On Two Variants of Rainbow Connection

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Abstract: A vertex-colored graph G is rainbow vertex-connected if any two vertices are connected by a path whose internal vertices have distinct colors. The rainbow vertex-connection number of a connected graph G, denoted by $rvc(G)$, is the smallest number of colors that are needed in order to make G rainbow vertex-connected. A path P connecting two vertices u and v in a total-colored graph G is called a rainbow total-path between u and v if all elements in $V(P) \cup E(P)$, except for u and v, are assigned distinct colors. The total-colored graph G is rainbow total-connected if it has a rainbow total-path between every two vertices. The rainbow total-connection number, denoted by $rtc(G)$, of a graph G is the minimum colors such that G is rainbow total-connected. In this paper, we will obtain some results for these two variants of rainbow connection. For rainbow vertex-connection, we will first investigate the rainbow vertex-connection number of a graph according to some structural conditions of its complementary graph $\overline{G}$. Next, we will investigate graphs with large rainbow vertex-connection numbers. We then derive a sharp upper bound for rainbow vertex-connection numbers of line graphs. For rainbow total-connection, we will determine the precise values for rainbow total-connection numbers of some special graph classes, including complete graphs, trees, cycles and wheels.

Key–Words: vertex-coloring, total-coloring, rainbow vertex-connection number, rainbow total-connection number, rainbow connection number, complementary graph

1 Introduction

The graphs considered in this paper are finite, undirected and simple graphs. We follow the notations of Bondy and Murty [1], unless otherwise stated. For a graph G, let $V(G)$, $E(G)$, $n(G)$, $m(G)$ and $\overline{G}$, respectively, be the set of vertices, the set of edges, the order, the size and the complement of G.

Let G be a nontrivial connected graph on which an edge-coloring $c : E(G) \to \{1, 2, \cdots, n\}$, $n \in \mathbb{N}$, is defined, where adjacent edges may be colored the same. A path is $rainbow$ if no two edges of it are colored the same. An edge-colored graph G is $rainbow\ connected$ if any two vertices are connected by a rainbow path. Clearly, if a graph is rainbow connected, it must be connected, whereas any connected graph has a trivial edge-coloring that makes it rainbow connected; just color each edge with a distinct color. Thus, in [5] Chartrand et al. defined the $rainbow\ connection\ number$ of a connected graph G, denoted by $rc(G)$, as the smallest number of colors that are needed in order to make G rainbow connected. Clearly, $rc(G) \geq diam(G)$ where $diam(G)$ denotes the diameter of G.

The rainbow connection number is not only a natural combinatorial measure, but also has applications to the secure transfer of classified information between agencies. In addition, the rainbow connection number can also be motivated by its interesting interpretation in the area of networking(see [4]): Suppose that G represents a network (e.g., a cellular network). We wish to route messages between any two vertices in a pipeline, and require that each link on the route between the vertices (namely, each edge on the path) is assigned a distinct channel (e.g. a distinct frequency). Clearly, we want to minimize the number of distinct channels that we use in our network. This number is precisely $rc(G)$. There are more and more researchers investigating this new topic. The readers can see [16] for a survey and [17] for a new monograph on it.

The concept of rainbow connection has several interesting variants, one of them is rainbow vertex-connection which was first proposed by Krivelevich and Yuster in [11]. A vertex-colored graph G is *rainbow vertex-connected* if two vertices are connected by a path whose internal vertices have distinct colors. The *rainbow vertex-connection number* of a connected graph G, denoted by $rvc(G)$, is the smallest number of colors that are needed in order to make

G rainbow vertex-connected. Notice that $rvc(G) \geq diam(G) - 1$ with equality if the diameter is 1 or 2. Note that $rvc(G)$ may be much smaller than $rc(G)$ for some graph G. For example, we consider the star graph $K_{1,n}$, we have $rvc(K_{1,n}) = 1$ while $rc(K_{1,n}) = n$. $rvc(G)$ may also be much larger than $rc(G)$ for some graph G. For example(see [11]), take n vertex-disjoint triangles and, by designating a vertex from each of them, add a complete graph on the designated vertices. This graph has n cut vertices and hence $rvc(G) \geq n$. In fact, $rvc(G) = n$ by coloring only the cut vertices with distinct colors. On the other hand, it is not difficult to see that $rc(G) \leq 4$. Just color the edges of K_n with color 1, and color the edges of each triangle with the colors $2, 3, 4$.

Recently, Uchizawa, Aoki, Ito, Suzuki, and Zhou [19] introduced a new variant of rainbow connection, named rainbow total-connection. For a graph $G = (V, E)$, let $c: V \cup E \longrightarrow C$ be a total-coloring of G which is not necessarily proper. A path P in G connecting two vertices u and v in V is called a *rainbow total-path* between u and v if all elements in $V(P) \cup E(P)$, except for u and v, are assigned distinct colors by c. Similarly as in the vertex-coloring version, we do not care about the colors assigned to the end-vertices u and v of P. The total-colored graph G is *rainbow total-connected* if G has a rainbow total-path between every two vertices in V. Now we define the *rainbow total-connection number*, denoted by $rtc(G)$, of a graph G as the minimum colors such that G is rainbow total-connected.

For a set S, let $|S|$ denote the cardinality of S. A k-subset of a set S is a subset of S whose cardinality is k where $k \leq |S|$. An *inner* vertex of a graph G is a vertex of degree at least 2 in G and we use V_2 to denote the set of inner vertices of G and let $n_2 = |V_2|$. We use V_c to denote the set of cut vertices of the graph G and let $n_c = |V_c|$. Clearly, $V_c \subseteq V_2$ and $n_c \leq n_2$. For a subset X of $V(G)$, we use $G[X]$ to denote the induced subgraph of X in G. For $U \subseteq V(G)$, we denote $G \backslash U$ the subgraph by deleting the vertices of U and its adjacent edges from G. If $E(W)$ is the edge subset of G, then $G \backslash E(W)$ denote the subgraph by deleting the edges of $E(W)$. For any two vertex sets U and V, let $E[U, V]$ denote the set of edges between U and V in G. The *distance* between two vertices u and v in a connected graph G, denoted by $dist_G(u, v)$, is the length of a shortest path between them in G. The eccentricity of a vertex x, denoted by $ecc_G(x)$, in a connected graph G is defined as $ecc_G(x) = \max_{v \in G}\{dist_G(x, v)\}$. For a graph G, we define the *degree-sum* as $\sigma_k(G) = min\{d(u_1) + d(u_2) + \cdots + d(u_k)|\ u_1, u_2, \ldots, u_k \in V(G), u_iu_j \notin E(G), i \neq j, i, j \in \{1, \cdots, k\}\}$.

We first list some recent results on these two variants of rainbow connection, then we will introduce our results. The complexity of determining rainbow vertex-connection of a given graph was first settled by Chen, Li and Shi [7]. For the introduction of complexity theory, see [10]. They derived the following two results.

Theorem 1 *[7] Given a graph G, deciding if $rvc(G) = 2$ is NP-complete. In particular, computing $rvc(G)$ is NP-hard.*

Theorem 2 *[7] The following problem is NP-complete: given a vertex-colored graph G, check whether the given coloring makes G rainbow vertex-connected.*

By theorem 1, we know it is hard to compute the value of rainbow vertex-connection number for a connected graph G. Thus, people aim to give nice upper bounds for this parameter, especially sharp upper bounds, according to some parameters of the graph G.

Krivelevich and Yuster [11] first gave an upper bound for $rvc(G)$ according to the minimum degree δ of G by the technique of dominating set.

Theorem 3 *[11] A connected graph G with n vertices has $rvc(G) < \frac{11n}{\delta(G)}$.* □

Motivated by the method of Theorem 3, Li and Shi derived a result, which greatly improved Theorem 3.

Theorem 4 *[13] A connected graph G of order n with minimum degree δ has $rvc(G) \leq 3n/(\delta+1)+5$ for $\delta \geq \sqrt{n-1} - 1, n \geq 290$, while $rvc(G) \leq 4n/(\delta + 1) + 5$ for $16 \leq \delta \leq \sqrt{n-1} - 2$, $rvc(G) \leq 4n/(\delta + 1) + C(\delta)$ for $6 \leq \delta \leq 16$ where $C(\delta) = e^{\frac{3\log(\delta^3+2\delta^2+3)-3(\log 3-1)}{\delta-3}} - 2$, $rvc(G) \leq n/2 - 2$ for $\delta = 5$, $rvc(G) \leq 3n/5 - 8/5$ for $\delta = 4$,$rvc(G) \leq 3n/4 - 2$ for $\delta = 3$. Moreover, an example shows that when $\delta \geq \sqrt{n-1} - 1$, and $\delta = 3, 4, 5$ the bounds are seen to be tight up to additive factors.*

It is also tried to look for some other better parameters to replace δ. Such a natural parameter is σ_k. Observe that σ_k is monotonically increasing in k. Motivated by the method of Theorem 3, Dong and Li [8] also obtained a result analogous to Theorem 3 for the rainbow vertex-connection version according to the degree-sum condition σ_2, which is stated as the following theorem.

Theorem 5 *[8] Let G be a connected graph of order n. Then $rvc(G) \leq \frac{8n}{\sigma_2+2} + 8$ for $2 \leq \sigma_2 \leq 6$ and $\sigma_2 \geq 28$, $rvc(G) \leq \frac{10n}{\sigma_2+2} + 8$ for $7 \leq \sigma_2 \leq 8$ and*

$16 \le \sigma_2 \le 27$, *and* $rvc(G) \le \frac{10n}{\sigma_2+2} + A(\sigma_2)$ *for* $9 \le \sigma_2 \le 15$, *where* $A(\sigma_2) = 63, 41, 27, 20, 16, 13, 11$, *respectively.*

Dong and Li in [9] also showed a theorem for the rainbow vertex-connection number according to the degree-sum condition σ_k, which is stated as follows.

Theorem 6 *[9] Let* G *be a connected graph of order* n *with* k *independent vertices. Then* $rvc(G) \le \frac{(4k+2k^2)n}{\sigma_k+k} + 5k$ *if* $\sigma_k \le 7k$ *and* $\sigma_k \ge 8k$*; whereas* $rvc(G) \le \frac{(\frac{38k}{9}+2k^2)n}{\sigma_k+k} + 5k$ *if* $7k < \sigma_k < 8k$.

Nordhaus-Gaddum-type results are related to complementary graphs. Chen, Li and Lian [6] investigated Nordhaus-Gaddum-type results. A Nordhaus-Gaddum-type result is a (sharp) lower or upper bound on the sum or product of the values of a parameter for a graph and its complement. The name "Nordhaus-Gaddum-type" is so given because it is Nordhaus and Gaddum [18] who first established the following type of inequalities for chromatic numbers of graphs in 1956. Chen, Li and Lian derived the following results.

Theorem 7 *[6] When* G *and* $\overline{G}$ *are both connected, then* $2 \le rvc(G) + rvc(\overline{G}) \le n - 1$. *Both the upper and the lower bounds are best possible for all* $n \ge 5$.

For rainbow total-connection, Uchizawa, Aoki, Ito, Suzuki, and Zhou [19] obtained some hardness and algorithmic results. For a given total-coloring c of a graph G, the $Rainbow\ Total\text{-}Connectivity$ problem is to determine whether G is rainbow total-connected. A graph G is a $cactus$ if every edge is part of at most one cycle in G. Uchizawa, Aoki, Ito, Suzuki, and Zhou gave the following theorem from the viewpoints of diameter and graph classes, respectively.

Theorem 8 *[19]*
(i) $Rainbow\ Total\text{-}Connectivity$ *is in P for graphs of diameter 1, while is strongly NP-complete for graphs of diameter 2.*
(ii) $Rainbow\ Total\text{-}Connectivity$ *is strongly NP-complete even for outerplanar graphs.*
(iii) $Rainbow\ Total\text{-}Connectivity$ *is solvable in polynomial time for cacti.*

They also considered the FPT algorithms for rainbow total-connection.

Theorem 9 *[19] For a total-coloring of a graph* G *using* k *colors, one can determine whether the total-colored graph* G *is rainbow total-connected in time* $O(k2^kmn)$ *using* $O(k2^kn)$ *space, where* n *and* m *are the numbers of vertices and edges in* G*, respectively.*

In this paper, we will obtain some results for these two variants of rainbow connection. For rainbow vertex-connection, in Section 3, we will investigate the rainbow vertex-connection number of a graph according to some structural conditions of its complement graph $\overline{G}$ (Theorems 15 and 17). In Section 4, we will investigate graphs with large rainbow vertex-connection numbers, that is, graphs whose rainbow vertex-connection numbers are close to n_2 (see Proposition 18 and Theorem 19). In Section 5, we will consider an important graph class, line graph (see Theorem 20). For rainbow total-connection, in Section 6, we will determine the precise values for rainbow total-connection numbers of some special graph classes, including complete graphs, trees, cycles and wheels (see Proposition 21, Theorems 22 and 23).

2 Preliminaries

We need several basic results to obtain our conclusions. The following two propositions give the precise values for rainbow connection number and rainbow vertex-connection number of a cycle.

Proposition 10 *[5] For each integer* $n \ge 4$, $rc(C_n) = \lceil \frac{n}{2} \rceil$, *where* C_n *is a cycle of length* n.

Proposition 11 *[12] Let* G *be a 2-connected graph of order* $n (n \ge 3)$. *Then*

$$rvc(G) \le \begin{cases} 0 & \text{if } n = 3; \\ 1 & \text{if } n = 9; \\ \lceil \frac{n}{2} \rceil - 1 & \text{if } n = 6, 7, 8, 10, 11, 12, 13, 15; \\ \lceil \frac{n}{2} \rceil & \text{if } n \ge 16 \text{ or } n = 14, \end{cases}$$

and the upper bound can be achieved by the cycle C_n.

From the proposition, we know that

$$rvc(C_n) = \begin{cases} 0 & \text{if } n = 3; \\ 1 & \text{if } n = 4, 5; \\ 3 & \text{if } n = 9; \\ \lceil \frac{n}{2} \rceil - 1 & \text{if } n = 6, 7, 8, 10, 11, 12, 13, 15; \\ \lceil \frac{n}{2} \rceil & \text{if } n \ge 16 \text{ or } n = 14, \end{cases}$$

it will be useful in the sequel.

3 Upper bounds according to complementary graphs

By the definition of rainbow vertex-connection number, the following proposition is clear.

Proposition 12 *For a connected graph* G, *rvc(G)=1 if and only if* $diam(G) = 2$.

Let G be a complete k-partite graph which is not a complete graph where $k \geq 2$, we have $rvc(G) = 1$ from the above proposition. We know if $\overline{G}$ is disconnected, then G is a complete graph or contains a complete k-partite graph as a spanning subgraph, where $k \geq 2$. From the above discussion, we have:

Proposition 13 *For a graph G, if $\overline{G}$ is disconnected, then $rvc(G) = 0$ or 1.*

Furthermore, we know that for a graph G, if $\overline{G}$ is disconnected, then $rvc(G) = 0$ if and only if G is a complete graph, if and only if each vertex of $\overline{G}$ is an isolated vertex.

In the following lemma, we will investigate the rainbow vertex-connection number of a connected complement graph.

Lemma 14 *If G is a connected graph with $diam(G) \geq 3$, then*

$$rvc(\overline{G}) = \begin{cases} 1, & if \ \ diam(G) \geq 4; \\ 1 \ or \ 2, & if \ \ diam(G) = 3. \end{cases}$$

Moreover, there are graphs G such that $diam(G) = 3$ and $rvc(\overline{G}) = 2$.

Proof: First of all, we see that $\overline{G}$ must be connected, since otherwise, $diam(G) \leq 2$, contradicting the condition $diam(G) \geq 3$. Thus, $rvc(\overline{G}) \geq 1$.

We choose a vertex x with $ecc_G(x) = diam(G) = d \geq 3$. Let $N^i_G(x) = \{v : dist_G(x, v) = i\}$ where $0 \leq i \leq d$. Clearly, $N^0_G(x) = \{x\}, N^1_G(x) = N_G(x)$ as usual. We know $\bigcup_{0 \leq i \leq d} N^i_G(x)$ is a vertex partition of $V(G)$ with $|N^i_G(x)| = n_i$. Let $A = \bigcup_{i \ is \ even} N^i_G(x)$, $B = \bigcup_{i \ is \ odd} N^i_G(x)$. For example, see Figure 1, a graph G with $diam(G) = 4$.

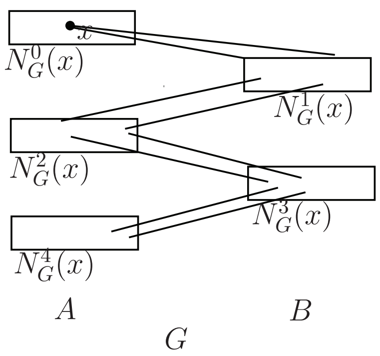

Figure 1: A graph G with diameter 4.

We know that if $d = 2k (k \geq 2)$, then $A = \bigcup_{0 \leq i \leq d \ is \ even} N^i_G(x)$ and $B = \bigcup_{1 \leq i \leq d-1 \ is \ odd} N^i_G(x)$; if $d = 2k + 1 (k \geq 1)$, then $A = \bigcup_{0 \leq i \leq d-1 \ is \ even} N^i_G(x)$ and $B = \bigcup_{1 \leq i \leq d \ is \ odd} N^i_G(x)$. By the definition of a complement graph, we know that $\overline{G}[A] (\overline{G}[B])$ contains a spanning complete k_1-partite subgraph (complete k_2-partite subgraph) where $k_1 = \lceil \frac{d+1}{2} \rceil (k_2 = \lceil \frac{d}{2} \rceil)$. For example, see Figure 1, $\overline{G}[A]$ contains a spanning complete tripartite subgraph K_{n_0,n_2,n_4}, $\overline{G}[B]$ contains a spanning complete bipartite subgraph K_{n_1,n_3}.

Case 1. $d \geq 4$. Now we have $k_1 \geq 3, k_2 \geq 2$. We will show that $diam(\overline{G}) = 2$ in this case. As G is connected, the complement graph $\overline{G}$ is not a complete graph, and $diam(\overline{G}) \geq 2$. Thus, we need to show that for any two vertices $u, v \in V(\overline{G})$, we have $dist_{\overline{G}}(u, v) \leq 2$.

We will consider the following two subcases:

Subcase 1.1. $u, v \in A$ or $u, v \in B$.

If $u, v \in A$, then u and v are contained in the spanning complete k_1-partite subgraph of $\overline{G}[A]$. Thus $dist_{\overline{G}}(u, v) \leq 2$. The result is also true for the subcase that $u, v \in B$.

Subcase 1.2. $u \in A$ and $v \in B$.

If $u = x$, $v \in B$, then u is adjacent to all vertices in $\overline{G}[B] \setminus N^1_G(x)$, so $dist_{\overline{G}}(u, v) = 1$ for $v \in \overline{G}[B] \setminus N^1_G(x)$. For $v \in N^1_G(x)$, let $P := u, x_3, v$, where $x_3 \in N^3_G(x)$, clearly, $dist_{\overline{G}}(u, v) \leq 2$.

If $u \neq x$, without loss of generality, we assume that $u \in N^2_G(x)$ and $v \in N^1_G(x)$. Let $Q := u, x_4, v$, where $x_4 \in N^4_G(x)$, clearly, $dist_{\overline{G}}(u, v) \leq 2$.

From the above discussion, we conclude that $diam(\overline{G}) = 2$, by Proposition 12, we have $rvc(\overline{G}) = 1$.

Case 2. $d = 3$, that is, $A = N^0_G(x) \cup N^2_G(x)$, $B = N^1_G(x) \cup N^3_G(x)$. Now $\overline{G}[A]$ contains a spanning complete bipartite subgraph K_{n_0,n_2}. We give $\overline{G}$ a vertex-coloring as follows: assign vertex x the color 1 and all vertices of $N^3_G(x)$ the color 2.

We choose any pair of vertices $(u, v) \in (N^i_G(x), N^j_G(x))$ where $i, j \in \{0, 1, 2, 3\}$, without loss of generality, we assume that $i = 2, j = 1$. It is easy to see that there is a $u - v$ path $P = u, x, x_3, v$ whose inner vertices have distinct colors in $\overline{G}$, where $x_3 \in N^3_G(x)$. Thus, $dist_{\overline{G}}(u, v) \leq 3$ and the above coloring is a rainbow vertex-coloring of $\overline{G}$, so $diam(\overline{G}) \leq 3$ and $1 \leq rvc(\overline{G}) \leq 2$ in this case.

Moreover, for the case that $diam(G) = 3$, if there is a vertex x_0 with $ecc_G(x_0) = 3$ such that there is a vertex $y_0 \in N^2_G(x_0)$ which is adjacent to all vertices of $N^1_G(x_0) \cup N^3_G(x_0) \cup (N^2_G(x_0) \setminus \{y_0\})$, we choose $x = x_0$ in the above discussion, for example, see Figure 2. We know that $\overline{G}$ is connected, and in $\overline{G}$, y_0 is not adjacent to any vertex of $N^1_G(x_0) \cup N^3_G(x_0) \cup (N^2_G(x_0) \setminus \{y_0\})$. Clearly, we have $dist_{\overline{G}}(y_0, y_1) = 3$.

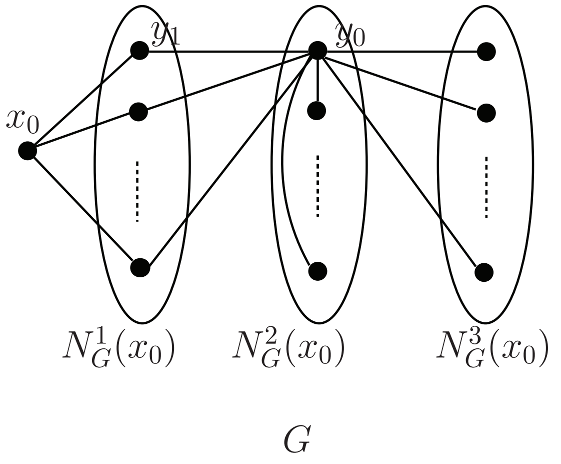

Figure 2: A graph G with diameter 3 whose complement has diameter 3.

As shown above $diam(\overline{G}) \leq 3$, we have $diam(\overline{G}) = 3$. Thus, $rvc(\overline{G}) \geq diam(\overline{G})-1 = 2$, as $rvc(\overline{G}) \leq 2$, we have $rvc(\overline{G}) = 2$. □

As the complement graph of $\overline{G}$ is G, from Proposition 13 and Lemma 14, we drive the following theorem.

Theorem 15 *For a graph G, we have:*

(i) *if $\overline{G}$ is disconnected, then $rvc(G) = 0$ or 1;*

(ii) *if $diam(\overline{G}) \geq 4$, then $rvc(G) = 1$;*

(iii) *if $diam(\overline{G}) = 3$, then $rvc(G) = 1$ or 2; moreover, there are graphs G such that $diam(\overline{G}) = 3$ and $rvc(G) = 2$.*

The above theorem investigate the rainbow connection number of a graph G under the condition that $diam(\overline{G}) \neq 2$.

For the case that $diam(\overline{G}) = 2$, $rvc(G)$ can be very large since $diam(G)$ may be very large. For example, Let $\overline{G} = K_n \setminus E(C_n)$, where C_n is a cycle of length n in K_n. Then $G = C_n$ and $rvc(G) \geq diam(G) - 1 = \lceil \frac{n}{2} \rceil - 1$ for a sufficiently large n.

Thus, we will add a condition, that is, let $\overline{G}$ be triangle-free. We need to show the following lemma at first.

Lemma 16 *For a triangle-free graph G with diameter 2, if $\overline{G}$ is connected, then $rvc(\overline{G}) \leq 3$.*

Proof: Since $d = 2$, we choose a vertex x with $ecc_G(x) = 2$, and let $A = N_G^0(x) \cup N_G^2(x)$, $B = N_G^1(x)$. Then $\overline{G}[A]$ contains a spanning complete bipartite subgraph K_{n_0,n_2}.

Since G is triangle-free, $N_G^1(x)$ is a stable set in G and a clique in $\overline{G}$. There is at least one edge, denoted by $e = uv$, between $N_G^1(x)$ and $N_G^2(x)$ in $\overline{G}$, since $\overline{G}$ is connected, where $u \in N_G^1(x)$ and $v \in N_G^2(x)$.

We now give $\overline{G}$ a vertex-coloring as follows: color vertex x with 1, color u with 2 and color v with 3. For any $x_1 \in N_G^1(x)$, $x_2 \in N_G^2(x)$, path x_1, u, v, x_2 is a $x_1 - x_2$ path whose inner vertices have distinct colors. Thus, $rvc(\overline{G}) \leq 3$. □

From Theorem 15 and Lemma 16, we derive the following theorem.

Theorem 17 *For a connected graph G, if $\overline{G}$ is triangle-free, then $rvc(G) \leq 3$.*

4 Graphs with large rainbow vertex-connection numbers

Recall that a *block* of a connected subgraph without a cut vertex. Thus, every block of a connected graph G is either a maximal 2-connected subgraphs, or a bridge together with its ends. Conversely, every such subgraph is a block. Here a 2-*connected block* of G is a block which is a maximal 2-connected subgraph of G.

We know that $0 \leq rvc(G) \leq n_2$. It is interesting to study graphs with extremal rainbow vertex-connection numbers, that is, graphs with small (large) rainbow vertex-connection numbers. As noted before, $rvc(G) = 0$ if and only if $diam(G) = 1$, $rvc(G) = 1$ if and only if $diam(G) = 2$. Thus, we now investigate graphs with large rainbow vertex-connection numbers, especially n_2 and derive the following result.

Proposition 18 *For a connected graph G, $rvc(G) = n_2$ if and only if $n_2 = n_c$.*

Proof: It is easy to show that, in a rainbow vertex-coloring, any two cut vertices must obtain distinct colors. Thus, $rvc(G) \geq n_c$. If $n_2 = n_c$, then $rvc(G) = n_2$.

Now we prove the other direction. We know each cut vertex is an inner vertex, so $n_c \leq n_2$. Suppose that $n_c < n_2$, that is, there exists some inner vertex, say u, which is not a cut vertex. Clearly, u must belong to some 2-connected block, say B_u. We now give G a vertex-coloring as follows: We first assign a distinct color to each inner vertex except u, then assign any color which has been used to the remaining vertices, that is, u and all leaves.

We now show that G is rainbow vertex-connected with the above coloring. For any two vertex v, w. If $v = u$ or $w = u$, then each $v - w$ path is a path whose inner vertices receive distinct colors. If $v, w \neq u$, then we choose any $v - w$ path in the subgraph $G \setminus$

$\{u\}$ (this path must exist since $G \setminus \{u\}$ is connected), clearly, it must be a path whose inner vertices receive distinct colors. Thus, G is rainbow vertex-connected and $rvc(G) \leq n_2 - 1$, this produces a contradiction. Furthermore, we have $n_c = n_2$. □

We now consider the graphs with $rvc(G) = n_2 - 1$, but at first we need to introduce the following two new graph classes:

$\mathcal{G}_1 = \{G : |V_2 \setminus V_c| = 1$ for graph G.$\}$

$\mathcal{G}_2 = \{G : V_2 \setminus V_c \subseteq B$ and each 2-subset of $V_2 \setminus V_c$ is a vertex cut of G, where B is a 2-connected block of G.$\}$.

In the following theorem, we will consider graphs G with $rvc(G) = n_2 - 1$.

Theorem 19 *For a connected graph G, if $rvc(G) = n_2 - 1$, then $G \in \mathcal{G}_1 \cup \mathcal{G}_2$.*

Proof: For any connected graph $G \in \mathcal{G}_1$, we know now there exists one inner vertex, say u, which is not a cut vertex of G. As now $n_c = n_2 - 1$, we have $rvc(G) \geq n_2 - 1$. We give G a vertex-coloring as follows: We first assign each vertex of $V_2 \setminus V_c$ a fresh color, then assign any old color to the remaining vertices. It is easy to show that, with the above vertex-coloring, G is rainbow vertex-connected. Thus, $rvc(G) \leq n_2 - 1$. By the above discussion, we know $rvc(G) = n_2 - 1$ for the case that $G \in \mathcal{G}_1$.

Let G be a connected graph with $rvc(G) = n_2 - 1$ such that $G \notin \mathcal{G}_1$. Then in this case, we have $|V_2 \setminus V_c| \geq 2$.

If there are two vertices of $V_2 \setminus V_c$, say u_1 and v_1, which belong to distinct blocks, say B_1 and B_2, respectively. Clearly, both B_1 and B_2 are 2-connected blocks and so $G' = G \setminus \{u_1, v_1\}$ is a connected graph. Now we give the graph G a vertex-coloring with $n_2 - 2$ colors as follows: We first assign a fresh color to each vertex of V_2 except u_1 and v_1, then assign an old color to the remaining vertices. We now show that, with the above coloring, G is rainbow vertex-connected. It suffices to show that for any two vertices u and v, there is a $u - v$ path whose internal vertices have distinct colors. For the case that $u \neq u_1, v_1$ and $v \neq u_1, v_1$, as G' is connected, then any $u - v$ path in G' is a desired path. The remaining cases are similar and easier. Thus, $rvc(G) \leq n_2 - 2$, this produces a contradiction.

Now we know that for a connected graph with $rvc(G) = n_2 - 1$ such that $G \notin \mathcal{G}_1$, we have $V_2 \setminus V_c \subseteq B$, where B is a 2-connected block of G. If there exists two vertices of $V_2 \setminus V_c$, say u_2 and v_2, such that $\{u_2, v_2\}$ is not a vertex cut of G, then the graph $G'' = G \setminus \{u_2, v_2\}$ is a connected graph. Now we give the graph G a vertex-coloring with $n_2 - 2$ colors as follows: We first assign a fresh color to each vertex of V_2 except u_2 and v_2, then assign an old color to the remaining vertices. We now show that, with the above coloring, G is rainbow vertex-connected. It suffices to show that for any two vertices u, v, there is a $u - v$ path whose internal vertices have distinct colors. For the case that $u \neq u_2, v_2$ and $v \neq u_2, v_2$, as G'' is connected, then any $u - v$ path in G' is a desired path. The remaining cases are similar and easier. And $rvc(G) \leq n_2 - 2$, this produces a contradiction. Thus, any 2-subset of $V_2 \setminus V_c$ is a vertex cut of G.

From the above discussion, we know that for a a connected graph with $rvc(G) = n_2 - 1$, we have $G \in \mathcal{G}_1 \cup \mathcal{G}_2$. □

5 A sharp upper bound for rainbow vertex-connection numbers of line graphs

In [14, 15], the authors investigated the rainbow connection number of the line graph $L(G)$ of a graph G. They derived several upper bounds for $rc(L(G))$ in terms of some parameters of the original graph G. In this section, we continue to investigate the rainbow vertex-connection numbers of line graphs and give a sharp upper bound for $rvc(L(G))$ in terms of $rc(G)$.

Theorem 20 *For a connected graph G, we have $rvc(L(G)) \leq rc(G)$. Moreover, the bound is sharp.*

Proof: Let $rc(G) = k$, we first assign the graph G a rainbow k-edge-coloring $c : E(G) \to \{1, 2, \cdots, k\}$. Recall that $V(L(G)) = E(G)$, that is, there is an one-to-one corresponding between vertex set of $L(G)$ and edge set of G. We assign the line graph $L(G)$ a k-vertex-coloring c' such that $c'(e) = c(e)$ where $e \in E(G)$, it suffices to show $L(G)$ is rainbow vertex-connected under this vertex-coloring.

We choose any two vertices $e_1, e_2 \in V(L(G))$, suppose $e_1 = u_1u_2, e_2 = v_1v_2$, where $u_i, v_i \in V(G)$ for $i \in \{1, 2\}$. We know there is a rainbow (edge) path connecting u_i and v_j in graph G, where $i, j \in \{1, 2\}$. We choose the shortest one, say P, among these rainbow paths, and without loss of generality, let $P = a_1, a_2, \cdots, a_\ell$ be a rainbow $u_1 - v_1$ path in graph G, where $a_1 = u_1, a_\ell = v_1$. The path P clearly does not contain the edges e_1 and e_2. Recall that $c'(a_ia_{i+1}) = c(a_ia_{i+1})$, then the path $P' = e_1, a_1a_2, \cdots, a_{\ell-1}a_\ell, e_2$ is a $e_1 - e_2$ path whose inner vertices have distinct colors in $L(G)$. By definition, we know $L(G)$ is rainbow vertex-connected under this coloring.

For the sharpness of the bound, we can consider the cycle $C_n (n \geq 16)$. By Propositions 10 and 11, we know that $rvc(L(C_n)) = rvc(C_n) = \lceil \frac{n}{2} \rceil = rc(C_n)$. The conclusion now holds. □

6 Some rainbow total-connection numbers of graphs

In this section, we will do some basic research for rainbow total-connection and will derive the precise values of rainbow total-connection numbers for some special graph classes.

Proposition 21 *For a connected graph G, we have*
(i) $rtc(G) = 1$ *if and only if G is a complete graph.*
(ii) $rtc(G) \neq 2$ *for any noncomplete graph G.*
(iii) $rtc(G) = m + n_2$ *if and only if G is a tree.*

Proof: We now verify (i). If G is a complete graph, then the coloring that assign a color 1 to every edge and vertex of G is a rainbow total-coloring, and so $rtc(G) = 1$. If $rtc(G) = 1$, then $diam(G) = 1$, since otherwise there exist two vertices u and v with $dist(u, v) \geq 2$. So the number of inner vertices and edges of any $u - v$ path must be at least three, so $rtc(G) \geq 3$. This produces a contradiction. Thus, $diam(G) = 1$ and G is a complete graph.

For (ii), if $rtc(G) = 2$ for a connected graph G, then G is not a complete graph, by the above discussion, we have $rtc(G) \geq 3$, this produces a contradiction.

For (iii), let $rtc(G) = m + n_2$, we will show that G is a tree. Suppose first G is not a tree, then G contains a cycle $C : v_1, v_2, \cdots, v_k, v_1$, where $k \geq 3$. We give graph G a $(m+n_2-1)$-total-coloring as follows: We first assign each edge except v_1v_2 and each inner vertex a distinct color, next we let $c(v_1v_2) = c(v_2v_3)$. It is easy to show that this coloring is a rainbow total-coloring because $G - v_1v_2$ is a connected graph. Thus, we have $rtc(G) \leq m + n_2 - 1$, this produces a contradiction.

Next, let G be a tree. Let $A(G)$ be the set of all inner vertices and edges of G, we have $|A(G)| = m + n_2$. We assign each element of $A(G)$ a distinct color and assign the leaves an old color. Clearly, the above coloring is a rainbow total-coloring. Thus, $rtc(G) \leq m+n_2$. Assume that $rtc(G) \leq m+n_2-1$. Let c be a rainbow $(m + n_2 - 1)$-total-coloring of G. Thus, there are two elements of $A(G)$ which receive the same color, say $c(a_1) = c(a_2)$ where $a_1, a_2 \in A(G)$. There are three cases to consider: both a_1 and a_2 are inner vertices; both a_1 and a_2 are edges; one of a_1, a_2 is an inner vertex. We only consider the last case, since the remaining two cases are similar. Without loss of generality, let a_1 be an inner vertex and $a_2 = uv$ be an edge of G. Clearly, there is a path $P : w, a_1, \cdots, u, v$ which contains both a_1 and a_2 in G. It is the unique $w - v$ path in G, so there is no rainbow total $w - v$ path, this produces a contradiction. □

We will determine the precise value for rainbow total-connection numbers of C_n with $n \geq 10$. For a path P, we use $\bar{l}(P)$ to denote the number of edges and inner vertices of P. Clearly, $\bar{l}(P) = 2l(P) - 1$ and we know that $rtc(G) \geq \bar{l}(P)$ for any path P.

Theorem 22 *For* $n \geq 10$*, the rainbow total-connection number of the cycle* C_n *is*

$$rtc(C_n) = \begin{cases} n & \text{if } n \geq 11,\ n \neq 12; \\ n-1 & \text{if } n = 10, 12. \end{cases}$$

Proof: Assume that $C_n = v_1, v_2, \cdots, v_n, v_{n+1} = v_1$. Let $E(C_n) = \{e_i | e_i = v_iv_{i+1}, 1 \leq i \leq n\}$ and $A = V(C_n) \cup E(C_n) = \{a_i | 1 \leq i \leq 2n\}$ with $a_{2j-1} = v_j$ and $a_{2j} = v_jv_{j+1}$ where $1 \leq j \leq n$.

We define a total-coloring c of C_n by $c(a_i) = c(a_{i+n})$ for $1 \leq i \leq n$. It is easy to show that this coloring is a rainbow n-total-coloring, then $rtc(C_n) \leq n$.

Next, we will show that $rtc(C_n) \geq n$ for the case that $n \geq 11$ and $n \neq 12$. Suppose that $rtc(C_n) \leq n - 1$. We give C_n a rainbow $(n - 1)$-total-coloring c. As $|A_n| = 2n$, there are at least three elements of A which have the same color. We will consider the following four cases.

Case 1. All these three elements are edges, say $e_1 = v_1v_2$, $e_i = v_iv_{i+1}$ and $e_j = v_jv_{j+1}(2 \leq i \leq j - 1)$. Clearly, one pair of vertices among $\{v_1, v_i, v_j\}$, say v_1 and v_i, satisfies that $d_{C_n}(v_1, v_i) \leq \lfloor \frac{n}{3} \rfloor$.

Subcase 1.1. The path $P : v_1, v_2, \cdots, v_{i-1}, v_i$ is a $v_1 - v_i$ path of length $d_{C_n}(v_1, v_i)$. Then $P' : v_1, v_2, \cdots, v_i, v_{i+1}$ is a $v_1 - v_{i+1}$ path of length $d_{C_n}(v_1, v_{i+1}) \leq \lfloor \frac{n}{3} \rfloor + 1$. As the two edges v_1v_2, v_iv_{i+1} have the same color, the rainbow total $v_1 - v_{i+1}$ path must be $P'' : v_1, v_n, \cdots, v_{i+2}, v_{i+1}$. Now $l(P'') \geq n - (\lfloor \frac{n}{3} \rfloor + 1)$.

If $n = 3k$ where $k \geq 3$, then $l(P'') \geq 3k - (k + 1) = 2k-1$ and $\bar{l}(P'') \geq 2(2k-1)-1 = 4k-3 \geq n$; If $n = 3k + 1$ where $k \geq 2$, then $l(P'') \geq (3k + 1) - (k+1) = 2k$ and $\bar{l}(P'') \geq 2(2k)-1 = 4k-1 \geq n$; If $n = 3k+2$ where $k \geq 1$, then $l(P'') \geq (3k+2)-(k+1) = 2k+1$ and $\bar{l}(P'') \geq 2(2k+1)-1 = 4k+1 \geq n$.

Subcase 1.2. The path $P : v_1, v_n, \cdots, v_{i+1}, v_i$ is a $v_1 - v_i$ path of length $d_{C_n}(v_1, v_i)$. Then the path $P' : v_1, v_2, \cdots, v_i$ is a rainbow total $v_1 - v_i$ path as the two edges v_iv_{i+1}, v_jv_{j+1} receive the same color, and $l(P') \geq n - \lfloor \frac{n}{3} \rfloor$.

If $n = 3k$ where $k \geq 1$, then $l(P') \geq n - \lfloor \frac{n}{3} \rfloor = 3k - k = 2k$ and $\bar{l}(P') \geq 2(2k) - 1 = 4k - 1 \geq n$; If $n = 3k + 1$ where $k \geq 1$, then $l(P') \geq n - \lfloor \frac{n}{3} \rfloor =$

$(3k+1)-k = 2k+1$ and $\bar{l}(P') \geq 2(2k+1)-1 = 4k+1 \geq n$; If $n = 3k+2$ where $k \geq 1$, then $l(P') \geq n - \lfloor \frac{n}{3} \rfloor = (3k+2)-k = 2k+2$ and $\bar{l}(P') \geq 2(2k+2)-1 = 4k+3 > n$.

From Subcases 1.1 and 1.2, we know that $rtc(G) \geq \bar{l}(P'') \geq n$ for the case $n \geq 5$ except that $n = 6$.

Case 2. Exactly two of these three elements are edges. Assume that this three elements are v_1, $e_i = v_iv_{i+1}$ and $e_j = v_jv_{j+1}$. Clearly, there is one pair of vertices among $\{v_1, v_i, v_j\}$ such that the distance between these two vertices is at most $\lfloor \frac{n}{3} \rfloor$. We will consider the following three subcases.

Subcase 2.1. $dist_{C_n}(v_1, v_i) \leq \lfloor \frac{n}{3} \rfloor$.

If the path $P : v_1, v_2, \cdots, v_{i-1}, v_i$ is the $v_1 - v_i$ path of length $dist_{C_n}(v_1, v_i)$, then the path $P' : v_1, v_2, \cdots, v_i, v_{i+1}$ is the $v_1 - v_{i+1}$ path of length $dist_{C_n}(v_1, v_{i+1})$. For the two vertices v_n and v_{i+1}. We know that the path $P'' : v_n, v_1, \cdots, v_i, v_{i+1}$ is not a rainbow total $v_n - v_{i+1}$ path since v_1 and v_iv_{i+1} have the same color. The path $P''' : v_n, v_{n-1}, \cdots, v_{i+1}, v_i$ must be a rainbow total $v_n - v_i$ path. Now we have $l(P''') \geq n - (\lfloor \frac{n}{3} \rfloor + 2)$. If $n = 3k$ where $k \geq 5$, then $l(P''') \geq 3k - (k+2) = 2k-2$ and $\bar{l}(P''') \geq 2(2k-2)-1 = 4k-5 \geq n$; If $n = 3k+1$ where $k \geq 4$, then $l(P''') \geq (3k+1)-(k+2) = 2k-1$ and $\bar{l}(P''') \geq 2(2k-1)-1 = 4k-3 \geq n$; If $n = 3k+2$ where $k \geq 3$, then $l(P''') \geq (3k+2)-(k+2) = 2k$ and $\bar{l}(P''') \geq 2(2k)-1 = 4k-1 \geq n$.

Otherwise, the path $P : v_1, v_n, \cdots, v_{i+1}, v_i$ is the $v_1 - v_i$ path of length $dist_{C_n}(v_1, v_i)$, then the path $P' : v_1, v_2, \cdots, v_{i-1}, v_i$ must be the rainbow total $v_1 - v_i$ path since v_jv_{j+1} and v_iv_{i+1} receive the same color. Now we have $l(P') \geq n - \lfloor \frac{n}{3} \rfloor$. With a similar argument to that of Subcase 1.2, we have $\bar{l}(P') \geq n$.

From the above discussion, we know that $rtc(G) \geq \bar{l}(P'') \geq n$ for the case $n \geq 11$ except that $n = 12$.

Subcase 2.2. $dist_{C_n}(v_i, v_j) \leq \lfloor \frac{n}{3} \rfloor$.

With a similar argument to that of Case 1, we derive that $rtc(G) \geq \bar{l}(P'') = n$ for the case $n \geq 5$ except that $n = 6$.

Subcase 2.3. $dist_{C_n}(v_1, v_j) \leq \lfloor \frac{n}{3} \rfloor$.

If the path $P : v_1, v_2, \cdots, v_j$ is the $v_1 - v_j$ path of length $dist_{C_n}(v_1, v_j)$, then the path $P' : v_1, v_2, \cdots, v_{j+1}$ must be the $v_1 - v_{j+1}$ path of length $dist_{C_n}(v_1, v_{j+1})$. And the path $P'' : v_1, v_{n+1}, \cdots, v_{j+2}, v_{j+1}$ must be the rainbow total $v_1 - v_{j+1}$ path since v_iv_{i+1} and v_jv_{j+1} have the same color. Now we have $l(P'') \geq n - (\lfloor \frac{n}{3} \rfloor + 1)$. With a similar argument to that of Subcase 1.1, we obtain that if $n = 3k$ where $k \geq 3$, then $\bar{l}(P'') \geq n$; if $n = 3k+1$ where $k \geq 2$, then $\bar{l}(P'') \geq n$; if $n = 3k+2$ where $k \geq 1$, then $\bar{l}(P'') \geq n$.

Otherwise, the path $P : v_1, v_n, \cdots, v_{j+1}, v_j$ is the $v_1 - v_j$ path of length $dist_{C_n}(v_1, v_j)$, then the path $P' : v_2, v_1, \cdots, v_{j+1}, v_j$ must be the $v_2 - v_j$ path of length $dist_{C_n}(v_2, v_j)$ and the path $P'' : v_2, v_3, \cdots, v_{j-1}, v_j$ must be the rainbow total $v_2 - v_j$ path since v_1 and v_jv_{j+1} have the same color. Similarly, we obtain that if $n = 3k$ where $k \geq 3$, then $\bar{l}(P'') \geq n$; if $n = 3k+1$ where $k \geq 2$, then $\bar{l}(P'') \geq n$; if $n = 3k+2$ where $k \geq 1$, then $\bar{l}(P'') \geq n$.

By Subcases 2.1, 2.2 and 2.3, we know that $rtc(G) \geq \bar{l}(P'') \geq n$ for the case $n \geq 11$ except that $n = 12$.

Case 3. Exactly one of these three elements is an edge. Assume that these three elements are v_1, v_i and $e_j = v_jv_{j+1}$. Clearly, there is one pair of vertices among $\{v_1, v_i, v_j\}$ such that the distance between these two vertices is at most $\lfloor \frac{n}{3} \rfloor$. We will consider the following three subcases.

Subcase 3.1. $dist_{C_n}(v_1, v_i) \leq \lfloor \frac{n}{3} \rfloor$.

If the path $P : v_1, v_2, \cdots, v_i$ is the $v_1 - v_i$ path of length $dist_{C_n}(v_1, v_i)$, then the rainbow total path between v_n and v_{i+1} must be $P' : v_n, v_{n-1}, \cdots, v_{i+1}$ since v_1 and v_i have the same colors. Now we have $l(P') \geq n - (\lfloor \frac{n}{3} \rfloor + 2)$. If $n = 3k$ where $k \geq 5$, then $\bar{l}(P') \geq n$; If $n = 3k+1$ where $k \geq 4$, then $\bar{l}(P') \geq n$; If $n = 3k+2$ where $k \geq 3$, then $\bar{l}(P') \geq n$.

Otherwise, the path $P : v_1, v_n, \cdots, v_{i+1}, v_i$ is the $v_1 - v_i$ path of length $dist_{C_n}(v_1, v_i)$, then the path $P' : v_2, v_3, \cdots, v_{i-1}, v_i$ must be the rainbow total path connecting v_2 and v_i. Now we have $l(P') \geq n - (\lfloor \frac{n}{3} \rfloor + 1)$. Similarly, we obtain that if $n = 3k$ where $k \geq 3$, then $\bar{l}(P'') \geq n$; if $n = 3k+1$ where $k \geq 2$, then $\bar{l}(P'') \geq n$; if $n = 3k+2$ where $k \geq 1$, then $\bar{l}(P'') \geq n$.

By the above discussion, we know that $rtc(G) \geq \bar{l}(P'') = n$ for the case $n \geq 11$ except that $n = 12$.

Subcase 3.2. $dist_{C_n}(v_i, v_j) \leq \lfloor \frac{n}{3} \rfloor$.

If the path $P : v_i, v_{i+1}, \cdots, v_j$ is the $v_i - v_j$ path of length $dist_{C_n}(v_i, v_j)$, then the rainbow total-path connecting v_{i-1} and v_{j+1} must be $P' : v_{i-1}, v_{i-2}, \cdots, v_{j+2}, v_{j+1}$, since v_i and v_jv_{j+1} have the same color. Now we have $l(P') \geq n - (\lfloor \frac{n}{3} \rfloor + 2)$. Similarly, we derive that if $n = 3k$ where $k \geq 5$, then $\bar{l}(P') \geq n$; if $n = 3k+1$ where $k \geq 4$, then $\bar{l}(P') \geq n$; if $n = 3k+2$ where $k \geq 3$, then $\bar{l}(P') \geq n$.

Otherwise, the path $P'' : v_i, v_{i-1}, \cdots, v_{j+1}, v_j$ is the $v_i - v_j$ path of length $dist_{C_n}(v_i, v_j)$. Then the path $P''' : v_i, v_{i+1}, \cdots, v_{j-1}, v_j$ must be the rainbow total-path connecting v_i and v_j since v_1 and v_jv_{j+1} have the same color, and $l(P''') \geq n - \lfloor \frac{n}{3} \rfloor$. Similarly, we obtain that if $n = 3k$ where $k \geq 1$, then $\bar{l}(P') \geq n$; if $n = 3k + 1$ where $k \geq 1$, then $\bar{l}(P') \geq n$; if $n = 3k + 2$ where $k \geq 1$, then $\bar{l}(P') > n$.

From the above discussion, we know that $rtc(G) \geq \bar{l}(P'') = n$ for the case $n \geq 11$ except that $n = 12$.

Subcase 3.3. $dist_{C_n}(v_1, v_j) \leq \lfloor \frac{n}{3} \rfloor$. The discussion is similar to that of Subcase 2.3.

By Subcase 3.1, 3.2 and 3.3, we know that $rtc(G) \geq \bar{l}(P'') \geq n$ for the case $n \geq 11$ except that $n = 12$.

Case 4. All these three elements are vertices. Assume that these three elements are v_1, v_i and v_j. Clearly, there is one pair of vertices among $\{v_1, v_i, v_j\}$ such that the distance between these two vertices is at most $\lfloor \frac{n}{3} \rfloor$. Without loss of generality, we assume that $dist_{C_n}(v_1, v_i) \leq \lfloor \frac{n}{3} \rfloor$.

If the path $P : v_1, v_2, \cdots, v_i$ is the $v_1 - v_i$ path of length $dist_{C_n}(v_1, v_i)$, then the path $P' : v_n, v_{n-1}, \cdots, v_{i+2}, v_{i+1}$ must be a rainbow total $v_n - v_{i+1}$ path since v_1 and v_i have the same color. Now we have $l(P') \geq n - (\lfloor \frac{n}{3} \rfloor + 2)$. Similarly, we derive that if $n = 3k$ where $k \geq 5$, then $\bar{l}(P') \geq n$; if $n = 3k+1$ where $k \geq 4$, then $\bar{l}(P') \geq n$; if $n = 3k + 2$ where $k \geq 3$, then $\bar{l}(P') \geq n$.

Otherwise, the path $P'' : v_1, v_n, \cdots, v_{i+1}, v_i$ is the path of length $dist_{C_n}(v_1, v_i)$, then the path $P''' : v_1, v_2, \cdots, v_{i-2}, v_{i-1}$ is the rainbow total $v_1 - v_{i-1}$ path since v_i and v_j have the same color. Now we have $l(P''') \geq n - (\lfloor \frac{n}{3} \rfloor + 1)$. With a similar argument to that of Subcase 2.3, we derive that if $n = 3k$ where $k \geq 3$, then $\bar{l}(P''') \geq n$; if $n = 3k + 1$ where $k \geq 2$, then $\bar{l}(P''') \geq n$; if $n = 3k + 2$ where $k \geq 1$, then $\bar{l}(P''') \geq n$.

From the above four cases, we know that $rtc(G) \geq \bar{l}(P'') = n$ for the case $n \geq 11$ except that $n = 12$, this produces a contradiction. Thus, $rtc(C_n) = n$ for the case that $n \geq 11$ and $n \neq 12$.

For the case $n = 12$, we know that $rtc(C_n) \geq 2diam(C_n) - 1 = 11$. We also give C_{12} a rainbow-total coloring with 11 colors as shown in Figure 3. Thus, $rtc(C_{12}) = 11$.

For the case $n = 10$, we know that $rtc(C_n) \geq 2diam(C_n) - 1 = 9$. We also give C_{10} a rainbow-total coloring with 9 colors as shown in Figure 3. Thus, $rtc(C_{10}) = 9$. □

A well-known class of graphs constructed from cycles are the wheels. For $n \geq 3$, the *wheel* W_n is defined as $C_n + K_1$, the join of C_n and K_1, constructed by joining a new vertex to every vertex of C_n. We will determine the precise values of rainbow total-connection numbers of wheels.

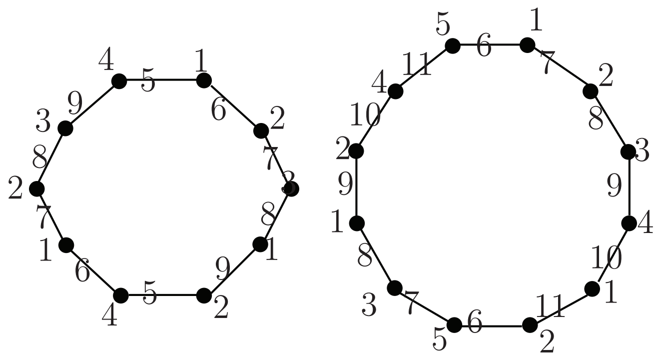

Figure 3: The total-colorings for C_{10} and C_{12}.

Theorem 23 *For $n \geq 3$, the rainbow total-connection number of the wheel W_n is*

$$rtc(W_n) = \begin{cases} 1 & \textit{if } n = 3; \\ 3 & \textit{if } n = 4, 5, 6; \\ 4 & \textit{if } n = 7, 8, 9; \\ 5 & \textit{if } n \geq 10. \end{cases}$$

Proof: Suppose that W_n consists of an n-cycle $C_n : v_1, v_2, \cdots, v_n, v_{n+1} = v_1$ and another vertex v joined to every vertex of C_n. Since $W_3 = K_4$, it follows by Proposition 21 that $rtc(W_3) = 1$.

For $4 \leq n \leq 6$, the wheel W_n is not complete and $rtc(W_n) \geq 3$ by Proposition 21. From Figure 4, there are rainbow total-colorings with 3 colors for W_n. Thus, in this case, we have $rtc(W_n) = 3$.

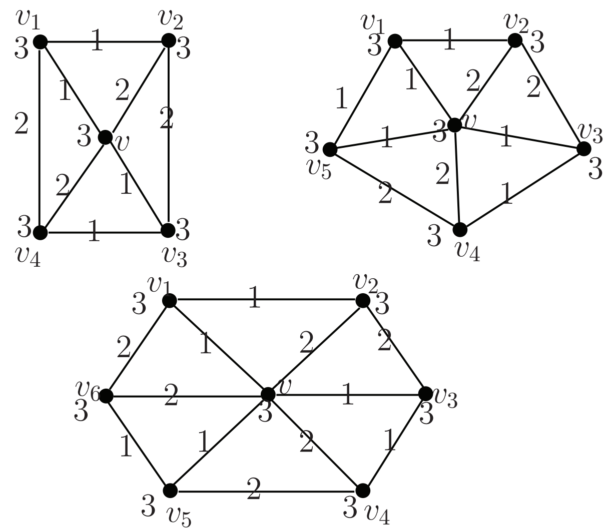

Figure 4: The total-colorings for W_4, W_5 and W_6.

For $7 \leq n \leq 9$, by Figure 5, there are rainbow total-colorings with 4 colors for W_n. So $rtc(W_n) \leq$

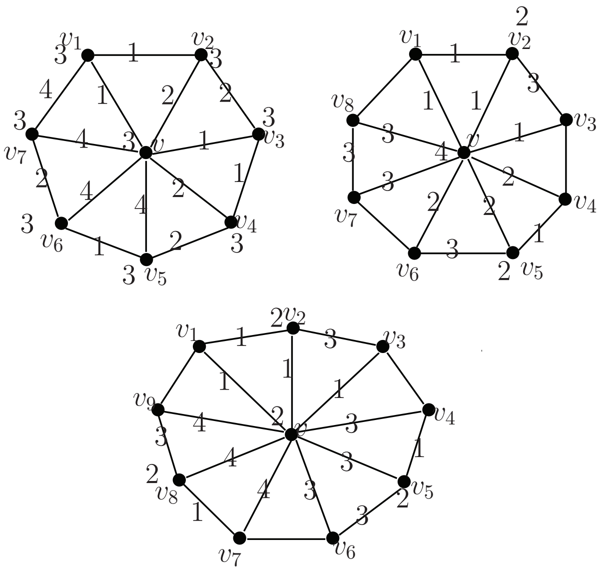

Figure 5: The total-colorings for W_7, W_8 and W_9.

4. Suppose $rtc(W_n) \leq 3$, then there is a rainbow total 3-coloring of W_n. Let $c(v) = 1$. If there is some edge vv_i, say vv_1, with $c(vv_1) = c(v)$, then there is no rainbow total $v_1 - v_5$ path, this produces a contradiction. Thus, $c(vv_i) \neq c(v)$ for $1 \leq i \leq n$ and $c(vv_i) \in \{2, 3\}$. Then there are at least four edges, say $vv_1, vv_{i_1}, vv_{i_2}, vv_{i_3}$, with $c(vv_1) = c(vv_{i_1}) = c(vv_{i_2}) = c(vv_{i_3})$ where $1 < i_1 < i_2 < i_3 \leq n$. Clearly, there exist two elements of $\{1, i_1, i_2, i_3\}$, say i_1 and i_2, such that the distance of v_{i_1} and v_{i_2} in the cycle C_n is at least 3. As now $c(vv_{i_1}) = c(vv_{i_2})$, there is no rainbow total path connecting v_{i_1}, v_{i_2}, this produces a contradiction. Thus, in this case, we have $rtc(W_n) = 4$.

For $n \geq 10$, we give W_n a total-coloring with 5 colors as follows: $c(vv_i) = 1$ if i is odd, $c(vv_i) = 2$ if i is even, $c(e) = 3$ for each $e \in E(C_n)$, $c(v) = 4$ and $c(v_i)$ for $1 \leq i \leq n$. It is easy to show that this total-coloring is rainbow, we have $rtc(W_n) \leq 5$. We will show that $rtc(G) \geq 5$. Suppose $rtc(G) \leq 4$, then there is a rainbow total 4-coloring of W_n. Let $c(v) = 1$. If there is some edge vv_i, say vv_1, with $c(vv_1) = c(v)$, then there is no rainbow total $v_1 - v_5$ path, this produces a contradiction. Thus, $c(vv_i) \neq c(v)$ for $1 \leq i \leq n$ and $c(vv_i) \in \{2, 3, 4\}$. Then there are at least four edges, say $vv_1, vv_{i_1}, vv_{i_2}, vv_{i_3}$, with $c(vv_1) = c(vv_{i_1}) = c(vv_{i_2}) = c(vv_{i_3})$ where $1 < i_1 < i_2 < i_3 \leq n$. Clearly, there exist two elements of $\{1, i_1, i_2, i_3\}$, say i_1 and i_2, such that the distance of v_{i_1} and v_{i_2} in the cycle C_n is at least 3. As now $c(vv_{i_1}) = c(vv_{i_2})$, there is no rainbow total path connecting v_{i_1}, v_{i_2}, this produces a contradiction. Thus, in this case, we have $rtc(W_n) = 5$. □

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