(1 - \beta_n)(x_n - x^*) + \beta_n \langle f(x_n) - f(x^*) \rangle\|^2 + 2\beta_n \langle f(x_n) - x^*, y_n - x^* \rangle \|x_n - x^*\|^2 + 2\beta_n \langle f(x_n) - x^*, y_n - x^* \rangle \|x_{n+1} - x^*\|^2 + 2\beta_n \langle f(x_n) - x^*, y_{n+1} - x^* \rangle \|x_{n+1} - x^*\|^2 + 2\beta_n \langle f(x_n) - x^*, y_{n+1} - x^* \rangle \|y_{n+1} - x^*\|^2.
\end{align*}
\tag{28}
\]
Substituting (28) into (27), we have
\[
\begin{align*}
\|x_{n+1} - x^*\|^2 &\leq \alpha_n \|x_n - x^*\|^2 + (1 - \alpha_n) \|1 - (1 - \alpha^2)\beta_n\| \|x_n - x^*\|^2 + 2\beta_n \langle 1 - (1 - \alpha^2)\beta_n \langle f(x^*) - x^*, y_n - x^* \rangle + 2\alpha_n \langle Sx_n - x^*, x_{n+1} - x^* \rangle
\end{align*}
\tag{29}
\]
Therefore, we can apply Lemma 3 to (29) to conclude that $x_n \to x^*$ as $n \to \infty$. This completes the proof. □
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References:


