Polar Sets for Slices of Generalized Brownian Sheet

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Abstract: Let $\widetilde{W}$ be a two-parameter, $\mathbb{R}^d$-valued generalized Brownian sheet. We can view $\widetilde{W}$ as a sequence of interacting generalized Brownian motions or slices. The sample path properties for the slices of $\widetilde{W}$ are studied, and the connections between the polar sets for the slices of $\widetilde{W}$ and capacity are also presented. A common feature of our results is that exhibit phase transition.

Key–Words: Generalized Brownian sheet, capacity, polar set, dimension

1 Introduction

Given $0 < \alpha \leq 1,\ 1 \leq \beta < \infty$, let $Y = \{Y(s,t) : s,t \in [0,\infty)\}$ be the real-valued, centered gaussian random field with covariance function

$$\mathbb{E}[Y(s,t)Y(u,v)] = F_1(0, s \wedge u]F_2(0, t \wedge v], \quad (1)$$

where F_1, F_2 denote Lebesgue-Stieljes measure that are absolutely continuous with respect to Lebesgue measure. Suppose that for each $\ell = 1,2$, F_ℓ satisfies the following condition: there exist positive constants $c,\ \delta$ such that for all $s,t \in [0,\infty)$ with $|t-s| < \delta$,

$$c^{-1}|s-t|^\beta \leq |F_\ell(0,s] - F_\ell(0,t]| \leq c|s-t|^\alpha. \quad (2)$$

Consider the gaussian random field $\widetilde{W} = \{\widetilde{W}(s,t), s,t \in [0,\infty)\}$ in $\mathbb{R}^d$ defined by

$$\widetilde{W}(s,t) = \big(\widetilde{W}_1(s,t), \dots, \widetilde{W}_d(s,t)\big),$$

where $\widetilde{W}_1(s,t), \dots, \widetilde{W}_d(s,t)$ are independent copies of Y. Then $\widetilde{W}$ is called a two-parameter, $\mathbb{R}^d$-valued generalized Brownian sheet with indexes α, β. A typical example of a generalized Brownian sheet is the Brownian sheet taking values in $\mathbb{R}^d$ with the Lebesgue measure, that is $W(s,t) = \big(W_1(s,t), \dots, W_d(s,t)\big)$, where $W_1, \dots, W_d$ are independent copies of the centered real valued gaussian random field X with covariance function

$$\mathbb{E}[X(s,t)X(u,v)] = (s \wedge u)(t \wedge v).$$

It is easy to see that $\widetilde{W}$ is an independent increment process; see [1]. [2] gave some reasonable remarks on $0 < \alpha \leq 1,\ 1 \leq \beta < \infty$ in (2). Even if $\alpha = \beta = 1$, we can say that $\widetilde{W}$ is also wider than the Brownian sheet; see the example in Section 5 for a precise statement .

For the generalized Brownian sheet, there are some excellent arguments such as [1-7]. However, contrast to the extensive studies on the Brownian sheet, there has been little systematic investigation on the generalized Brownian sheet. The main reasons for this, in my opinion, are the complexity of dependence structures and the non-availability of convenient scaling property for these more general independent increment gaussian processes. The objective of this paper is to further study the sample path properties of the slices of $\widetilde{W}$.

Choose and fix some number $s > 0$. The *slice* of $\widetilde{W}$ along s is the stochastic process $\{\widetilde{W}(s,t),\ t \geq 0\}$. It is easy to see that if s is non-random then the slice of $\widetilde{W}$ is a one parameter generalized Brownian sheet. More precisely, $t \mapsto F_1^{-1/2}(0,s]\widetilde{W}(s,t)$ is a standard generalized d-dimensional Brownian motion. It is not difficult to see that if s is random, then the slice of $\widetilde{W}$ along s need not be a generalized Brownian motion. For example, when W is a two-dimensional Brownian sheet in $\mathbb{R}^d$, the slice of W along a non-random s hits points if and only if $d = 1$. However, there are random values of s such that the slice of W along s hits zero up to dimension $d = 3$ (cf. [8]). Nonetheless, one may expect the slice of $\widetilde{W}$ along s to look like a generalized Brownian motion in some sense, even for some random values of s.

A common question in infinite-dimensional stochastic analysis is to ask if there are slices that behave differently from d-dimensional Brownian motion predescribed manner. There is a large literature on this subject (cf. [9]). On the other hand, some new examples where there is, generally, a cutoff phenomenon or phase transition are presented on the sheet (cf. [10]). In this paper, we present some results about the slices of the generalized Brownian sheet. The relationships between the polar sets for the slices of $\widetilde{W}$ and capacity are also obtained. A common feature of our results is that exhibit phase transition. Our approach is based on the results on the slices of the Brownian sheet in [10].

Our first result is related to the zero-set of the generalized Brownian sheet. Orey and Pruit have proven that $W^{-1}\{0\}$ is non-trivial if and only if the spatial dimension d is three or less in [8]. See also [11] and [12]. Khoshnevisan has studied the slices of the Brownian sheet in [10] and the relationship between Brownian sheet images and Bessel-Riesz capacity in [14] respectively. Chen has proven the following refinement in [6]: For all non-random, compact sets $E, F \subset (0,\infty)$,

$$A_1 \mathrm{Cap}_{\beta d/2}(E\times F) \le \mathbb{P}\{\widetilde{W}^{-1}\{0\}\cap(E\times F)\neq\emptyset\} \le A_2 \mathrm{Cap}_{\alpha d/2}(E\times F), \quad (3)$$

where A_1, A_2 are finite positive constants, and for all $\gamma > 0$, $\mathrm{Cap}_\gamma(E\times F)$ denotes the γ-dimensional Bessel-Riesz capacity of $E \times F$ based on the d-dimensional energy form I_γ; i.e.

$$\mathrm{Cap}_\gamma(E\times F) = \frac{1}{\inf_{\mu\in\mathcal{P}(E\times F)} I_\gamma(\mu)},$$

where

$$I_\gamma(\mu) \hat{=} \int\int \frac{1}{|s-t|^\gamma}\mu(ds)\mu(dt),$$

and $\mathcal{P}(E\times F)$ denotes the collection of all probability measures that are supported in $E \times F$. The results in [8] and [14] follows immediately from (3) and Taylor's theorem.

Now consider the projection $\mathcal{L}_d$ of $\widetilde{W}^{-1}\{0\}$ onto x-axis. Choose and fix two number $0 < a < b < \infty$ with $b - a < \delta$. Let

$$\mathcal{L}_d \hat{=} \{s \ge 0 : \widetilde{W}(s,t) = 0 \text{ for some t} \in \text{[a, b]}\}. \quad (4)$$

Thus, $s \in \mathcal{L}_d$ if and only if the slice of $\widetilde{W}$ along s hits zero. Of course, zero is always in $\mathcal{L}_d$. The following results characterize the polar sets of $\mathcal{L}_d$.

Theorem 1 *For all non-random compact sets* $F \subset (0,\infty)$ *with* $\mathrm{diam} F < \delta$, *then*

$$K_1 \mathrm{Cap}_{(\beta d-2)/2}(F) \le \mathbb{P}\{\mathcal{L}_d \cap F \neq \emptyset\} \le K_2 \mathrm{Cap}_{(\alpha d-2)/2}(F), \quad (5)$$

where K_1 *and* K_2 *are finite positive constants which only depend on the parameters* a, b, α, β *and* d.

The following is a consequence of Theorem 1.

Corollary 2 *For all non-random compact sets* $F \subset (0,\infty)$ *with* $\mathrm{diam} F < \delta$, *then*

$$\mathrm{Cap}_{(\beta d-2)/2}(F) > 0 \Rightarrow \mathbb{P}\{\mathcal{L}_d \cap F \neq \emptyset\} > 0 \Rightarrow \mathrm{Cap}_{(\alpha d-2)/2}(F) > 0.$$

Furthermore, we can apply Theorem 1 and a codimension argument ([15], Theorem 4.7.1, p.436) to find that

$$1 \wedge \left(2 - \frac{\beta d}{2}\right)^+ \le \dim_{\mathrm{H}} \mathcal{L}_d \le 1 \wedge \left(2 - \frac{\alpha d}{2}\right)^+ \text{a.s.},$$

where $\dim_{\mathrm{H}}$ denotes Hausdorff dimension.

Next, for all $0 < a < b < \infty$, we consider the following random set,

$$\mathcal{D}_d \hat{=} \{a \le s \le b : \widetilde{W}(s,t_1) = \widetilde{W}(s,t_2) \text{ for some } t_2 > t_1 > 0\}. \quad (6)$$

We can note that $s \in \mathcal{D}_d$ if and only if the slice of $\widetilde{W}$ along s has a double point.

For the Brownian sheet, Lyons has proven that $\mathcal{D}_d$ is non-trivial if and only if $d \le 5$ in [16]. That is

$$\mathbb{P}\{\mathcal{D}_d \neq \{0\}\} \ge 0 \text{ if and only if } d \le 5. \quad (7)$$

See also [17]. Lyons's theorem (10) is an improvement to an earlier theorem in [11] which asserts the necessity of the condition $d \le 6$. Our next result characterizes the polar sets of $\mathcal{D}_d$ for the generalized Brownian sheet.

Theorem 3 *Choose and fix two numbers* $0 < a < b < \infty$ *with* $b - a < \delta$. *Then, there exist positive constants* K_3 *and* K_4 *such that for all compact, non-random sets* $F \subset [a, b]$,

$$K_3 \mathrm{Cap}_{\beta(d-4)/2}(F) \le \mathbb{P}\{\mathcal{D}_d \cap F \neq \emptyset\} \le K_4 \mathrm{Cap}_{\alpha(d-4)/2}(F). \quad (8)$$

The following corollary is a consequence of Theorem 3.

Corollary 4 *Let* $F \subset (0, \infty)$ *be a non-random compact set with diam*$F < \delta$ *. Then*

$$\mathrm{Cap}_{\beta(d-4)/2}(F) > 0 \Rightarrow \mathbb{P}\{\mathcal{D}_d \cap F \neq \emptyset\} > 0 \Rightarrow \mathrm{Cap}_{\alpha(d-4)/2}(F) > 0. \quad (9)$$

Lyons's theorem (10) follows at once from this and Taylor's theorem. In addition, a codimension argument reveals that almost surely

$$1 \wedge \Big(3 - \frac{\beta d}{2}\Big)^+ \leq \dim_{\mathrm{H}} \mathcal{D}_d \leq 1 \wedge \Big(3 - \frac{\alpha d}{2}\Big)^+.$$

Finally, we consider the random sets

$$\hat{\mathcal{D}}_d := \{(s, t_1, t_2) \in [a, b]^3 : \widetilde{W}(s, t_1) = \widetilde{W}(s, t_2)\}$$
$$\bar{\mathcal{D}}_d := \{(s, t_1) \in [a, b]^2 : \widetilde{W}(s, t_1) = \widetilde{W}(s, t_2) \text{ for some } t_2 > 0\}. \tag{10}$$

The following theorem characterizes the polar sets of $\bar{\mathcal{D}}_d$ and $\hat{\mathcal{D}}_d$. This method is not sufficiently delicate, but I also believe such a characterization is within reach of the existing technology (cf. [6]).

Theorem 5 *Choose and fix two numbers $0 < a < b < \infty$ with $b - a < \delta$. Then, there exist constants K_5 and K_6 such that for all non-random compact set $E \subset [a, b]^2$ and $G \subset [a, b]^3$,*

$$K_5^{-1} \mathrm{Cap}_{\beta d/2}(G) \leq \mathbb{P}\{\hat{\mathcal{D}}_d \cap G \neq \emptyset\} \leq K_5 \mathcal{H}_{\alpha d/2}(G), \tag{11}$$
$$K_6^{-1} \mathrm{Cap}_{\beta(d-2)/2}(E) \leq \mathbb{P}\{\bar{\mathcal{D}}_d \cap E \neq \emptyset\} \leq K_6 \mathcal{H}_{\alpha(d-2)/2}(E), \tag{12}$$

where $\mathcal{H}_\gamma$ denotes the γ-dimensional Hausdorff measure.

Corollary 6 *For all non-random compact sets $E \subset [a, b]^2$ and $G \subset [a, b]^3$ with $b - a \leq \delta$,*

$$\mathrm{Cap}_{\beta d/2}(G) > 0 \Rightarrow \mathbb{P}\{\hat{\mathcal{D}}_d \cap G \neq \emptyset\} > 0 \Rightarrow \mathcal{H}_{\alpha d/2}(G) > 0,$$
$$\mathrm{Cap}_{\beta(d-2)/2}(E) > 0 \Rightarrow \mathbb{P}\{\bar{\mathcal{D}}_d \cap E \neq \emptyset\} > 0 \Rightarrow \mathcal{H}_{\alpha(d-2)/2}(E) > 0.$$

The remainder of this paper is organized as follows. In Section 2, we introducing some basic real-variable computations by analyzing the properties of three classes of functions. Theorems 3 and 5 are respectively proved in Sections 3 and 4. Section 5 contains the proof of Theorem 1, some corollaries of Theorems 1, 3 and 5, and a few related remarks. Even if $\alpha = \beta = 1$, we can say that $\widetilde{W}$ is also wider than Brownian sheet in the above theorem . The results that we obtain are sharper and more general than those of the Brownian sheet.

Throughout, any n-vector x is written, coordinatewise, as $x = (x_1, \ldots, x_n)$. Moreover, $|x|$ will always denote the ℓ^1-norm of $x \in \mathbb{R}^n$; i.e.,

$$|x| \hat{=} |x_1| + \cdots + |x_n|.$$

Generic constants that do not depend on anything interesting are denoted by $c, c_1, c_2, \ldots ; K, K_1, K_2, \ldots$; they are always assumed to be positive and finite, and their values may change between, as well as within, lines. Let A denote a Borel set in $\mathbb{R}^n$. The collection of all Borel probability measures on A is always denoted by $\mathcal{P}(A)$.

2 Some Estimates

Our analysis depends on the properties of three classes of functions. We develop the requisite estimates here in this section. We define for all ϵ, $h > 0$ and $x \in \mathbb{R}$,

$$f_\epsilon(x) \hat{=} \Big(\frac{\epsilon}{|x|^{\frac{1}{2}}} \wedge 1\Big)^d,$$
$$g_\epsilon(x) \hat{=} \int_0^h f_\epsilon(y + |x|) dy,$$
$$G_\epsilon(x) \hat{=} \int_0^h g_\epsilon(y + |x|) dy.$$

Our first result is a simple fact which can be deduced from the covariance of the generalized Brownian sheet.

Lemma 7 *There exist constants c_1, $c_2 > 0$ such that for all $\epsilon > 0$ and $s, t \in (0, \infty)$,*

$$c_1 f_\epsilon(F_1(0, s] F_2(0, t]) \leq \mathbb{P}\{|\widetilde{W}(s, t)| \leq \epsilon\} \leq c_2 f_\epsilon(F_1(0, s] F_2(0, t]). \tag{13}$$

Proof: Note that $|\widetilde{W}(s, t)| = |\widetilde{W}_1(s, t)| + \cdots + |\widetilde{W}_d(s, t)|$. Therefore,

$$\begin{aligned}
&\mathbb{P}\{|\widetilde{W}(s, t)| \leq \epsilon\} \\
\leq &\mathbb{P}\{|\widetilde{W}_1(s, t)| \leq \epsilon, \ldots, |\widetilde{W}_d(s, t)| \leq \epsilon\} \\
= &\Big[\Big(\int_{-\epsilon}^{\epsilon} \frac{1}{\sqrt{2\pi F_1(0, s] F_2(0, t]}} \times \exp\Big(-\frac{u^2}{2F_1(0, s] F_2(0, t]}\Big) du\Big) \wedge 1\Big]^d \\
\leq &2^d \Big(\frac{\epsilon}{\sqrt{F_1(0, s] F_2(0, t]}} \wedge 1\Big)^d.
\end{aligned} \tag{14}$$

The upper bound of the lemma follows from (14). To derive the lower bound, we can find that when $\epsilon^2 \le F_1(0,s]F_2(0,t]$,

$$\begin{aligned}
&\mathbb{P}\{|\widetilde{W}(s,t)| \le \epsilon\} \\
\ge&\mathbb{P}\{|\widetilde{W}_1(s,t)| \le \frac{\epsilon}{d}, \dots, |\widetilde{W}_d(s,t)| \le \frac{\epsilon}{d}\} \\
=&\Big(\int_{-\epsilon/d}^{\epsilon/d} \frac{1}{\sqrt{2\pi F_1(0,s]F_2(0,t]}} \\
&\times \exp\Big(-\frac{u^2}{2F_1(0,s]F_2(0,t]}\Big)du\Big)^d \\
\ge&\Big(\frac{2}{\pi d^2}\Big)^{d/2}\exp\Big(-\frac{1}{2d^2}\Big)\Big(\frac{\epsilon}{\sqrt{F_1(0,s]F_2(0,t]}}\Big)^d \\
=&c_3 f_\epsilon(F_1(0,s]F_2(0,t]). \qquad (15)
\end{aligned}$$

The same reasoning shows that when $\epsilon^2 \ge F_1(0,s]F_2(0,t]$,

$$\begin{aligned}
&\mathbb{P}\{|\widetilde{W}(s,t)| \le \epsilon\} \\
=&\Big(\int_{-\epsilon}^{\epsilon} \frac{1}{\sqrt{2\pi F_1(0,s]F_2(0,t]}} \\
&\times \exp\Big(-\frac{u^2}{2F_1(0,s]F_2(0,t]}\Big)du\Big)^d \\
=&\Big(\int_{-\epsilon/\sqrt{F_1(0,s]F_2(0,t]}}^{\epsilon/\sqrt{F_1(0,s]F_2(0,t]}} \frac{1}{\sqrt{2\pi}}\exp\Big(-\frac{u^2}{2}\Big)du\Big)^d \\
\ge&\Big(\int_{-1}^{1} \frac{1}{\sqrt{2\pi}}\exp\Big(-\frac{u^2}{2}\Big)du\Big)^d f_\epsilon(F_1(0,s]F_2(0,t]) \\
=&c_4 f_\epsilon(F_1(0,s]F_2(0,t]). \qquad (16)
\end{aligned}$$

Taking $c_1 = \min\{c_3, c_4\}$, we finished the proof of Lemma 7 by (15) and (16). □

Let $\gamma \in \mathbb{R}$ and $x \in \mathbb{R}\backslash\{0\}$ define

$$U_\gamma(x) \hat{=} \begin{cases} 1, & \text{if } \gamma < 0, \\ \log_+(1/|x|), & \text{if } \gamma = 0, \\ |x|^{-\gamma}, & \text{if } \gamma > 0. \end{cases} \qquad (17)$$

Next, we derive the upper and lower bounds for g_ϵ in terms of the above function U_γ that is defined in (2.5).

Lemma 8 *There exist constants $c_5,\ c_6 > 0$ such that for all $\epsilon > 0$ and $0 < y \le \delta$,*

$$g_\epsilon(F_\ell(0,y]) \le c_5 \epsilon^d U_{\beta(d-2)/2}(y),\ \ell = 1,\ 2, \qquad (18)$$

and for all $F(0,y] \ge \epsilon^2$,

$$g_\epsilon(F_\ell(0,y]) \ge c_6 \epsilon^d U_{\alpha(d-2)/2}(y),\ \ell = 1,\ 2. \qquad (19)$$

Proof: It follows from (2) and $0 < y \le \delta$ that

$$c^{-1}y^\beta \le F_\ell(0,y] \le c y^\alpha. \qquad (20)$$

By the definition of the function g_ϵ, we have

$$\begin{aligned}
&g_\epsilon(F_\ell(0,y]) \\
\le&\epsilon^d \int_0^h \frac{dx}{(x+F_\ell(0,y])^{d/2}} \\
=&\epsilon^d \int_{F_\ell(0,y]}^{h+F_\ell(0,y]} \frac{dx}{x^{d/2}} \\
\le&\begin{cases} \frac{2\epsilon^d}{2-d}(h+F_\ell(0,\delta])^{1-d/2}, & \text{if } d < 2, \\ \epsilon^d \ln\frac{(h+F_\ell(0,\delta])}{F_\ell(0,y]}, & \text{if } d = 2, \\ \frac{2\epsilon^d}{d-2}(F_\ell(0,y])^{1-d/2}, & \text{if } d > 2. \end{cases} \qquad (21)
\end{aligned}$$

For all $F(0,y] \ge \epsilon^2$, we use the same reasoning to show that

$$\begin{aligned}
&g_\epsilon(F_\ell(0,y]) \\
=&\epsilon^d \int_{F_\ell(0,y]}^{h+F_\ell(0,y]} \frac{dx}{x^{d/2}} \\
\ge&\begin{cases} \frac{2\epsilon^d}{2-d}h^{1-d/2} \\ \quad\times\Big[1-\Big(\frac{F_\ell(0,\delta]}{h+F_\ell(0,\delta]}\Big)^{1-d/2}\Big], & \text{if } d < 2, \\ \epsilon^d \ln\frac{h}{F_\ell(0,y]}, & \text{if } d = 2, \\ \frac{2\epsilon^d}{d-2}(F_\ell(0,y])^{1-d/2} \\ \quad\times[1-(1+hF_\ell^{-1}(0,\delta])^{1-d/2}], & \text{if } d > 2. \end{cases} \\
&\qquad (22)
\end{aligned}$$

The lemma follows from (20), (21) and (22). □

By the definition of the function G_ϵ, we also note that

$$\begin{aligned}
G_\epsilon(x) \hat{=}& \int_0^h g_\epsilon(y+|x|)dy \\
=& \int_0^h \int_0^h f_\epsilon(|x|+y_1+y_2)dy_1dy_2. \qquad (23)
\end{aligned}$$

The following lemma follows from Lemma 8 and some elementary estimates.

Lemma 9 *Choose and fix a number* $0 < b < \infty$. *Then there exist constants* $c_7,\ c_8 > 0$ *such that for all* $\epsilon > 0$ and $0 < y \le x \le b$ with $x - y \le \delta$,

$$G_\epsilon(F_\ell(0,x] - F_\ell(0,y]) \le c_7 \epsilon^d U_{\beta(d-4)/2}(x-y), \tag{24}$$

and for all $F_\ell(0,x] - F_\ell(0,y] \ge \epsilon^2$,

$$G_\epsilon(F_\ell(0,x] - F_\ell(0,y]) \ge c_8 \epsilon^d U_{\alpha(d-4)/2}(x-y), \tag{25}$$

where $\ell = 1,\ 2$.

Proof: It follows from (2) and $x - y \le \delta$ that

$$c^{-1}|x-y|^\beta \le |F_\ell(0,x] - F_\ell(0,y]| \le c|x-y|^\alpha. \tag{26}$$

We first prove (24) for $\epsilon > 0$ and $0 < y \le x \le b$. By using (17) and a similar computation as in the proof of Lemma 8, we have

$$\begin{aligned} & G_\epsilon(F_\ell(0,x] - F_\ell(0,y]) \\ = & \int_0^h g_\epsilon(z + |F_\ell(0,x] - F_\ell(0,y]|)dz \\ \le & \begin{cases} c_9\epsilon^d h(2h + F_\ell(0,b])^{(2-d)/2}, & \text{if } d < 2, \\ c_9\epsilon^d (h \ln 2 + h), & \text{if } d = 2, \\ c_9\epsilon^d \dfrac{2}{4-d}(h + F_\ell(0,b])^{(4-d)/2}, & \text{if } 2 < d < 4, \\ c_{10}\epsilon^d \ln \dfrac{h + F_\ell(0,b]}{|F_\ell(0,x] - F_\ell(0,y]|}, & \text{if } d = 4, \\ c_{10}\epsilon^d |F_\ell(0,x] - F_\ell(0,y]|^{(4-d)/2}, & \text{if } d > 4. \end{cases} \end{aligned}$$

This, together with (26), implies the upper in (24). The lower in (25) is proved along the same lines. □

Lemma 10 *For all* $\epsilon > 0$ *and* $x \ge 0$,

$$G_\epsilon(F_\ell(0,x]) \ge \frac{1}{2}\int_0^{2h} g_\epsilon(y + F_\ell(0,x])dy.$$

Proof: By the change of the variance with $y = t/2$ and the monotonicity, we have

$$\begin{aligned} G_\epsilon(F_\ell(0,x]) &= \int_0^h g_\epsilon(y + F_\ell(0,x])dy \\ &= \frac{1}{2}\int_0^{2h} g_\epsilon(\frac{t}{2} + F_\ell(0,x])dt \\ &\ge \frac{1}{2}\int_0^{2h} g_\epsilon(y + F_\ell(0,x])dy. \end{aligned}$$

This finishes the proof of the lemma. □

3 Polar sets of $\mathcal{D}_d$

In this section, we characterizes the polar sets of $\mathcal{D}_d$ generated by the generalized Brownian sheet. Because of the complicated covariance, the proof of the polar sets of $\mathcal{D}_d$ is quite involved. Therefore, we split the proof into several lemmas, which are also of their own interest.

Choose and fix two number $0 < a < b < \infty$. Let $\widetilde{W}^{(1)}$ and $\widetilde{W}^{(2)}$ be two independent generalized Brownian sheets in $\mathbb{R}^d$, and define for all $\mu \in \mathcal{P}(\mathbb{R}_+)$,

$$J_\epsilon(\mu) \hat{=} \frac{1}{\epsilon^d}\int_a^b\int_a^b\int \mathbf{1}_{\mathbf{A}(\epsilon;s,t)}\mu(ds)dF_2(0,t_1]dF_2(0,t_2], \tag{27}$$

where $\mathbf{A}(\epsilon; s,t)$ is the event

$$\mathbf{A}(\epsilon; s,t) \hat{=} \big\{|\widetilde{W}^{(2)}(s,t_2) - \widetilde{W}^{(1)}(s,t_1)| \le \epsilon\big\}, \tag{28}$$

for all $a \le s, t_1, t_2 \le b$ and $\epsilon > 0$.

Lemma 11 *Choose and fix two numbers* $0 < a < b < \infty$ *. Then,*

$$\inf_{0<\epsilon<1}\ \inf_{\mu\in\mathcal{P}([a,b])} \mathbb{E}[J_\epsilon(\mu)] > 0. \tag{29}$$

Proof: For all $s, t_1, t_2 \in [a,b]$, let

$$\sigma^2 \hat{=} \mathrm{Var}\big(\widetilde{W}_j^{(2)}(s,t_2) - \widetilde{W}_j^{(1)}(s,t_1)\big), \ \ 1 \le j \le d.$$

The independence of $\widetilde{W}^{(1)}$ and $\widetilde{W}^{(2)}$ implies

$$2F_1(0,a]F_2(0,a] \le \sigma^2 \le 2F_1(0,b]F_2(0,b]. \tag{30}$$

By (28) and (30), we have

$$\begin{aligned} & \mathbb{P}\big\{\mathbf{A}(\epsilon; s,t)\big\} \\ \ge & \Big[\int_{-\epsilon/d}^{\epsilon/d} \frac{1}{\sqrt{2\pi}\sigma}\exp\Big(-\frac{u^2}{2\sigma^2}\Big)du\Big]^d \\ \ge & \Big(\frac{\varepsilon^2}{\pi F_1(0,b]F_2(0,b]d^2}\Big)^{d/2} \\ & \times \exp\Big(-\frac{\epsilon^2}{4dF_1(0,a]F_2(0,a]}\Big). \end{aligned} \tag{31}$$

By (27) and (31), we have

$$\begin{aligned} & \mathbb{E}[J_\epsilon(\mu)] \\ = & \frac{1}{\epsilon^d}\int_a^b\int_a^b\int \mathbb{P}\big\{\mathbf{A}(\epsilon; s,t)\big\}\mu(ds)dF_2(0,t_1]dF_2(0,t_2] \\ \ge & \frac{(F_2(0,b] - F_2(0,a])^2\mu([a,b])}{\big(\pi F_1(0,b]F_2(0,b]d^2\big)^{d/2}} \\ & \times \exp\Big(-\frac{\epsilon^2}{4dF_1(0,a]F_2(0,a]}\Big). \end{aligned}$$

This proves the lemma. □

Next we present a bound for the second moment of $J_\epsilon(\mu)$. For technical reason, we first define

$$\hat{J}_\epsilon(\mu) \hat{=} \frac{1}{\epsilon^d}\int_a^c\int_a^c\int \mathbf{1}_{\mathbf{A}(\epsilon;s,t)}\mu(ds)dF_2(0,t_1]dF_2(0,t_2], \tag{32}$$

Lemma 12 *Choose and fix three numbers $0 < a < b < c < \infty$ with $c - a \le \delta$. Then, there exist positive constants c_9 and c_{10} such that for all Borel probability measures μ on $\mathbb{R}_+$ and all $0 < \epsilon < 1$,*

$$\begin{aligned}\mathbb{E}[\hat{J}_\epsilon(\mu)]^2 \le & \frac{c_9}{\epsilon^d}\int\int G_\epsilon(F_1(0,s]-F_1(0,u])\mu(ds)\mu(du)\\ \le & c_{10} I_{\beta(d-4)/2}(\mu).\end{aligned} \tag{33}$$

Proof: Define $\widehat{W}^{(i)}$ to be the generalized white noise that corresponds to the generalized Brownian sheet $\widetilde{W}^{(i)}$ $(i = 1, 2)$. It is convenient to think of $\widetilde{W}^{(i)}$ as the distribution function of a d-dimensional generalized white noise $\widehat{W}^{(i)}$ on $\mathbb{R}_+^2$; see the survey paper [21]. For all $\epsilon > 0$, $s, u \in [a, b]$, and $t = (t_1, t_2)$, $v = (v_1, v_2) \in [a, b] \times [b, c]$, let

$$\mathbb{P}_\epsilon(s,u;t,v) \hat{=} \mathbb{P}\big\{\mathbf{A}(\epsilon;s,t)\cap\mathbf{A}(\epsilon;u,v)\big\}. \tag{34}$$

By symmetry, we assume that $s \le u$. We give an estimate of (34) by analyzing two different cases.

The first case is that $t_1 \le v_1$ and $t_2 \le v_2$. Consider

$$\begin{aligned}
H_1^{(1)} &\hat{=} \widehat{W}^{(1)}([0,s]\times[0,t_1]),\\
H_2^{(1)} &\hat{=} \widehat{W}^{(1)}([0,s]\times[t_1,v_1]),\\
H_3^{(1)} &\hat{=} \widehat{W}^{(1)}([s,u]\times[0,v_1]);\\
H_1^{(2)} &\hat{=} \widehat{W}^{(2)}([0,s]\times[0,t_2]),\\
H_2^{(2)} &\hat{=} \widehat{W}^{(2)}([0,s]\times[t_2,v_2]),\\
H_3^{(2)} &\hat{=} \widehat{W}^{(2)}([s,u]\times[0,v_2]).
\end{aligned}$$

Thus, the $H's$ are all totally independent Gaussian random vectors. By the triangle inequality,

$$\begin{aligned}
&\mathbb{P}_\epsilon(s,u;t,v)\\
=&\mathbb{P}\big\{|H_1^{(2)}-H_1^{(1)}|\le\epsilon; |H_1^{(2)}+H_2^{(2)}\\
&+H_3^{(2)}-H_1^{(1)}-H_2^{(1)}-H_3^{(1)}|\le\epsilon\big\}\\
\le&\mathbb{P}\big\{|H_1^{(2)}-H_1^{(1)}|\le\epsilon\big\}\\
&\times\mathbb{P}\big\{|H_2^{(2)}+H_3^{(2)}-H_2^{(1)}-H_3^{(1)}|\le 2\epsilon\big\}\\
\hat{=}&I\times J.
\end{aligned} \tag{35}$$

As $\Gamma_1 \hat{=} H_1^{(2)} - H_1^{(1)}$ and $\Gamma_2 \hat{=} H_2^{(2)} + H_3^{(2)} - H_2^{(1)} - H_3^{(1)}$ both have i.i.d. mean 0 coordinates , one can check the coordinatewise variances and find that for all $1 \le i \le d$,

$$\begin{aligned}
\mathrm{Var}(\Gamma_1^{(i)}) =& F_1(0,s](F_2(0,t_1]+F_2(0,t_2])\\
\ge& 2F_1(0,a]F_2(0,a],
\end{aligned} \tag{36}$$

$$\begin{aligned}
\mathrm{Var}(\Gamma_2^{(i)}) =& F_1(0,s](F_2(0,v_2]-F_2(0,t_2])\\
&+(F_1(0,u]-F_1(0,s])F_2(0,v_2]\\
&+F_1(0,s](F_2(0,v_1]-F_2(0,t_1])\\
&+(F_1(0,u]-F_1(0,s])F_2(0,v_1]\\
\ge& \min\{F_1(0,a],2F_2(0,a]\}\big(|F_1(0,u]\\
&-F_1(0,s]|+|F_2(0,v_1]-F_2(0,t_1]|\\
&+|F_2(0,v_2]-F_2(0,t_2]|\big).
\end{aligned} \tag{37}$$

By (35), (36), (37) and Lemma 7, we have $I \le c_{11}\epsilon^d$ and

$$\begin{aligned}
J \le& c_{12}f_\epsilon\big(|F_1(0,u]-F_1(0,s]|+|F_2(0,v_1]\\
&-F_2(0,t_1]|+|F_2(0,v_2]-F_2(0,t_2]|\big).
\end{aligned} \tag{38}$$

By (35) and (38), we have

$$\begin{aligned}
\mathbb{P}_\epsilon(s,u;t,v) \le& c_{13}\epsilon^d f_\epsilon\big(|F_1(0,u]-F_1(0,s]|\\
&+|F_2(0,v_1]-F_2(0,t_1]|\\
&+|F_2(0,v_2]-F_2(0,t_2]|\big).
\end{aligned} \tag{39}$$

The second case is that $t_2 \ge v_2$ and $t_1 \le v_1$. We can replace the $H_i^{(j)}$'s of (35) with the following:

$$\begin{aligned}
&H_1^{(1)} \hat{=} \widehat{W}^{(1)}([0,s]\times[0,t_1]),\ H_2^{(1)} \hat{=} \widehat{W}^{(1)}([0,s]\times[t_1,v_1]),\\
&H_3^{(1)} \hat{=} \widehat{W}^{(1)}([s,u]\times[0,v_1]); H_1^{(2)} \hat{=} \widehat{W}^{(2)}([0,s]\times[0,v_2]),\\
&H_2^{(2)} \hat{=} \widehat{W}^{(2)}([0,s]\times[v_2,t_2]),\ H_3^{(2)} \hat{=} \widehat{W}^{(2)}([s,u]\times[0,v_2]).
\end{aligned}$$

It follows then that

$$\begin{aligned}
&\mathbb{P}_\epsilon(s,u;t,v)\\
=&\mathbb{P}\big\{|H_1^{(2)}+H_2^{(2)}-H_1^{(1)}|\le\epsilon; |H_1^{(2)}+H_3^{(2)}\\
&-H_1^{(1)}-H_2^{(1)}-H_3^{(1)}|\le\epsilon\big\}.
\end{aligned} \tag{40}$$

As the density function of $H_1^{(2)} - H_1^{(1)}$ is bound above by $K \hat{=} (2\pi F_1(0,a]F_2(0,a])^{-d/2}$. Therefore, by the unimodality of centered multivariate Gaussian distributions,

$$\begin{aligned}&\mathbb{P}_\epsilon(s,u;t,v)\\ =&c_{14}\int_{\mathbb{R}^d}\mathbb{P}\big\{|H_2^{(2)}+z|\le\epsilon;|H_3^{(2)}-H_2^{(1)}\\ &\quad-H_3^{(1)}+z|\le\epsilon\big\}dz\\ \le&c_{14}\int_{\{|\omega|\le\epsilon\}}\mathbb{P}\big\{|H_3^{(2)}-H_2^{(2)}-H_2^{(1)}\\ &\quad-H_3^{(1)}+\omega|\le\epsilon\big\}d\omega\\ \le&c_{14}(2\epsilon)^d\mathbb{P}\big\{|H_3^{(2)}-H_2^{(2)}-H_2^{(1)}\\ &\quad-H_3^{(1)}|\le\epsilon\big\}.\end{aligned}\tag{41}$$

As $\Gamma_3\hat{=}H_3^{(2)}-H_2^{(2)}-H_2^{(1)}-H_3^{(1)}$ has i.i.d. mean 0 coordinates , we can show that for all $1\le i\le d$,

$$\begin{aligned}\mathrm{Var}(\Gamma_3^{(i)})=&(F_1(0,u]-F_1(0,s])F_2(0,v_2]\\ &+F_1(0,s](F_2(0,t_2]-F_2(0,v_2])\\ &+F_1(0,s](F_2(0,v_1]-F_2(0,t_1])\\ &+(F_1(0,u]-F_1(0,s])F_2(0,v_1]\\ \ge&\min\{F_1(0,a],2F_2(0,a]\}\big(|F_1(0,u]\\ &-F_1(0,s]|+|F_2(0,v_1]-F_2(0,t_1]|\\ &+|F_2(0,t_2]-F_2(0,v_2]|\big).\end{aligned}\tag{42}$$

By (41), (42), and Lemma 7, we have

$$\begin{aligned}\mathbb{P}_\epsilon(s,u;t,v)\le&c_{15}\epsilon^d f_\epsilon\big(|F_1(0,u]-F_1(0,s]|\\ &+|F_2(0,v_1]-F_2(0,t_1]|\\ &+|F_2(0,t_2]-F_2(0,v_2]|\big).\end{aligned}\tag{43}$$

Symmetry considerations, together with Cases 1 and 2, prove that (39) and (43) holds for all possible configurations of (s,u,t,v). It follows from (39), (43), and Fubini-Tonelli theorem,

$$\begin{aligned}&\mathbb{E}[\hat{J}_\epsilon(\mu)]^2\\ \le&\frac{c_{16}}{\epsilon^d}\int\int\Big[\int\int_{[a,c]^2\times[a,c]^2}f_\epsilon\big(|F_1(0,u]-F_1(0,s]|\\ &+|F_2(0,v_1]-F_2(0,t_1]|+|F_2(0,t_2]-F_2(0,v_2]|\big)\\ &dF_2(0,t_1]dF_2(0,t_2]dF_2(0,v_1]dF_2(0,v_2]\Big]\mu(ds)\mu(du)\\ \le&\frac{c_9}{\epsilon^d}\int\int G_\epsilon\big(F_1(0,u]-F_1(0,s]\big)\mu(ds)\mu(du).\end{aligned}$$

This is the first inequality of the lemma. By the above inequality and Lemma 9, we have

$$\begin{aligned}\mathbb{E}[\hat{J}_\epsilon(\mu)]^2\le c_{10}&\int\int U_{\beta(d-4)/2}(u-s)\mu(ds)\mu(du)\\ &=c_{10}I_{\beta(d-4)/2}(\mu).\end{aligned}$$

This completes our proof. □

For all $s,t,u,v\in[0,\infty)$, define the partial order as follows: $(s,t)\preccurlyeq(u,v)$ if and only if $s\le u$ and $t\le v$. For all $i\in\{1,2\}$ and $s,t\in[0,\infty)$, define

$$\mathcal{F}_{s,t}^{(i)}\hat{=}\sigma\big(\widetilde{W}^{(i)}(u,v):0\le u\le s;0\le v\le t\big).$$

We can assume that $\mathcal{F}^{(i)}$'s are complete and right-continuous in the partial order " $\preccurlyeq$ ". If not, then complete $\mathcal{F}^{(i)}$ and then make it $\preccurlyeq$-right continuous. Based on $\mathcal{F}^{(1)}$ and $\mathcal{F}^{(2)}$, we define

$$\mathcal{F}_{s;t,v}\hat{=}\mathcal{F}_{s,t}^{(1)}\vee\mathcal{F}_{s,v}^{(2)}\quad\text{for all } s,t,v\ge0.$$

The following is the Cairoli's maximal inequality with respect to the family of $\mathcal{F}_{s;t,v}$'s.

Lemma 13 *Choose and fix a number $p>1$. Then for almost surely nonnegative random variable $Y\in\mathcal{L}^p\hat{=}L^p(\Omega,\vee\mathcal{F}_{s;t,v\ge0},\mathbb{P})$,*

$$\mathbb{E}\Big[\sup_{s,t,v\in\mathbb{Q}_+}\mathbb{E}\big[Y|\mathcal{F}_{s;t,v}\big]\Big]^p\le\Big(\frac{p}{p-1}\Big)^{3p}\mathbb{E}\big[Y^p\big].$$

Proof: By using Corollary 3.5.1 in [15, p.37], it suffices to show that for all $s,s',t,t',v,v'\ge0$, and all bounded random variables Y that are $\mathcal{F}_{s',t',v'}$-measurable,

$$\mathbb{E}\Big[Y|\mathcal{F}_{s;t,v}\Big]=\mathbb{E}\Big[Y|\mathcal{F}_{s\wedge s';t\wedge t',v\wedge v'}\Big]\ \ a.s.\tag{44}$$

This proves that the three-parameter filtration $\{F_{s;t,v}:s,t,v\in\mathbb{Q}_+\}$ is commuting in the sense of Khoshnevisan [15, p. 35]. By a density argument, it suffices to prove (44) in the case that $Y=Y_1Y_2$, where Y_1 and Y_2 are bounded, and measurable with respect to $\mathcal{F}_{s',t'}^{(1)}$ and $\mathcal{F}_{s',v'}^{(2)}$, respectively. By the independence of $\mathcal{F}_{s',t'}^{(1)}$ and $\mathcal{F}_{s',v'}^{(2)}$,

$$\mathbb{E}\Big[Y|\mathcal{F}_{s;t,v}\Big]=\mathbb{E}\Big[Y_1|\mathcal{F}_{s,t}^{(1)}\Big]\mathbb{E}\Big[Y_2|\mathcal{F}_{s,v}^{(2)}\Big]\ \ a.s.\tag{45}$$

By the Cairoli-Walsh commutation theorem [15, Theorem 2.4.1, p. 237], $F^{(1)}$ and $F^{(2)}$ are each two-parameter, commuting filtrationes. It follows from Theorem 3.4.1 in [15, p.36] that

$$\begin{aligned}\mathbb{E}\Big[Y_1|\mathcal{F}_{s,t}^{(1)}\Big]&=\mathbb{E}\Big[Y_1|\mathcal{F}_{s\wedge s',t\wedge t'}^{(1)}\Big]\ \ a.s.\\ \mathbb{E}\Big[Y_2|\mathcal{F}_{s,t}^{(2)}\Big]&=\mathbb{E}\Big[Y_2|\mathcal{F}_{s\wedge s',v\wedge v'}^{(2)}\Big]\ \ a.s.\end{aligned}\tag{46}$$

Combining (45) and (46), we obtain (44) in the case that Y has the special form $Y=Y_1Y_2$ as described above. The general form of (44) follows from the mentioned special case and density. □

Lemma 14 *Choose and fix a number* $p > 1$. *Then for almost surely nonnegative random variable* $Y \in \mathcal{L}^p \hat{=} L^p(\Omega, \vee\mathcal{F}_{s;t,v\geq 0}, \mathbb{P})$*, there exists a continuous modification of the three-parameter process* $\{\mathbb{E}[Y|\mathcal{F}_{s;t,v}], s, t, v \geq 0\}$*, such that*

$$\mathbb{E}\Big[\sup_{s,t,v\in\mathbb{R}_+} \mathbb{E}\big[Y|\mathcal{F}_{s;t,v}\big]\Big]^p \leq \Big(\frac{p}{p-1}\Big)^{3p}\mathbb{E}\big[Y^p\big].$$

Proof: By using (44), Lemma 9 and a similar argument as in the proof of Lemma 2,3 of Chen and Liu [7], we can also prove Lemma 14. □

Lemma 15 *Choose and fix three numbers* $0 < a < b < c < \infty$ *with* $c - a \leq \delta$ *. Then, there exists a constant* c_{17} *such that the following holds outside a single null set: For all* $0 < \epsilon < 1,\ a \leq a_1, b_1, b_2 \leq b$*, and* $\mu \in \mathcal{P}(\mathbb{R}_+)$*,*

$$\mathbb{E}[\hat{J}_\epsilon(\mu)|\mathcal{F}_{a_1;b_1,b_2}] \geq \frac{c_{17}}{\epsilon^d}\int_{F\cap[a_1,b]} G_\epsilon(F_1(0,s] - F_1(0,a_1]) \times \mu(ds)\mathbf{1}_{A(\epsilon/2;a_1,b_1,b_2)}. \quad (47)$$

Proof: It follows from (32) that

$$\begin{aligned}&\mathbb{E}[\hat{J}_\epsilon(\mu)|\mathcal{F}_{a_1;b_1,b_2}]\\ \geq& \frac{1}{\epsilon^d}\int_{b_1}^c\int_{b_2}^c\int_{F\cap[a_1,b]} \mathbb{P}\big(\mathbf{A}(\epsilon;s,t)|\mathcal{F}_{a_1;b_1,b_2}\big)\\ &\times \mu(ds)dF_2(0,t_2]dF_2(0,t_1]. \qquad (48)\end{aligned}$$

A generalized white-noise decomposition implies that following: For all $s \geq a_1$, $t_1 \geq b_1$, and $t_2 \geq b_2$,

$$\begin{aligned}\widetilde{W}^{(1)}(s,t_1) =& \widetilde{W}^{(1)}(a_1,b_1) + \widehat{W}^{(1)}([a_1,s]\times[0,b_1])\\ &+ \widehat{W}^{(1)}([0,a_1]\times[b_1,t_1])\\ &+ \widehat{W}^{(1)}([a_1,s]\times[b_1,t_1]), \qquad (49)\\ \widetilde{W}^{(2)}(s,t_2) =& \widetilde{W}^{(2)}(a_1,b_2) + \widehat{W}^{(2)}([a_1,s]\times[0,b_2])\\ &+ \widehat{W}^{(2)}([0,a_1]\times[b_2,t_2])\\ &+ \widehat{W}^{(2)}([a_1,s]\times[b_2,t_2]). \qquad (50)\end{aligned}$$

Here, the $\widehat{W}^{(i)}$'s are generalized Brownian sheets. The collection $\{\widehat{W}_j^{(i)}, \widetilde{W}^{(i)}(a_1,b_i) : i,j = 1,2\}$ is totally independent. By (48), (49) and (50), we can infer that the following is a lower bound for the right of (48), almost surely on the event $\mathbf{A}(\epsilon/2, a_1; b_1, b_2)$,

$$\begin{aligned}&\frac{1}{\epsilon^d}\int_{b_1}^c\int_{b_2}^c\int_{F\cap[a_1,b]}\mu(ds)dF_2(0,t_2]dF_2(0,t_1]\\ &\times\mathbb{P}\Big\{\Big|\widehat{W}^{(1)}([a_1,s]\times[0,b_1]) + \widehat{W}^{(1)}([0,a_1]\\ &\times[b_1,t_1]) + \widehat{W}^{(1)}([a_1,s]\times[b_1,t_1])\\ &- \widehat{W}^{(2)}([a_1,s]\times[0,b_2]) - \widehat{W}^{(2)}([0,a_1]\\ &\times[b_2,t_2]) - \widehat{W}^{(2)}([a_1,s]\times[b_2,t_2])\Big| \leq \frac{\epsilon}{2}\Big\}\\ =& \frac{1}{\epsilon^d}\int_{b_1}^c\int_{b_2}^c\int_{F\cap[a_1,b]}\mathbb{P}\Big\{\sigma|\widetilde{W}(\langle 1\rangle)| \leq \frac{\epsilon}{2}\Big\}. \quad (51)\end{aligned}$$

Here,

$$\begin{aligned}\sigma^2 =& (F_1(0,s] - F_1(0,a_1])(F_2(0,t_1] - F_2(0,b_1])\\ &+ (F_1(0,s] - F_1(0,a_1])F_2(0,b_1]\\ &+ F_1(0,a_1](F_2(0,t_1] - F_2(0,b_1])\\ &+ (F_1(0,s] - F_1(0,a_1])(F_2(0,t_2] - F_2(0,b_2])\\ &+ (F_1(0,s] - F_1(0,a_1])F_2(0,b_2]\\ &+ F_1(0,a_1](F_2(0,t_2] - F_2(0,b_2]).\end{aligned}$$

The range of possible values of a_1, b_1 and b_2 is respectively $[a,b]$ and $[a,b]^2$. Therefore, there exists a constant, which only depends on the parameters a and b, such that

$$\begin{aligned}\sigma^2 \leq& \min\{F_1(0,b], 2F_2(0,b]\}(|F_1(0,s]\\ &- F_1(0,a_1]| + |F_2(0,t_1] - F_2(0,b_1]|\\ &+ |F_2(0,t_2] - F_2(0,b_2]|). \qquad (52)\end{aligned}$$

Applying the bound of (52) and Lemma 9, we can find that (47) holds a.s., but the null set could feasibly depend on $(a_1, b_1, b_2, \epsilon)$.

To ensure that the null set can be chosen independently from $(a_1, b_1, b_2, \epsilon)$, we first note that the integral on the right-hand side of (47) is:

(i) Continuous in $\epsilon > 0$;

(ii) independent of $b_1, b_2 \in [a,b]$;

(iii) lower semi-continuous in $a_1 \in [a,b]$.

Similarly, $(a_1, b_1, b_2, \epsilon) \mapsto \mathbf{1}_{\mathbf{A}(\epsilon;a_1,b_1,b_2)}$ is left-continuous in $\epsilon > 0$ and lower semi-continuous in $(a_1, b_1, b_2) \in [a,b]^3$. Therefore, it suffices to prove that the left-hand side of (47) is a.s. continuous in $(a_1, b_1, b_2) \in [a,b]^3$, and left-continuous in $\epsilon > 0$. The left-continuity assertion about $\epsilon > 0$ is evident; continuous in (a_1, b_1, b_2) follows if we could prove that for all bounded random variables Y, $(a_1, b_1, b_2) \mapsto \mathbb{E}[Y|\mathcal{F}_{a_1;b_1,b_2}]$ has an a.s.-continuous modification. But this follows from Lemma 14. □

Lemma 16 *Choose and fix two numbers* $0 < a < b < \infty$ *with* $b - a \leq \delta$, *and*

$$D(\omega) \hat{=} \Big\{ a \leq s \leq b : \inf_{t_1, t_2 \in [a,b]} |\widetilde{W}^{(2)}(s, t_2) - \widetilde{W}^{(1)}(s, t_1)|(\omega) = 0 \Big\}.$$

Then, there exist positive constants c_{18} *and* c_{19} *such that for all compact, non-random sets* $F \subset [a, b]$,

$$c_{18} \mathrm{Cap}_{\beta(d-4)/2}(F) \leq \mathbb{P}\{D \cap F \neq \emptyset\} \leq c_{19} \mathrm{Cap}_{\alpha(d-4)/2}(F). \tag{53}$$

Proof: For all $0 < \epsilon < 1$, define the closed random sets

$$D_\epsilon(\omega) \hat{=} \Big\{ a \leq s \leq b : \inf_{t_1, t_2 \in [a,b]} |\widetilde{W}^{(2)}(s, t_2) - \widetilde{W}^{(1)}(s, t_1)|(\omega) \leq \epsilon \Big\}.$$

Choose and fix a probability measure $\mu \in \mathcal{P}(F)$ such that D_ϵ intersects F almost surely on the event $\{J_\epsilon(\mu) > 0\}$. By the Cauchy-Schwarz inequality,

$$\Big(\mathbb{E}[J_\epsilon(\mu)] \Big)^2 = \Big(\mathbb{E}[J_\epsilon(\mu); D_\epsilon \cap F \neq \emptyset] \Big)^2 \leq \mathbb{E}[J_\epsilon(\mu)]^2 \mathbb{P}(D_\epsilon \cap F \neq \emptyset). \tag{54}$$

It follows from (27), (32) and (54) that

$$\mathbb{P}(D_\epsilon \cap F \neq \emptyset) \geq \frac{\big(\mathbb{E}[J_\epsilon(\mu)]\big)^2}{\mathbb{E}[J_\epsilon(\mu)]^2} \geq \frac{\big(\mathbb{E}[J_\epsilon(\mu)]\big)^2}{\mathbb{E}[\hat{J}_\epsilon(\mu)]^2}.$$

Let $\epsilon \downarrow 0$ and appeal to compactness to find that

$$\mathbb{P}(D \cap F \neq \emptyset) \geq \frac{\liminf_{\epsilon \to 0} \big(\mathbb{E}[J_\epsilon(\mu)]\big)^2}{K I_{\beta(d-4)/2}(\mu)}.$$

Here, we have used Lemma 12. According to Lemma 11, the numerator is bounded below by a strictly positive number that does not depend on μ. Therefore, the lower bound of Lemma 16 follows from optimizing over all $\mu \in \mathcal{P}(F)$.

In order to obtain the upper bound, without any loss in generality, we can assume that $\mathbb{P}(D_\epsilon \cap F \neq \emptyset) > 0$. For all $0 < \epsilon < 1$, define

$$\tau_\epsilon \hat{=} \inf \Big\{ s \in F : \inf_{t_1, t_2 \in [a,b]} |\widetilde{W}^{(2)}(s, t_2) - \widetilde{W}^{(1)}(s, t_1)| \leq \epsilon \Big\}.$$

As usual, $\inf \emptyset \hat{=} \infty$. Define

$$\mathcal{F}_s \hat{=} \bigvee_{t,v \in [0,\infty)^2} \mathcal{F}_{s;t,v} \quad \text{for all } s \geq 0.$$

We note that τ_ϵ is a stopping time with respect to the one-parameter filtration $\{\mathcal{F}_s : s \geq 0\}$. It is also easy to see that there exist $[0, \infty]$-valued random variables τ'_ϵ and τ''_ϵ such that: (i) $\tau'_\epsilon \vee \tau''_\epsilon = \infty$ if and only if $\tau_\epsilon = \infty$; and (ii) almost surely on $\{\tau_\epsilon < \infty\}$,

$$|\widetilde{W}^{(2)}(\tau_\epsilon, \tau'_\epsilon) - \widetilde{W}^{(1)}(\tau_\epsilon, \tau''_\epsilon)| \leq \epsilon.$$

Define

$$p_\epsilon \hat{=} \mathbb{P}\{\tau_\epsilon < \infty\}, \text{ and } \nu_\epsilon(\bullet) \hat{=} \mathbb{P}\{\tau_\epsilon \in \bullet | \tau_\epsilon < \infty\}.$$

By the assumption and the definition of the classical conditional probability,

$$\inf_{\epsilon > 0} p_\epsilon \geq \mathbb{P}(D \cap F \neq \emptyset) > 0, \text{ and } \nu_\epsilon \in \mathcal{P}(F). \tag{55}$$

Now consider the process $\{M^\epsilon, 0 < \epsilon < 1\}$ defined by

$$M^\epsilon_{a_1;b_1,b_2} \hat{=} \mathbb{E}[\hat{J}_\epsilon(\nu_\epsilon) | \mathcal{F}_{a_1;b_1,b_2}].$$

By Lemmas 14, 15, and the Cauchy-Schwarz inequality,

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{a_1, b_1, b_2 \in \mathbb{R}_+} \big(M^\epsilon_{a_1;b_1,b_2} \big)^2 \Big] \\
\geq &\mathbb{E}\Big[\big(M^\epsilon_{\tau_\epsilon, \tau''_\epsilon, \tau'_\epsilon} \big)^2 \Big] \\
= &\mathbb{E}\Big[\mathbb{E}\big[\hat{J}_\epsilon(\nu_\epsilon) | \mathcal{F}_{\tau_\epsilon, \tau''_\epsilon, \tau'_\epsilon} \big] \Big]^2 \\
\geq &\frac{c_{20} p_\epsilon}{\epsilon^{2d}} \mathbb{E}\Big[\Big(\int_{F \cap [\tau_\epsilon, b]} G_\epsilon(F_1(0, s] - F_1(0, \tau_\epsilon]) \nu_\epsilon(ds) \Big)^2 | \tau_\epsilon < \infty \Big] \\
\geq &\frac{c_{20} p_\epsilon}{\epsilon^{2d}} \Big(\mathbb{E}\Big[\int_{F \cap [\tau_\epsilon, b]} G_\epsilon(F_1(0, s] - F_1(0, \tau_\epsilon]) \nu_\epsilon(ds) | \tau_\epsilon < \infty \Big] \Big)^2.
\end{aligned} \tag{56}$$

In addition,

$$\begin{aligned}
&\mathbb{E}\Big[\int_{F \cap [\tau_\epsilon, b]} G_\epsilon(F_1(0, s] - F_1(0, \tau_\epsilon]) \nu_\epsilon(ds) | \tau_\epsilon < \infty \Big] \\
= &\int \int_{\{s \in F \cap [u, 2]\}} G_\epsilon(F_1(0, s] - F_1(0, u]) \nu_\epsilon(ds) \nu_\epsilon(du) \\
\geq &\frac{1}{2} \int \int G_\epsilon(F_1(0, s] - F_1(0, u]) \nu_\epsilon(ds) \nu_\epsilon(du).
\end{aligned} \tag{57}$$

Combining Lemmas 12, 13, (56) and (57),

$$\frac{c_{21}p_\epsilon}{4\epsilon^{2d}}\Big(\int\int G_\epsilon(F_1(0,s]-F_1(0,u])\nu_\epsilon(ds)\nu_\epsilon(du)\Big)^2$$
$$\le\mathbb{E}\Big[\sup_{a_1,b_1,b_2\in\mathbb{Q}_+}\big(M^\epsilon_{a_1;b_1,b_2}\big)^2\Big]$$
$$\le 2^6\mathbb{E}\Big[\big(\hat{J}_\epsilon(\nu_\epsilon)\big)^2\Big]$$
$$\le\frac{c_{22}}{\epsilon^d}\int\int G_\epsilon(F_1(0,s]-F_1(0,u])\nu_\epsilon(ds)\nu_\epsilon(du). \tag{58}$$

Thanks to (23), (55) and (58),

$$\mathbb{P}(D\cap F\neq\emptyset)$$
$$\le c_{23}\epsilon^d\Big[\int\int G_\epsilon(F_1(0,s]-F_1(0,u])\nu_\epsilon(ds)\nu_\epsilon(du)\Big]^{-1}$$
$$\le c_{24}\Big[\int\int U_{\alpha(d-4)/2}(s-u)\nu_\epsilon(ds)\nu_\epsilon(du)\Big]^{-1}$$
$$=\frac{c_{25}}{I_{\alpha(d-4)/2}(\nu_\epsilon)}. \tag{59}$$

Choose and fix a number $\eta>0$. For all $0<\epsilon<\eta^{1/2}$,

$$I_{\alpha(d-4)/2}(\nu_\epsilon)\ge\int\int_{\{|s-u|\ge\eta\}}U_{\alpha(d-4)/2}(s-u)\times\nu_\epsilon(ds)\nu_\epsilon(du). \tag{60}$$

Recall that $\{\nu_\epsilon,\ \epsilon>0\}$ is a net of probability measures on F. Because F is compact, Prohorov's theorem ensures that there exists a subsequential weak limit $\nu_0\in\mathcal{P}(F)$ of $\{\nu_\epsilon,\ \epsilon>0\}$, as $\epsilon\to 0$. Therefore, we can apply Fatou's lemma to find that

$$\liminf_{\epsilon\to 0} I_{\alpha(d-4)/2}(\nu_\epsilon)$$
$$\ge\lim_{\eta\to 0}\int\int_{\{|s-u|\ge\eta\}}U_{\alpha(d-4)/2}(s-u)\nu_0(ds)\nu_0(du)$$
$$=I_{\alpha(d-4)/2}(\nu_0). \tag{61}$$

By (59), (60) and (61),

$$\mathbb{P}(D\cap F\neq\emptyset)\le\frac{c_{25}}{I_{\alpha(d-4)/2}(\nu_0)}\le c_{25}\mathrm{Cap}_{\alpha(d-4)/2}(F).$$

This finished the proof of Lemma 16. □

Proof of Theorem 3. Let I and J be disjoint, closed intervals in $(0,\infty)$ with the following property that $x<y$ for all $x\in I$ and $y\in J$. Define

$$D_d(I,J)\hat{=}\Big\{a\le s\le b:\widetilde{W}(s,t_1)=\widetilde{W}(s,t_2)\text{ for some }t_1\in I\text{ and }t_2\in J\Big\}.$$

It suffices to prove that

$$K_3\mathrm{Cap}_{\beta(d-4)/2}(F)\le\mathbb{P}\{D_d(I,J)\cap F\neq\emptyset\}\le K_4\mathrm{Cap}_{\alpha(d-4)/2}(F). \tag{62}$$

Choose and fix two number $0<a<b<\delta$ with $b=2a$. Moreover, without loss of much generality, we may assume that

$$I=[\frac{b-a}{2},\frac{b+a}{2}],\ J=[\frac{3b+a}{2},\frac{4b+a}{2}]\subset[0,\delta]$$

and $F\subseteq[a,b]$. Now define the random fields,

$$\widetilde{W}^{(1)}(s,t)\hat{=}\widetilde{W}\Big(s,\tfrac{b+2a}{2}-t\Big)-\widetilde{W}\Big(s,\tfrac{b+2a}{2}\Big),$$
$$\widetilde{W}^{(2)}(s,t)\hat{=}\widetilde{W}\Big(s,\tfrac{b+2a}{2}+t\Big)-\widetilde{W}\Big(s,\tfrac{b+2a}{2}\Big)$$

for $0\le s,t\le\frac{b+2a}{2}$. By checking two covariances, we can show that

$$\big\{\widetilde{W}\big(s,\frac{b+2a}{2}-t\big)-\widetilde{W}\big(s,\frac{b+2a}{2}\big):\ a\le s,t\le b\big\},$$
$$\big\{\widetilde{W}\big(s,\frac{b+2a}{2}+t\big)-\widetilde{W}\big(s,\frac{b+2a}{2}\big):\ a\le s,t\le b\big\}$$

are independent generalized Brownian sheets. On the other hand, the following two results are equivalent:

(i) There exists $s,t_1,t_2\in[a,b]^3$ such that $\widetilde{W}^{(1)}(s,t_1)=\widetilde{W}^{(2)}(s,t_2)$;

(ii) There exists $(s,t_1,t_2)\in[a,b]\times I\times J$ such that $\widetilde{W}(s,t_1)=\widetilde{W}(s,t_2)$.

Therefore, (62) follows from Lemma 16. This finished the proof of Theorem 3. □

4 Polar sets of $\hat{\mathcal{D}}_d$ and $\bar{\mathcal{D}}_d$

An N-parameter, $\mathbb{R}^d$-valued generalized Brownian sheet has the following stochastic integral representation

$$\widetilde{W}(t)=\int_0^{t_1}\cdots\int_0^{t_N}f_1(s_1)\cdots f_N(s_N)dW(s)\hat{=}\int_{[0,t]}f(s)dW(s), \tag{63}$$

where $W=\{W(s),s\in R^N_+\}$ is a Brownian sheet, and for any $k=1,\ldots,N$, $f_k(x)=\sqrt{\frac{dF_k}{dx}}$ is the

Radon-Nikodym derived function of F_k which is square integrable (cf. [1]).

Lemma 17 *For all $s,t \in \mathbb{R}_+^N$ with $s \preceq t$, and $r \in \mathbb{R}^d$. Then,*

$$\mathbb{E}\big\{\langle r, \widetilde{W}(t)\rangle | \mathcal{F}_u,\ 0 \preceq u \preceq s\big\} = \langle r, \widetilde{W}(s)\rangle. \quad (64)$$

Proof: For all $s,t \in \mathbb{R}_+^N$ with $s \preceq t$, we decompose the rectangle $[0,t]$ into the following disjoint union:

$$[0,t] = [0,s] \cup \bigcup_{j=1}^{N} R(t_j) \cup \triangle(s,t), \quad (65)$$

where $R(t_j) = \{(v_1,\cdots,v_N) : 0 \le v_i \le s_i,$ if $i \ne j,\ s_j < v_j \le t_j\}$ and $\triangle(s,t)$ can be written as a union of $2^N - N - 1$ sub-rectangles of $[0,t]$; see [18]. It follows from (63) and (65) that for all $s,t \in \mathbb{R}_+^N$ with $s \preceq t$,

$$\begin{aligned}\widetilde{W}(t) =& \int_{[0,s]} f(v)W(dv) + \sum_{j=1}^{N} \int_{R(t_j)} f(v)W(dv) \\ &+ \int_{\triangle(s,t)} f(v)W(dv) \\ \hat{=}& \widetilde{W}(s) + \sum_{j=1}^{N} Y_j(t) + Z(s,t).\end{aligned}$$

Since the processes $\widetilde{W}(s),\ Y_j(t)\ (1 \le j \le N)$ and $Z(s,t)$ are defined by the stochastic integrals over disjoint sets, they are independent. Therefore, $Y_j(t)\ (1 \le j \le N)$ and $Z(s,t)$ are independent with $\mathcal{F}(s)$. This completes the proof of Lemma 17. □

Let $\widetilde{W}^{(1)}$ and $\widetilde{W}^{(2)}$ be two independent, two-parameter generalized Brownian sheets in $\mathbb{R}^d$. For all $0 < \epsilon < 1$, $0 \le s, t_1, t_2 < \infty$ and $x = (x_1,x_2,x_3) \in \mathbb{R}_+^3$, define

$$\begin{aligned}&\Delta(s,t_1,t_2) \hat{=} \widetilde{W}^{(2)}(s,t_2) - \widetilde{W}^{(1)}(s,t_1), \\ &\flat(x;\epsilon) \hat{=} [x_1, x_1+\epsilon] \times [x_2, x_2+\epsilon] \times [x_3, x_3+\epsilon].\end{aligned}$$

Lemma 18 *Choose and fix two numbers $0 < a < b < \infty$. Then, there exists a constant c_{26} such that for all $0 < \epsilon < \delta,\ x = (x_1,x_2,x_3) \in [a,b]^3$,*

$$\mathbb{E}\Big[\sup_{y \in \flat(x;\epsilon)} |\Delta(x) - \Delta(y)|\Big]^d \le c_{26}\epsilon^{\frac{\alpha d}{2}}. \quad (66)$$

Proof: By some elementary estimates, we have

$$\begin{aligned}&\mathbb{E}\Big[\sup_{y \in \flat(x;\epsilon)} |\Delta(x) - \Delta(y)|\Big]^d \\ &\le 2^{d-1}\Big(\mathbb{E}\Big[\sup_{y \in \flat(x;\epsilon)} |\widetilde{W}^{(1)}(x_1,x_2) - \widetilde{W}^{(1)}(y_1,y_2)|^d\Big] \\ &\quad + \mathbb{E}\Big[\sup_{y \in \flat(x;\epsilon)} |\widetilde{W}^{(2)}(x_1,x_3) - \widetilde{W}^{(2)}(y_1,y_3)|^d\Big]\Big) \\ &\hat{=} 2^{d-1}(I_1 + I_2). \quad (67)\end{aligned}$$

Let $S^d := \{r \in \mathbb{R}^d : |r| = 1\}$. It follows from the Hölder inequality that

$$\begin{aligned}I_1 =& \mathbb{E}\Big[\sup_{y \in \flat(x;\epsilon)} \Big(\sum_{i=1}^{d} \Big(\widetilde{W}_i^{(1)}(x_1,x_2) - \widetilde{W}_i^{(1)}(y_1,y_2)\Big)^2\Big)^{\frac{d}{2}}\Big] \\ \le& d^{\frac{d}{2}} \sup_{r \in S^d} \mathbb{E}\Big[\sup_{y \in \flat(x;\epsilon)} \Big|\langle r, \widetilde{W}^{(1)}(x_1,x_2) - \widetilde{W}^{(1)}(y_1,y_2)\rangle\Big|^d\Big].\end{aligned} \quad (68)$$

By (68), Lemma 17 and the Doob-Cairoli inequality, we can deduce that

$$\begin{aligned}I_1 \le& d^{\frac{d}{2}}\Big(\frac{d}{d-1}\Big)^{2d} \sup_{r \in S^d} \mathbb{E}\Big[\Big|\langle r, \Xi(x_1,x_2;\epsilon)\rangle\Big|^d\Big] \\ =& d^{\frac{d}{2}}\Big(\frac{d}{d-1}\Big)^{2d} \mathbb{E}\Big[|\Xi(x_1,x_2;\epsilon)|\Big]^d, \quad (69)\end{aligned}$$

where $\Xi(x_1,x_2;\epsilon) \hat{=} \widetilde{W}^{(1)}(x_1,x_2) - \widetilde{W}^{(1)}(x_1+\epsilon, x_2+\epsilon)$.

In accord with the inequality [7, Lemma 3.2], there exists a constant $c_{27}(a,b,\delta)$ such that for all $0 < \epsilon < \delta,\ (x_1,x_2,x_3) \in [a,b]^3$,

$$\sigma^2 \hat{=} \mathbb{E}\big[\Xi_j^2(x_1,x_2;\epsilon)\big] \le c_{27}\epsilon^{\alpha}. \quad (70)$$

Since

$$\sum_{j=1}^{d} \Big(\frac{\Xi_j(x_1,x_2;\epsilon)}{\sigma}\Big)^2 \sim \chi^2(d).$$

Then there exists a constant $c_{28}(a,b,d)$ such that

$$\mathbb{E}\left[\sum_{j=1}^{d} \Big(\frac{\Xi_j(x_1,x_2;\epsilon)}{\sigma}\Big)^2\right]^{\frac{d}{2}} \le c_{28}. \quad (71)$$

By (69), (70) and (71),

$$\begin{aligned}I_1 \le& d^{\frac{d}{2}}\Big(\frac{d}{d-1}\Big)^{2d} (\sigma^2)^{\frac{d}{2}} \mathbb{E}\left[\sum_{j=1}^{d} \Big(\frac{\Xi_j(x_1,x_2;\epsilon)}{\sigma}\Big)^2\right]^{\frac{d}{2}} \\ \le& \big(c_{27}c_{28}d\big)^{\frac{d}{2}} \Big(\frac{d}{d-1}\Big)^{2d} \epsilon^{\frac{\alpha d}{2}}. \quad (72)\end{aligned}$$

Similarly, we can apply the same method above to obtain that

$$I_2 \leq \left(c_{27}c_{28}d\right)^{\frac{d}{2}}\Big(\frac{d}{d-1}\Big)^{2d}\epsilon^{\frac{\alpha d}{2}}. \tag{73}$$

Therefore, (66) follows from (67), (72), and (73). This finished the proof of Lemma 18. □

Let $\widetilde{W}^{(1)}$ and $\widetilde{W}^{(2)}$ be two independent generalized Brownian sheets in $\mathbb{R}^d$, and define for all $\mu \in \mathcal{P}(\mathbb{R}^2_+)$,

$$\mathcal{I}_\epsilon(\mu) \hat{=} \frac{1}{\epsilon^d}\int_a^b \int\int \mathbf{1}_{\mathbf{A}(\epsilon;s,t)}\mu(dsdt_1)dF_2(0,t_2], \tag{74}$$

where $\mathbf{A}(\epsilon;s,t)$ is the event

$$\mathbf{A}(\epsilon;s,t) \hat{=} \big\{|\widetilde{W}^{(2)}(s,t_2) - \widetilde{W}^{(1)}(s,t_1)| \leq \epsilon\big\},$$

for all $a \leq s, t_1, t_2 \leq b$ and $\epsilon > 0$.

Lemma 19 *Choose and fix two numbers* $0 < a < b < \infty$ *. Then,*

$$\inf_{0<\epsilon<1}\inf_{\mu\in\mathcal{P}([a,b])}\mathbb{E}[\mathcal{I}_\epsilon(\mu)] > 0. \tag{75}$$

Proof: For all $s, t_1, t_2 \in [a,b]$, let

$$\sigma^2 \hat{=} \mathrm{Var}\big(\widetilde{W}^{(2)}_j(s,t_2) - \widetilde{W}^{(1)}_j(s,t_1)\big),\ 1 \leq j \leq d.$$

The independence of $\widetilde{W}^{(1)}$ and $\widetilde{W}^{(2)}$ implies

$$2F_1(0,a]F_2(0,a] \leq \sigma^2 \leq 2F_1(0,b]F_2(0,b]. \tag{76}$$

By (76), we have

$$\begin{aligned}\mathbb{P}\big\{\mathbf{A}(\epsilon;s,t)\big\} &\geq \Big[\int_{-\epsilon/d}^{\epsilon/d}\frac{1}{\sqrt{2\pi}\sigma}\exp\Big(-\frac{u^2}{2\sigma^2}\Big)du\Big]^d \\ &\geq \Big(\frac{\varepsilon^2}{\pi F_1(0,b]F_2(0,b]d^2}\Big)^{d/2} \\ &\quad\times\exp\Big(-\frac{\epsilon^2}{4dF_1(0,a]F_2(0,a]}\Big). \end{aligned}\tag{77}$$

By (74) and (77), we have

$$\begin{aligned}\mathbb{E}[\mathcal{I}_\epsilon(\mu)] &= \frac{1}{\epsilon^d}\int_a^b\iint \mathbb{P}\big\{\mathbf{A}(\epsilon;s,t)\big\}\mu(dsdt_1)dF_2(0,t_2] \\ &\geq \frac{(F_2(0,b]-F_2(0,a])\mu([a,b]^2)}{\big(\pi F_1(0,b]F_2(0,b]d\big)^{d/2}} \\ &\quad\times\exp\Big(-\frac{\epsilon^2}{4dF_1(0,a]F_2(0,a]}\Big).\end{aligned}$$

This proves the lemma. □

Next we present a bound for the second moment of $\mathcal{I}_\epsilon(\mu)$. For technical reason, we first define

$$\hat{\mathcal{I}}_\epsilon(\mu) \hat{=} \frac{1}{\epsilon^d}\int_a^c\int\int \mathbf{1}_{\mathbf{A}(\epsilon;s,t)}\mu(dsdt_1)dF_2(0,t_2],$$

Lemma 20 *Choose and fix three numbers* $0 < a < b < c < \infty$ *with* $c - a \leq \delta$ *. Then, there exists a constant* c_{29} *such that for all Borel probability measures* μ *on* $\mathbb{R}^2_+$ *and all* $0 < \epsilon < 1$,

$$\begin{aligned}\mathbb{E}[\hat{\mathcal{I}}_\epsilon(\mu)]^2 \leq& \frac{c_{29}}{\epsilon^d}\int\int g_\epsilon(|F_1(0,s]-F_1(0,u]| \\ &+|F_2(0,t_1]-F_2(0,v_1]|)\mu(dsdt_1)\mu(dudv_1) \\ \leq& c_{29}I_{\beta(d-2)/2}(\mu).\end{aligned}\tag{78}$$

Proof: For all $\epsilon > 0$, $s,u \in [a,b]$, and $t = (t_1,t_2)$, $v = (v_1,v_2) \in [a,b]\times[b,c]$, it follows from (43) that

$$\begin{aligned}&\mathbb{P}_\epsilon(s,u;t,v) \\ &\leq c_{30}\epsilon^d f_\epsilon\big(|F_1(0,u]-F_1(0,s]| + |F_2(0,v_1] \\ &\qquad - F_2(0,t_1]| + |F_2(0,t_2]-F_2(0,v_2]|\big).\end{aligned}$$

By the Fubini-Tonelli theorem and (2),

$$\begin{aligned}\mathbb{E}[\hat{\mathcal{I}}_\epsilon(\mu)]^2 \leq& \frac{c_{31}}{\epsilon^{2d}}\iint\Big[\int\limits_{[a,c]\times[a,c]}\mathbb{P}_\epsilon(s,u;t,v)dF_2(0,t_2] \\ &\times dF_2(0,v_2]\Big]\mu(dsdt_1)\mu(dudv_1) \\ \leq& \frac{c_{32}}{\epsilon^d}\iint g_\epsilon\big(|F_1(0,s]-F_1(0,u]|+|F_2(0,t_1] \\ &-F_2(0,v_1]|\big)\mu(dsdt_1)\mu(dudv_1) \\ \leq& \frac{c_{33}}{\epsilon^d}\int\int g_\epsilon\big((|s-u|+|t_1-v_1|)^\beta\big) \\ &\times\mu(dsdt_1)\mu(dudv_1).\end{aligned}\tag{79}$$

This is the first inequality of the lemma. By (79) and Lemma 12, we have

$$\begin{aligned}\mathbb{E}[\hat{\mathcal{I}}_\epsilon(\mu)]^2 \leq& c_{34}\int\int U_{\beta(d-2)/2}(|s-u|+|t_1-v_1|) \\ &\times\mu(dsdt_1)\mu(dudv_1) \\ &= c_{35}I_{\beta(d-2)/2}(\mu).\end{aligned}$$

This completes the lemma. □

Proof of Theorem 5 We first derive the lower bound in (11) and (12). Recall (74). Choose and fix $\mu \in \mathcal{P}(G)$, and define for all $\epsilon > 0$,

$$\mathcal{T}_\epsilon(\mu) \hat{=} \frac{1}{\epsilon^d}\int\int\int \mathbf{1}_{\mathbf{A}(\epsilon;s,t)}\mu(dsdt_1dt_2). \tag{80}$$

By the same argument as in the proof of Lemma 11, we can deduce that

$$\inf_{0<\epsilon<1} \inf_{\mu\in\mathcal{P}([a,b]^3)} \mathbb{E}[\mathcal{T}_\epsilon(\mu)] > 0. \tag{81}$$

Similarly, we can apply (3.19) to find that

$$\begin{aligned}\mathbb{E}[\mathcal{T}_\epsilon(\mu)]^2 \leq & \frac{c_{36}}{\epsilon^d} \int\int\int\int f_\epsilon\big(|F_1(0,u] - F_1(0,s]| \\ & + |F_2(0,v_1] - F_2(0,t_1]| + |F_2(0,v_2] \\ & - F_2(0,t_2]|\big)\mu(dsdt_1dt_2)\mu(dudv_1dv_2) \\ \leq & \frac{c_{37}}{\epsilon^d} \int\int\int\int f_\epsilon\big((|u-s| + |v_1 - t_1| \\ & + |v_2 - t_2|)^\beta\big)\mu(dsdt_1dt_2)\mu(dudv_1dv_2) \\ = & c_{38} I_{\beta d/2}(\mu). \end{aligned} \tag{82}$$

Here, we have used the inequality, $f_\epsilon(x) \leq \epsilon^d |x|^{-d/2}$. Define

$$\begin{aligned}\hat{\mathcal{D}}_\epsilon(\omega) \hat{=} \{ & (s,t_1,t_2) \in [a,b]^3 : |\widetilde{W}^{(1)}(s,t_1) \\ & - \widetilde{W}^{(2)}(s,t_2)|(\omega) \leq \epsilon\}, \\ \bar{\mathcal{D}}_\epsilon(\omega) \hat{=} \{ & (s,t_1) \in [a,b]^2 : \inf_{t_2\in[a,b]} |\widetilde{W}^{(1)}(s,t_1) \\ & - \widetilde{W}^{(2)}(s,t_2)|(\omega) \leq \epsilon\}.\end{aligned}$$

By (81), (82) and the Paley-Zygmund inequality to find that

$$\mathbb{P}(\hat{D}_\epsilon \cap G \neq \emptyset) \geq \frac{\big(\mathbb{E}[\mathcal{T}_\epsilon(\mu)]\big)^2}{\mathbb{E}[\mathcal{T}_\epsilon(\mu)]^2}.$$

Let $\epsilon \downarrow 0$ and appeal to compactness to find that

$$\mathbb{P}(\hat{\mathbb{D}}_d \cap F \neq \emptyset) \geq \frac{\liminf_{\epsilon\to 0} \big(\mathbb{E}[\mathcal{I}_\epsilon(\mu)]\big)^2}{K I_{\beta d/2}(\mu)}.$$

Therefore, the lower bound of (11) follows from optimizing over all $\mu \in \mathcal{P}(G)$. Applying Lemmas 19, 20, and repeating the same reason above, we can obtain the lower bound of (12).

Now we prove the upper bound of (11) and (12). For all $x \in [a,b]^3$ and $0 < \epsilon < \delta$, it follows from (66) that

$$\begin{aligned}&\mathbb{P}\big\{\hat{\mathbb{D}}_d \cap \flat(x;\epsilon) \neq \emptyset\big\} \\ \leq & \mathbb{P}\Big\{|\Delta(x)| \leq \sup_{y\in\flat(x;\epsilon)} |\Delta(y) - \Delta(x)|\Big\} \\ \leq & c_{39}\mathbb{E}\Big[\sup_{y\in\flat(x;\epsilon)} |\Delta(y) - \Delta(x)|\Big]^d \\ \leq & c_{40}\epsilon^{\alpha d/2}.\end{aligned} \tag{83}$$

In order to obtain the upper bound, without any loss in generality, we can assume that $\mathcal{H}_{\alpha d/2}(G) < \infty$. In this case we can find $x_1, x_2, \cdots \in [a,b]^3$ and $r_1, r_2, \cdots \in (0,\delta)$ such that $G \subseteq \cup_{i=1}^\infty \flat(x_i; r_i)$ and $\sum_{i=1}^\infty r_i^{\alpha d/2} \leq 2\mathcal{H}_{\alpha d/2}(G)$. Thus, by (83),

$$\begin{aligned}\mathbb{P}\big\{\hat{\mathbb{D}}_d \cap G \neq \emptyset\big\} &\leq \sum_{i=1}^\infty \mathbb{P}\big\{\hat{\mathbb{D}}_d \cap \flat(x_i;r_i) \neq \emptyset\big\} \\ &\leq c_{40} \sum_{i=1}^\infty r_i^{\alpha d/2} \\ &\leq 2c_{40}\mathcal{H}_{\alpha d/2}(G).\end{aligned}$$

This completes our proof of the upper bound of (11).

In order to prove the lower bound for $\bar{\mathbb{D}}_d$ note that $\bar{\mathbb{D}}_d$ intersects E if and only if $\hat{\mathbb{D}}_d$ intersects $[a,b]\times E$. To conclude, it suffices to prove that

$$\mathcal{H}_{\alpha d/2}([a,b]\times E) > 0 \to \mathcal{H}_{(\alpha d-2)/2}(E) > 0.$$

By the Frostman lemma, together with $\mathcal{H}_{\alpha d/2}([a,b]\times E) > 0$, then there exist a measure μ on $[a,b]\times E$ and a constant c_{41} such that for all three-dimensional balls $B(x,r)$ of radius $r > 0$,

$$\mu\big(B(x,r)\big) \leq c_{41} r^{\alpha d/2}.$$

For all Borel sets $C \subseteq \mathbb{R}^2$, define

$$\bar{\mu}(C) \hat{=} \mu([a,b]\times C).$$

Then, $\bar{\mu}$ is a measure supported on E. It follows from a covering argument, and the Frostman property of μ, that for all two-dimensional balls $B'(x,r)$ of radius $r > 0$,

$$\mu\big(B'(x,r)\big) \leq c_{42} r^{(\alpha d-2)/2}. \tag{84}$$

Applying the Frostman lemma again, and (84), we finish the proof of Theorem 5. □

5 Polar sets of $\mathcal{L}_d$

In this section, we consider the projection $\mathcal{L}_d$ of $\widetilde{W}^{-1}\{0\}$ onto x-axis.

Recall that a function $f : \mathbb{R}^n \to [0,\infty]$ is of strict positive type if (i) f is locally integrable away from $0 \in \mathbb{R}^n$; and (ii) the Fourier transform of f is strictly positive. Corresponding to such a function f we can define a function $\Pi_m f$ as follows:

$$(\Pi_m f)(x) \hat{=} \int_{[0,1]^m} f(x\otimes y)dy \;\text{ for all } x \in \mathbb{R}^{n-m}, \tag{85}$$

where $x\otimes y \hat{=} (x_1,\ldots,x_{n-m},y_1,\ldots,y_n) \in \mathbb{R}^n$ is the tensor product of x and y.

The following lemma follows from Theorem 3.1 [10, p.8].

Lemma 21 *(Projection theorem for capacities). Let $n > 1$ be an integer, and suppose that $f : \mathbb{R}^n \to [0, \infty]$ is of strict positive type and continuous on $\mathbb{R}^n \backslash \{0\}$. Then, for all all integers $1 \le m < n$ and compact sets $F \subset \mathbb{R}^{n-m}$,*

$$\mathrm{Cap}_f([0,1]^m \times F) = \mathrm{Cap}_{\Pi_m f}(F).$$

Theorem 1 characterizes the polar sets of $\mathcal{L}_d$. We provide its proof as follows.

Proof of Theorem 1 The function U_γ in (17) is of strict positive type for all $0 < \gamma < d$. We note also that U_γ is continuous away from the origin. In light of Theorem 1.1 [6, p.78], we can deduce the following results: For all non-random compact sets $E, F \subset (0, \infty)$ with $\mathrm{diam}(E \times F) < \delta$, then

$$A_1^{-1}\mathrm{Cap}_{\beta d/2}(E \times F) \le \mathbb{P}\{\widetilde{W}^{-1}\{0\} \cap (E \times F) \ne \emptyset\} \le A_1 \mathrm{Cap}_{\alpha d/2}(E \times F), \quad (86)$$

By Lemma 5.1, we have

$$\mathrm{Cap}_{\beta d/2}([0,1] \times F) = \mathrm{Cap}_{\Pi_1 U_{\beta d/2}}(F) \quad (87)$$

and

$$\mathrm{Cap}_{\alpha d/2}([0,1] \times F) = \mathrm{Cap}_{\Pi_1 U_{\alpha d/2}}(F). \quad (88)$$

For all $x \ge \varepsilon^2 > 0$,

$$\begin{aligned}(\Pi_1 U_{\beta d/2})(x) &\le \int_0^1 \frac{1}{|x+y|^{\beta d/2}} dy \\ &= \int_x^{x+1} \frac{1}{y^{\beta d/2}} dy \\ &= \frac{2}{\beta d - 2}[x^{1-\beta d/2} - (x+1)^{1-\beta d/2}] \\ &\le \frac{4}{\beta d - 2} U_{\beta d/2 - 1}(x). \end{aligned} \quad (89)$$

The same reasoning shows that when $x \ge \varepsilon^2 > 0$,

$$\begin{aligned}(\Pi_1 U_{\alpha d/2})(x) &\ge \int_0^1 \frac{1}{|x+y|^{\alpha d/2}} dy \\ &= \int_x^{x+1} \frac{1}{y^{\alpha d/2}} dy \\ &\ge \frac{2(1-2^{1-\alpha d/2})}{|\alpha d - 2|} x^{1-\alpha d/2} \\ &= \frac{2(1-2^{1-\alpha d/2})}{|\alpha d - 2|} U_{\alpha d/2 - 1}(x). \end{aligned} \quad (90)$$

By (87) and (89), we have

$$\mathrm{Cap}_{\Pi_1 U_{\beta d/2}}(F) \ge c_{43} \mathrm{Cap}_{\beta d/2}(F). \quad (91)$$

Using (88) and (90), we have

$$\mathrm{Cap}_{\Pi_1 U_{\alpha d/2}}(F) \le c_{44} \mathrm{Cap}_{\alpha d/2}(F). \quad (92)$$

Combining (86), (91) and (92), we prove the theorem. □

Now, we state some remarks and corollaries as follows. The following corollary follows from Theorem 1.

Corollary 22 *For all non-random compact sets $F \subset (0, \infty)$ with $\mathrm{diam} F < \delta$ and $\alpha = \beta = 1$, then*

$$\mathrm{Cap}_{(d-2)/2}(F) > 0 \Leftrightarrow \mathbb{P}\{\mathcal{L}_d \cap F \ne \emptyset\} > 0.$$

Furthermore, we can apply Theorem 3 to find the following lemma.

Corollary 23 *For all non-random compact sets $F \subset (0, \infty)$ with $\mathrm{diam} F < \delta$ and $\alpha = \beta = 1$, then*

$$\mathrm{Cap}_{(d-4)/2}(F) > 0 \Leftrightarrow \mathbb{P}\{\mathcal{D}_d \cap F \ne \emptyset\} > 0.$$

The last corollary follows from Theorem 5.

Corollary 24 *For all non-random compact sets $E \subset [a,b]^2$ and $G \subset [a,b]^3$ with $b - a < \delta$ and $\alpha = \beta = 1$, then*

$$\begin{aligned}&\mathrm{Cap}_{d/2}(G) > 0 \Rightarrow \mathbb{P}\{\hat{\mathcal{D}}_d \cap G \ne \emptyset\} > 0 \\ &\Rightarrow \mathcal{H}_{d/2}(G) > 0, \\ &\mathrm{Cap}_{(d-2)/2}(E) > 0 \Rightarrow \mathbb{P}\{\bar{\mathcal{D}}_d \cap E \ne \emptyset\} > 0 \\ &\Rightarrow \mathcal{H}_{(d-2)/2}(E) > 0.\end{aligned}$$

Even if $\alpha = \beta = 1$, we can say that $\widetilde{W}$ is much wider than Brownian sheet. In order to explain this, we give an example as following.

Example 25 *For any $t = (t_1, t_2) \in \mathbb{R}_+^2$, let*

$$F_\ell(0, t_\ell] = \int_0^{t_\ell} (2 - \sin x) dx, \quad \ell = 1, 2.$$

Clearly, F_ℓ is a Lebesgue-Stieljes measure , and for any $s_\ell, t_\ell \in \mathbb{R}_+$, we have

$$|s_\ell - t_\ell| \le |F_\ell(0, t_\ell] - F_\ell(0, s_\ell]| \le 3|s_\ell - t_\ell|, \quad \ell = 1, 2.$$

However, F_ℓ is not a Lebesgue measure.

The results above are sharp. They recover the corresponding results for Brownian sheet and Brownian motion.

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