Some New Dynamic Inequalities and Their Applications in the Qualitative Analysis of Dynamic Equations

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Abstract: In this paper, we establish some new Gronwall-Bellman-type dynamic inequalities in two independent variables containing integration on infinite intervals on time scales, which can be used as a handy tool in the boundedness analysis for solutions to some certain dynamic equations containing integration on infinite intervals on time scales. The presented inequalities are of new forms so far in the literature to our best knowledge.

Key–Words: Gronwall-Bellman-type inequality; Time scales; Dynamic equation; Qualitative analysis; Bound

1 Introduction

As is known to us, in the qualitative as well as quantitative analysis for solutions of differential equations, difference equations and dynamic equations on time scales, the Gronwall-Bellman inequality [1,2] play an important role as it provides explicit bounds for the unknown functions concerned. During the past few decades, various generalizations of the Gronwall-Bellman inequality have been developed (see [3–27] and the references therein). But we notice that in the analysis of boundedness for the solutions for some certain dynamic equations containing integration on infinite intervals on time scales, which some new bounds for the solutions for the two equations mentioned above are derived.

Throughout the paper, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}^+_0 = [0, \infty)$. $\mathbb{T}$ denotes an arbitrary time scale. $\mathbb{T}_0 = [x_0, \infty) \cap \mathbb{T}$, $\mathbb{T}_0 = [y_0, \infty) \cap \mathbb{T}$, where $x_0, y_0 \in \mathbb{T}$. On $\mathbb{T}$ we define the forward and backward jump operators $\sigma \in (\mathbb{T}, \mathbb{T})$ and $\rho \in (\mathbb{T}, \mathbb{T})$ such that $\sigma(t) = \inf\{s \in \mathbb{T}, s > t\}$, $\rho(t) = \sup\{s \in \mathbb{T}, s < t\}$. The graininess $\mu \in (\mathbb{T}, \mathbb{R}^+_0)$ is defined by $\mu(t) = \sigma(t) - t$.

Definition 1 A function $f \in (\mathbb{T}, \mathbb{R})$ is called rd-continuous if it is continuous in right-dense points and if the left-sided limits exist in left-dense points, while $f$ is called regressive if $1 + \mu(t)f(t) \neq 0$. $C_{rd}$ denotes the set of rd-continuous functions, while $\mathcal{R}$ denotes the set of all regressive and rd-continuous functions, and $\mathcal{R}^+ = \{f | f \in \mathcal{R}, 1 + \mu(t)f(t) > 0, \forall t \in \mathbb{T}\}$.

Definition 2 The cylinder transformation $\xi_h$ is defined by

$$
\xi_h(z) = \begin{cases} 
\frac{\text{Log}(1+hz)}{h}, & \text{if } h \neq 0 \text{ (for } z \neq -\frac{1}{h}), \\
\frac{z}{h}, & \text{if } h = 0,
\end{cases}
$$

where $\text{Log}$ is the principal logarithm function.
Definition 3 For \( p(x,y) \in \mathbb{R} \) with respect to \( y \), the exponential function is defined by

\[
e_p(y,s) = \exp\left( \int_s^y \xi_{\mu(\tau)}(p(x,\tau)) \Delta \tau \right)
\]

for \( s, y \in \mathbb{T} \).

Theorem 4 [28, Theorem 1.12]: If \( p(x,y) \in \mathbb{R} \) with respect to \( y \), then the following conclusions hold

(i) \( e_p(y,y) = 1 \), and \( e_0(s,y) = 1 \),
(ii) \( e_p(s,\sigma(y)) = (1 + \mu(y)p(x,y))e_p(s,y) \),
(iii) If \( p \in \mathbb{R}^+ \) with respect to \( y \), then \( e_p(s,y) > 0 \) for all \( s, y \in \mathbb{T} \),
(iv) If \( p \in \mathbb{R}^+ \) with respect to \( y \), then \( e_p \in \mathbb{R}^+ \),

\[
e_p(s,y) = \frac{1}{e_p(y,s)} = e_{\bar{e}_p}(y,s),
\]

where \( e_{\bar{e}_p}(x,y) = -\frac{p(x,y)}{1 + \mu(y)p(x,y)} \).

Theorem 5 [28, Theorem 1.13]: If \( p(x,y) \in \mathbb{R} \) with respect to \( y \), \( y_0 \in \mathbb{T} \) is a fixed number, then the exponential function \( e_p(y,y_0) \) is the unique solution of the following integral value problem

\[
\left\{ \begin{array}{l}
\Delta y\, (x,y) = p(x,y)z(x,y), \\
z(x,y_0) = 1.
\end{array} \right.
\]

2 Main Results

Lemma 6 Assume that \( u(x,\cdot), a(x,\cdot), b(x,\cdot), m(x,\cdot) \in C_{r\delta}(\mathbb{T}_0, \mathbb{R}_+) \) with respect to \( y \), \( \tilde{m}(x,y) = -m(x,y)b(x,y) \) and \( \tilde{m}(x,\cdot) \in \mathbb{R}_+ \) with respect to \( y \). Then for any fixed \( x \in \mathbb{T}_0 \),

\[
u(x,y) \leq a(x,y) + b(x,y) \int_y^\infty m(x,t)u(x,t) \Delta t,
\]

\( y \in \mathbb{T}_0 \) implies

\[
u(x,y) \leq a(x,y) + b(x,y) \int_y^\infty e_{\tilde{m}}(y,\sigma(t))m(x,t)a(x,t) \Delta t, \quad y \in \mathbb{T}_0.
\]

Proof: Denote \( v(x,y) = \int_y^\infty m(x,t)u(x,t) \Delta t \). Then

\[
\nu(x,y) = -m(x,y)u(x,y)
\]

\[
\geq -m(x,y)b(x,y)v(x,y) - m(x,y)a(x,y)
\]

\[
= \tilde{m}(x,y)v(x,y) - m(x,y)a(x,y).
\]

Since \( \tilde{m} \in \mathbb{R}^+ \), then from Theorem 4(iv) we have \( \tilde{m} \in \mathbb{R}^+ \), and furthermore from Theorem 4(iii) we obtain \( e_{\tilde{m}}(y,\sigma(t))a(x,t) \Delta t \geq 0 \) for all \( \sigma(t) \in \tilde{m}_0 \).

Moreover,

\[
[v(x,y)e_{\tilde{m}}(y,\alpha)]^y_\alpha = [e_{\tilde{m}}(y,\alpha)]^y_\alpha + e_{\tilde{m}}(\sigma(y,\alpha))v^y_\alpha y \Delta x(y,x).
\]

On the other hand, from Theorem 5 we have

\[
e_{\bar{e}_p}(y,\alpha) = (\tilde{m}(x,y)e_{\tilde{m}}(y,\alpha))\Delta y.
\]

A combination of (3), (4) and Theorem 4 yields

\[
[v(x,y)e_{\tilde{m}}(y,\alpha)]^y_\alpha = (\tilde{m}(x,y)e_{\tilde{m}}(y,\alpha))v(x,y) + e_{\tilde{m}}(\sigma(y,\alpha))v^y_\alpha y \Delta x(y,x).
\]

Substituting \( y \) with \( t \), and an integration for (5) with respect to \( t \) from \( \alpha \) to \( \infty \) yields

\[
v(x,\alpha)e_{\tilde{m}}(\alpha,\alpha) - v(x,\alpha)e_{\tilde{m}}(\alpha,\alpha) = \int_\alpha^\infty e_{\tilde{m}}(\sigma(t,\alpha))v^t_\alpha t \Delta t.
\]

Considering \( v(x,\infty) = 0 \), \( e_{\tilde{m}}(\alpha,\alpha) = 1 \), from (1) and (6) we have

\[
-v(x,\alpha) \geq - \int_\alpha^\infty e_{\tilde{m}}(\sigma(t,\alpha))m(x,t)a(x,t) \Delta t
\]

\[
= - \int_\alpha^\infty e_{\tilde{m}}(\alpha,\sigma(t))m(x,t)a(x,t) \Delta t,
\]

which is followed by

\[
v(x,\alpha) \leq \int_\alpha^\infty e_{\tilde{m}}(\alpha,\sigma(t))m(x,t)a(x,t) \Delta t.
\]

Since \( \alpha \in \tilde{m}_0 \) is arbitrary, after substituting \( \alpha \) with \( y \) we obtain the desired inequality.

Lemma 7 Under the conditions of Lemma 6, furthermore, assume \( a(x,y) \) is nonincreasing in \( y \) for every fixed \( x, b(x,y) \equiv 1 \). Then we have

\[
u(x,y) \leq a(x,y)e_{-m}(y,\infty).
\]

Proof: Since \( b(x,y) \equiv 1 \), and \( a(x,y) \) is nonincreasing on \( \mathbb{T}_0 \) with respect to \( y \), then \( \tilde{m} = -m \), and

\[
u(x,y) \leq a(x,y) + \int_y^\infty e_{-m}(y,\sigma(t))a(x,t)m(x,t) \Delta t
\]

\[
\leq a(x,y)[1 + \int_y^\infty e_{-m}(y,\sigma(t))m(x,t) \Delta t].
\]
From [29, Theorem 2.39 and 2.36 (i)], we have
\[
\int_y^\infty e^{-m(y, \sigma(y))} m(x, t) \Delta t = \lim_{\epsilon \to \infty} \int_y^\infty e^{-m(y, \sigma(t))} (-m(x, t)) \Delta t = \lim_{\epsilon \to \infty} e^{-m(y, \epsilon)} - e^{-m(y, \infty)} = e^{-m(y, \infty)} - 1.
\]
Combining the above information we can obtain the desired inequality. \hfill \square

**Lemma 8** [30] Assume that \(a \geq 0, p \geq q \geq 0, \) and \(p \neq 0, \) then for any \(K > 0, \)
\[
a^\frac{q}{p} \leq \frac{q}{p} K^{\frac{q-p}{p}} a + \frac{p-q}{p} K^{\frac{p}{p}}.
\]

**Theorem 9** Suppose \(u, f, g, h, a, b \in C_{rd}(\mathbb{T}_0 \times \mathbb{T}_0, \mathbb{R}_+), \) and \(a, b \) are nonincreasing. \(p, q, r, m \) are constants, and \(p \geq q \geq 0, p \geq r \geq 0, p \geq m \geq 0, p \neq 0. \) If for \((x, y) \in \mathbb{T}_0 \times \mathbb{T}_0, \) \(u(x, y) \) satisfies the following inequality:
\[
\begin{align*}
&u^p(x, y) \leq a(x, y) + b(x, y) \times \\
&\int_y^\infty \int_x^\infty \int_x^\infty \left[ f(s, t)u^q(s, t) + g(s, t)u^r(s, t) \right] \Delta s \Delta t \\
&+ b(x, y) \times \\
&\int_y^\infty \int_x^\infty \int_x^\infty \int_x^\infty h(\xi, \eta) u^m(\xi, \eta) \Delta \xi \Delta \eta \Delta s \Delta t,
\end{align*}
\] (8)
then
\[
\begin{align*}
u(x, y) & \leq \left[ B_1(x, y) + b(x, y) \times \\
&\int_y^\infty \int_x^\infty \int_x^\infty \int_x^\infty h(\xi, \eta) u^m(\xi, \eta) \Delta \xi \Delta \eta \Delta s \Delta t \right]^\frac{1}{p}, \\
(x, y) & \in \mathbb{T}_0 \times \mathbb{T}_0.
\end{align*}
\] (9)
provided that \(B_2(x, \cdot) \in \mathcal{H}^+, \) where
\[
\begin{align*}
B_1(x, y) &= a(x, y) + b(x, y) \times \\
&\int_y^\infty \int_x^\infty \left[ f(s, t)\frac{p-q}{p} K^{\frac{q-p}{p}} + g(s, t)\frac{p-r}{p} K^{\frac{p}{p}} \right] \Delta s \Delta t \\
&+ b(x, y) \times \\
&\int_y^\infty \int_x^\infty \int_x^\infty \int_x^\infty h(\xi, \eta) \frac{p-m}{p} K^{\frac{m}{p}} \Delta \xi \Delta \eta \Delta s \Delta t,
\end{align*}
\] (10)
\[
B_2(x, y) = \int_x^\infty \left[ f(s, y)\frac{q}{p} K^{\frac{q-p}{p}} + g(s, y)\frac{r}{p} K^{\frac{r-p}{p}} \right] \Delta s + b(x, y) \times \\
\int_y^\infty \int_x^\infty \int_x^\infty \int_x^\infty h(\xi, \eta) \frac{m}{p} K^{\frac{m}{p}} \Delta \xi \Delta \eta \Delta s, \forall K > 0,
\] (11)
and
\[
\overline{B}_2(x, y) = -b(x, y) B_2(x, y).
\] (12)

**Proof:** Denote the right side of (8) by \(v(x, y). \) Then we have
\[
u(x, y) \leq v^\frac{1}{p}(x, y), \quad (x, y) \in \mathbb{T}_0 \times \mathbb{T}_0.
\] (13)

Fix \(X \in \mathbb{T}_0. \) Then it follows that
\[
\begin{align*}
v(X, y) &= a(X, y) + b(X, y) \\
&\int_y^\infty \int_X \left[ f(s, t)u^q(s, t) + g(s, t)u^r(s, t) \right] \Delta s \Delta t \\
&+ b(X, y) \int_y^\infty \int_X \left[ g(s, t)\frac{p}{p} K^{\frac{q-p}{p}} + h(\xi, \eta) \frac{p}{p} K^{\frac{p}{p}} \right] \Delta s \Delta t
\end{align*}
\] (14)
Combining (14) with Lemma 8 we obtain
\[
\begin{align*}
v(X, y) & \leq a(X, y) + b(X, y) \\
&\int_y^\infty \int_X \left[ f(s, t) u^q(s, t) + g(s, t) u^r(s, t) \right] \Delta s \Delta t \\
&+ b(X, y) \int_y^\infty \int_X \left[ g(s, t)\frac{p}{p} K^{\frac{q-p}{p}} + h(\xi, \eta) \frac{p}{p} K^{\frac{p}{p}} \right] \Delta s \Delta t
\end{align*}
\] (15)
By use of Lemma 6, we obtain
\[
\begin{align*}
v(X, y) & \leq B_1(X, y) + b(X, y) \\
&\int_y^\infty \int_X \left[ f(s, y)\frac{q}{p} K^{\frac{q-p}{p}} + g(s, y)\frac{r}{p} K^{\frac{r-p}{p}} \right] \Delta s + b(X, y) \times \\
&\int_y^\infty \int_X \int_X \int_X \int_X h(\xi, \eta) \frac{m}{p} K^{\frac{m}{p}} \Delta \xi \Delta \eta \Delta s, \forall K > 0,
\end{align*}
\] (16)
Since \(X \in \mathbb{T}_0 \) is arbitrary, then in fact (16) holds for \(\forall x \in \mathbb{T}_0, \) that is,
\[
v(x, y) \leq B_1(x, y) + b(x, y)
\]
\[ \int_{y} e_{\mathcal{T}}(y, \sigma(t)) B_2(x, t) B_1(x, t) \Delta t, \quad (x, y) \in (\mathcal{T}_0 \times \mathcal{T}_0). \]  
(17)

Combining (13), (17) we obtain the desired inequality.

If we apply Lemma 7 instead of Lemma 6 at the end of the proof of Theorem 9, then we obtain the following theorem.

**Theorem 10** Suppose \( u, f, g, h, a, p, q, r, m \) are defined as in Theorem 2.1. If for \((x, y) \in \mathcal{T}_0 \times \mathcal{T}_0\), \( u(x, y) \) satisfies the following inequality:

\[
u^{p}(x, y) \leq a(x, y) + \int_{y}^{\infty} \int_{x}^{\infty} [f(s, t)u^{q}(s, t)
+g(s, t)u^{r}(s, t)] \Delta s \Delta t + \int_{y}^{\infty} \int_{x}^{\infty} \int_{s}^{\infty} h(x, y)u^{m}(x, \eta) \Delta x \Delta \eta \Delta s \Delta \eta, \]  
then
\[ u(x, y) \leq [B_1(x, y) + B_4(x, y)]^{\frac{1}{q}}, \quad (x, y) \in \mathcal{T}_0 \times \mathcal{T}_0, \]  
provided that \(-B_2(x, y) \in \mathbb{R}^+\), where

\[ B_1(x, y) = a(x, y) + \int_{y}^{\infty} \int_{x}^{\infty} [f(s, t)u^{q}(s, t)] \Delta s \Delta t + \int_{y}^{\infty} \int_{x}^{\infty} \int_{s}^{\infty} h(x, y)u^{m}(x, \eta) \Delta x \Delta \eta \Delta s \Delta \eta, \]  

\[ B_2(x, y) = \int_{y}^{\infty} \int_{s}^{\infty} h(x, \eta)u^{m}(x, \eta) \Delta x \Delta \eta + \int_{y}^{\infty} \int_{s}^{\infty} h(x, \eta)u^{m}(x, \eta) \Delta x \Delta \eta, \forall K > 0. \]  

**Theorem 11** Suppose \( u, f, g, h \) are defined as in Theorem 9. If for \((x, y) \in \mathcal{T}_0 \times \mathcal{T}_0\), \( u(x, y) \) satisfies the following inequality:

\[
u^{p}(x, y) \leq a(x, y) + \int_{y}^{\infty} \int_{x}^{\infty} [f(s, t)u^{q}(s, t)
+g(s, t)u^{r}(s, t)] \Delta s \Delta t + \int_{y}^{\infty} \int_{x}^{\infty} \int_{s}^{\infty} h(x, y)u^{m}(x, \eta) \Delta x \Delta \eta \Delta s \Delta \eta, \]  
then \( u(x, y) \equiv 0. \)

The proof for Theorem 11 is similar to Theorem 9, and we omit it here.

Based on Theorem 9, we establish a Gronwall-Bellman-Volterra-Fredholm type inequality containing integration on infinite intervals on time scales as follows.

**Theorem 12** Suppose \( u, f_i, g_i, h_i \in C_{rd}(\mathcal{T}_0 \times \mathcal{T}_0, \mathbb{R}_+), i = 1, 2, a, p, q, r, m \) are defined as in Theorem 9, and \( M \in \mathcal{T}_0, N \in \mathcal{T}_0 \) are two fixed numbers. If for \((x, y) \in ([M, \infty) \cap \mathcal{T}) \times ([N, \infty) \cap \mathcal{T})\), \( u(x, y) \) satisfies the following inequality:

\[
u^{p}(x, y) \leq a(x, y) + \int_{y}^{\infty} \int_{x}^{\infty} [f_1(s, t)u^{q}(s, t)
+g_1(s, t)u^{r}(s, t)] \Delta s \Delta t + \int_{y}^{\infty} \int_{x}^{\infty} \int_{s}^{\infty} h_1(x, \eta)u^{m}(x, \eta) \Delta x \Delta \eta \Delta s \Delta \eta, \]  
then we have

\[ u(x, y) \leq \left\{ \frac{\lambda + \tilde{B}_5}{1 - \tilde{B}_5} \right\}^{\frac{1}{q}}, \quad (x, y) \in ([M, \infty) \cap \mathcal{T}) \times ([N, \infty) \cap \mathcal{T}), \]  
provided that \( \tilde{B}_5 < 1 \), and \(-B_2(x, y) \in \mathbb{R}^+\), where

\[
l = \int_{N}^{\infty} \int_{M}^{\infty} [f_2(s, t)u^{q}(s, t) + g_2(s, t)u^{r}(s, t)] \Delta s \Delta t + \int_{N}^{\infty} \int_{M}^{\infty} \int_{s}^{\infty} h_2(x, \eta)u^{m}(x, \eta) \Delta x \Delta \eta \Delta s \Delta \eta, \]  

\[ \tilde{B}_1(x, y) = a(x, y) + \int_{y}^{\infty} \int_{x}^{\infty} [f_1(s, t)u^{q}(s, t)
+g_1(s, t)u^{r}(s, t)] \Delta s \Delta t + \int_{y}^{\infty} \int_{x}^{\infty} \int_{s}^{\infty} h_1(x, \eta)u^{m}(x, \eta) \Delta x \Delta \eta \Delta s \Delta \eta, \]  

then \( u(x, y) \equiv 0. \)
\[ \tilde{B}_5 = \int_N^\infty \int_M^\infty [f_2(s, t) \frac{q}{p} K^{\frac{p-q}{p}} \tilde{B}_3(s, t) + g_2(s, t) \frac{r}{p} K^{\frac{p-r}{p}} \tilde{B}_3(s, t)] \Delta s \Delta t + \int_N^\infty \int_M^\infty \int_t^\infty \int_s^\infty (h_2(\xi, \eta) \frac{m}{p} K^{\frac{q-m}{p}} \tilde{B}_3(\xi, \eta) \Delta \xi \Delta \eta \Delta s \Delta t, \quad (25) \]

\[ \tilde{B}_6 = \int_N^\infty \int_M^\infty [f_2(s, t) \frac{q}{p} K^{\frac{p-q}{p}} \tilde{B}_4(s, t) + g_2(s, t) \frac{r}{p} K^{\frac{p-r}{p}} \tilde{B}_4(s, t)] \Delta s \Delta t + \int_N^\infty \int_M^\infty \int_t^\infty \int_s^\infty (h_2(\xi, \eta) \frac{m}{p} K^{\frac{q-m}{p}} \tilde{B}_4(\xi, \eta) \Delta \xi \Delta \eta \Delta s \Delta t. \quad (26) \]

**Proof:** Let the right side of (18) be \( v(x, y) \), and

\[ \mu = \int_N^\infty \int_M^\infty [f_2(s, t) u^q(s, t) + g_2(s, t) u^r(s, t)] \Delta s \Delta t + \int_N^\infty \int_M^\infty \int_t^\infty \int_s^\infty (h_2(\xi, \eta) u^m(\xi, \eta) \Delta \xi \Delta \eta \Delta s \Delta t. \quad (27) \]

Then

\[ u(x, y) \leq \frac{1}{\nu} v(\sqrt{x, y}), \]

\((x, y) \in ([M, \infty) \cap \mathbb{T}) \times ([N, \infty) \cap \mathbb{T}), \quad (28) \)

Fix \( X \in [M, \infty) \cap \mathbb{T} \). Then

\[ v(x, y) = a(X, y) + \mu \]

\[ + \int_y^\infty \int_N^\infty \int_M^\infty [f_1(s, t) u^q(s, t) + g_1(s, t) u^r(s, t)] \Delta s \Delta t \]

\[ + \int_y^\infty \int_N^\infty \int_t^\infty \int_s^\infty h_1(\xi, \eta) u^m(\xi, \eta) \Delta \xi \Delta \eta \Delta s \Delta t \]

\[ \leq a(X, y) + \mu + \int_y^\infty \int_N^\infty \int_M^\infty [f_1(s, t) v^q(s, t) + g_1(s, t) v^r(s, t)] \Delta s \Delta t \]

\[ + \int_y^\infty \int_N^\infty \int_t^\infty \int_s^\infty h_1(\xi, \eta) v^m(\xi, \eta) \Delta \xi \Delta \eta \Delta s \Delta t. \quad (29) \]

Considering the structure of (29) is similar to (14), then following in a same manner as the process of (14)-(17) we can deduce that for \( y \in [N, \infty) \cap \mathbb{T} \),

\[ v(x, y) \leq \mu + \tilde{B}_1(X, y) \]

\[ + \int_y^\infty e^{-\tilde{B}_2(y, \sigma(t))} \tilde{B}_2(X, t) (\mu + \tilde{B}_1(X, t)) \Delta t \]

\[ = \mu [1 + \int_y^\infty e^{-\tilde{B}_2(y, \sigma(t))} \tilde{B}_2(X, t) \Delta t] + \tilde{B}_1(X, y) \]

\[ + \int_y^\infty e^{-\tilde{B}_2(y, \sigma(t))} \tilde{B}_2(X, t) \tilde{B}_1(X, t) \Delta t. \quad (30) \]

Since \( X \) is selected from \([M, \infty) \cap \mathbb{T}\) arbitrarily, then in fact (30) holds for \( \forall x \in [M, \infty) \cap \mathbb{T} \), that is

\[ v(x, y) \leq \mu [1 + \int_y^\infty e^{-\tilde{B}_2(y, \sigma(t))} \tilde{B}_2(X, t) \Delta t] + \tilde{B}_1(X, y) \]

\[ + \int_y^\infty e^{-\tilde{B}_2(y, \sigma(t))} \tilde{B}_2(X, t) \tilde{B}_1(X, t) \Delta t. \]

On the other hand, from Lemma 8, (27) and (28) we obtain

\[ \mu \leq \int_N^\infty \int_M^\infty [f_2(s, t) v^q(s, t) + g_2(s, t) v^r(s, t)] \Delta s \Delta t \]

\[ + \int_y^\infty \int_N^\infty \int_M^\infty h_2(\xi, \eta) v^m(\xi, \eta) \Delta \xi \Delta \eta \Delta s \Delta t \]

\[ \leq \int_N^\infty \int_M^\infty [f_2(s, t) \frac{q}{p} K^{\frac{p-q}{p}} v(s, t) + \frac{p-q}{p} K^{\frac{q}{p}}] \Delta s \Delta t \]

\[ + \int_N^\infty \int_M^\infty \int_t^\infty \int_s^\infty h_2(\xi, \eta) \frac{m}{p} K^{\frac{q-m}{p}} v(\xi, \eta) \Delta \xi \Delta \eta \Delta s \Delta t \]

\[ + \frac{p-m}{p} \int_N^\infty \int_M^\infty \int_t^\infty \int_s^\infty h_2(\xi, \eta) \frac{m}{p} K^{\frac{q-m}{p}} v(\xi, \eta) \Delta \xi \Delta \eta \Delta s \Delta t. \quad (32) \]

Then using (31) in (32) yields

\[ \mu \leq \lambda + \int_N^\infty \int_M^\infty [f_2(s, t) \frac{q}{p} K^{\frac{p-q}{p}} \mu \tilde{B}_3(s, t) + \tilde{B}_4 \]

\[ (s, t)] + g_2(s, t) \frac{r}{p} K^{\frac{p-r}{p}} [\mu \tilde{B}_3(s, t) + \tilde{B}_4(s, t))] \Delta s \Delta t \]

\[ + \int_N^\infty \int_M^\infty \int_t^\infty \int_s^\infty h_2(\xi, \eta) \frac{m}{p} K^{\frac{q-m}{p}} [\mu \tilde{B}_3(\xi, \eta) \]

\[ + \tilde{B}_3(\xi, \eta)) \Delta \xi \Delta \eta \Delta s \Delta t \]

\[ = \lambda + \mu [1 + \int_N^\infty \int_M^\infty [f_2(s, t) \frac{q}{p} K^{\frac{p-q}{p}} \tilde{B}_3(s, t) \]

\[ + \int_N^\infty \int_M^\infty [f_2(s, t) \frac{q}{p} K^{\frac{p-q}{p}} \tilde{B}_3(s, t) \]

\[ + g_2(s, t) \frac{r}{p} K^{\frac{p-r}{p}} \tilde{B}_3(s, t) \Delta s \Delta t. \]

On the other hand, from Lemma 8, (27) and (28) we obtain
\[
+ g_2(s, t) \frac{r}{p} K^{\frac{r}{p}} B_3(s, t) |\Delta s| |\Delta t|
+ \int_N \int_M \left. \left( \int_s^\infty h_2(\xi, \eta) \frac{m}{p} K^{\frac{m-p}{p}} \tilde{B}_3(\xi, \eta) \Delta \xi \Delta \eta \Delta s \Delta t \right) \right|_s^\infty
+ \int_N \int_M \left. \left( \int_s^\infty |f_2(s, t)| \frac{q}{p} K^{\frac{q}{p}} \tilde{B}_4(s, t) \Delta s \Delta t \right) \right|_s^\infty

\int_\infty^s \int_\infty^t \int_\infty^s h_2(\xi, \eta) \frac{m}{p} K^{\frac{m-p}{p}} \tilde{B}_3(\xi, \eta) \Delta \xi \Delta \eta \Delta s \Delta t
= \lambda + \mu \tilde{B}_5 + \tilde{B}_6,
\]

which is followed by
\[
\mu \leq \frac{\lambda + \tilde{B}_6}{1 - \tilde{B}_5}.
\]

Combining (28), (31) and (34) we can obtain the desired inequality (19).

In the proof of Theorem 12, if we let the right-hand side of (18) be \( a(x, y) + v(x, y) \) in the first statement, then following in a similar process as in Theorem 12 we obtain another bound of the function \( u(x, y) \), which is shown in the following theorem.

**Theorem 13** Under the conditions of Theorem 12, if \( (x, y) \in ([M, \infty] \cap \mathbb{T}) \times ([N, \infty] \cap \mathbb{T}) \), \( u(x, y) \) satisfies (18), then we have
\[
u(x, y) \leq \{ a(x, y) + \frac{\mu + J_1(M, N)}{1 - \lambda} e^{-J_2(y, \infty)} \}^{\frac{1}{p}},
\]
\( (x, y) \in ([M, \infty] \cap \mathbb{T}) \times ([N, \infty] \cap \mathbb{T}) \),

provided that \( \tilde{\lambda} < 1 \) and \(-J_2(x, \cdot) \in \mathfrak{R}^+\), where
\[
\tilde{\lambda} = \int_N \int_M \int_s^\infty \int_t^\infty |f_2(s, t)| \frac{q}{p} K^{\frac{q}{p}} e^{-J_2(t, \infty)} |\Delta s| |\Delta t|
+ \int_N \int_M \int_s^\infty \int_t^\infty h_2(\xi, \eta) \frac{m}{p} K^{\frac{m-p}{p}} e^{-J_2(\eta, \infty)} |\Delta \xi| |\Delta \eta| |\Delta s| |\Delta t|
+ \int_N \int_M \int_s^\infty \int_t^\infty |f_2(s, t)| \frac{q}{p} K^{\frac{q}{p}} a(s, t) + \frac{p - q}{p} K^{\frac{p}{p}}
+ g_2(s, t) \frac{r}{p} K^{\frac{r}{p}} a(s, t) + \frac{p - r}{p} K^{\frac{p}{p}}
\]

Finally, we establish a more general inequality than in Theorems 2.4-2.5. Consider the following inequality:
\[
u(x, y) \leq \{ a(x, y) + \int_N \int_M \left[ L(s, t, u(s, t)) \right] \Delta s \Delta t
+ \int_N \int_M \left[ J_1(x, y) u^q(\xi, \eta) \Delta \xi \Delta \eta \right] \Delta s \Delta t
+ \int_N \int_M \left[ J_2(x, y) u^q(\xi, \eta) \Delta \xi \Delta \eta \right] \Delta s \Delta t
\]

where \( u, a, p, q \) are defined as in Theorem 9, \( M \in \mathbb{T}_0, N \in \mathbb{T}_0 \) are two fixed numbers, \( L \in \mathbb{T}_0 \times \mathbb{T}_0 \times \mathbb{R}_+, \mathbb{R}_+ \), and \( 0 \leq L(s, t, x) - L(s, t, y) \leq A(s, t, y)(x - y) \) for \( x \geq y \geq 0 \), where \( A \in \mathbb{T}_0 \times \mathbb{T}_0 \times \mathbb{R}_+, \mathbb{R}_+ \).

**Theorem 14** If \( (x, y) \in ([M, \infty] \cap \mathbb{T}) \times ([N, \infty] \cap \mathbb{T}) \), \( u(x, y) \) satisfies (35), then the following inequality holds.
\[
u(x, y) \leq \{ \frac{\tilde{\lambda} + \tilde{B}_6}{1 - \tilde{B}_5} \tilde{B}_3(x, y) + \tilde{B}_4(x, y) \}^{\frac{1}{p}},
\]
\( (x, y) \in ([M, \infty] \cap \mathbb{T}) \times ([N, \infty] \cap \mathbb{T}) \),

provided that \( \tilde{B}_5 < 1 \) and \(-\tilde{B}_2(x, \cdot) \in \mathfrak{R}^+\), where
\[
\tilde{\lambda} = \int_N \int_M \int_s^\infty \int_t^\infty [L(s, t, p - \frac{1}{p})] \Delta s \Delta t.
\]
\begin{align*}
+ \int_t^\infty \int_s^\infty h_2(\xi, \eta) \frac{p-q}{p} K^{\frac{q}{p}} \Delta \xi \Delta \eta |\Delta s \Delta t, \\
\hat{B}_1(x, y) = a(x, y) + \int_y^\infty \int_x^\infty [L(s, t, \frac{p-1}{p} K^\frac{1}{p}) \\
+ \int_t^\infty \int_s^\infty h_1(\xi, \eta) \frac{p-q}{p} K^{\frac{q}{p}} \Delta \xi \Delta \eta |\Delta s \Delta t, \\
\hat{B}_2(x, y) = \int_x^\infty [A(s, y, \frac{p-1}{p} K^\frac{1}{p}) \frac{1}{p} K^{\frac{1}{p}} v(\xi, \eta) \\
+ \int_y^\infty \int_s^\infty h_1(\xi, \eta) \frac{q}{p} K^{\frac{q}{p}} \Delta \xi \Delta \eta |\Delta s, \forall K > 0, \\
\hat{B}_3(x, y) = 1 + \int_y^\infty e^{-\hat{B}_2}(y, \sigma(t)) \hat{B}_2(x, t) |\Delta t, \\
\hat{B}_4(x, y) = \hat{B}_1(x, y) \\
+ \int_y^\infty e^{-\hat{B}_2}(y, \sigma(t)) \hat{B}_2(x, t) \hat{B}_1(x, t) |\Delta t, \\
\hat{B}_5 = \int_N^\infty \int_M^\infty [A(s, t, \frac{p-1}{p} K^\frac{1}{p}) \frac{1}{p} K^{\frac{1}{p}} \hat{B}_3(s, t) \\
+ \int_t^\infty \int_s^\infty h_2(\xi, \eta) \frac{q}{p} K^{\frac{q}{p}} \hat{B}_3(\xi, \eta) \Delta \xi \Delta \eta |\Delta s \Delta t, \\
\hat{B}_6 = \int_N^\infty \int_M^\infty [A(s, t, \frac{p-1}{p} K^\frac{1}{p}) \frac{1}{p} K^{\frac{1}{p}} \hat{B}_4(s, t) \\
+ \int_t^\infty \int_s^\infty h_2(\xi, \eta) \frac{q}{p} K^{\frac{q}{p}} \hat{B}_4(\xi, \eta) \Delta \xi \Delta \eta |\Delta s \Delta t.
\end{align*}
Proof: Let the right side of (35) be \( v(x, y) \), and 
\[ \hat{\mu} = \int_N^\infty \int_M^\infty [L(s, t, u(s, t)) \\
+ \int_t^\infty \int_s^\infty h_2(\xi, \eta) u^q(\xi, \eta) \Delta \xi \Delta \eta |\Delta s \Delta t]. \] 
Then 
\[ u(x, y) \leq \frac{1}{v}(x, y), \] 
\((x, y) \in ([M, \infty) \cap T) \times ([N, \infty) \cap T), \]
Fix \( X \in [M, \infty) \cap T. \) Then 
\[ v(X, y) = a(X, y) + \hat{\mu} + \int_y^\infty \int_X^\infty [L(s, t, u(s, t)) \\
+ \int_t^\infty \int_s^\infty h_1(\xi, \eta) u^q(\xi, \eta) \Delta \xi \Delta \eta |\Delta s \Delta t \\
\leq a(X, y) + \hat{\mu} + \int_y^\infty \int_X^\infty [L(s, t, \frac{1}{v}(s, t)) \\
+ \int_t^\infty \int_s^\infty h_1(\xi, \eta) v^q(\xi, \eta) \Delta \xi \Delta \eta |\Delta s \Delta t]. \] 
Combining with Lemma 8 we have 
\[ v(X, y) \leq a(X, y) + \hat{\mu} + \int_y^\infty \int_X^\infty [L(s, t, -\frac{1}{p} K^\frac{1}{p} v(s, t) + \frac{p-1}{p} K^\frac{1}{p}) |\Delta s \Delta t \\
+ \int_y^\infty \int_X^\infty \int_t^\infty \int_s^\infty h_1(\xi, \eta) \frac{q}{p} K^{\frac{q}{p}} v(\xi, \eta) \\
+ \frac{p-q}{p} K^\frac{q}{p} \Delta \xi \Delta \eta |\Delta s \Delta t] \\
= a(X, y) + \hat{\mu} + \int_y^\infty \int_X^\infty [L(s, t, -\frac{1}{p} K^\frac{1}{p} v(s, t) + \frac{p-1}{p} K^\frac{1}{p}) \\
- L(s, t, \frac{p-1}{p} K^\frac{1}{p})] |\Delta s \Delta t \\
+ \int_y^\infty \int_X^\infty \int_t^\infty \int_s^\infty h_1(\xi, \eta) \frac{q}{p} K^{\frac{q}{p}} v(\xi, \eta) \\
+ \frac{p-q}{p} K^\frac{q}{p} \Delta \xi \Delta \eta |\Delta s \Delta t \\
\leq a(X, y) + \hat{\mu} + \int_y^\infty \int_X^\infty [A(s, t, -\frac{1}{p} K^\frac{1}{p}) \frac{1}{p} K^{\frac{1}{p}} v(s, t) \\
+ L(s, t, -\frac{1}{p} K^\frac{1}{p})] |\Delta s \Delta t + \int_y^\infty \int_X^\infty [\int_t^\infty \int_s^\infty h_1(\xi, \eta) \frac{q}{p} K^{\frac{q}{p}} v(X, t) |\Delta s \Delta t \\
+ \int_y^\infty \int_X^\infty \int_t^\infty \int_s^\infty h_1(\xi, \eta) \frac{p-q}{p} K^\frac{q}{p} \Delta \xi \Delta \eta |\Delta s \Delta t] \\
\leq a(X, y) + \hat{\mu} + \int_y^\infty \int_X^\infty [\int_t^\infty \int_s^\infty A(s, t, -\frac{1}{p} K^\frac{1}{p}) \frac{1}{p} K^{\frac{1}{p}} v(X, t) |\Delta s \Delta t \\
+ \int_y^\infty \int_X^\infty L(s, t, -\frac{1}{p} K^\frac{1}{p}) |\Delta s \Delta t + \int_y^\infty \int_X^\infty [\int_t^\infty \int_s^\infty h_1(\xi, \eta) \frac{q}{p} K^{\frac{q}{p}} \Delta \xi \Delta \eta |\Delta s \Delta t] \\
+ \int_y^\infty \int_X^\infty \int_t^\infty \int_s^\infty h_1(\xi, \eta) \frac{p-q}{p} K^\frac{q}{p} \Delta \xi \Delta \eta |\Delta s \Delta t] \\
= \hat{\mu} + \hat{B}_1(X, y) + \int_y^\infty \hat{B}_2(X, t) v(X, t) |\Delta t. \] 
We notice the structure of (47) is similar to (15). 
So following in a same manner as (15)-(17) we obtain 
\[ v(x, y) \leq \hat{\mu} |1 + \int_y^\infty e^{-\hat{B}_2}(y, \sigma(t)) \hat{B}_2(x, t) |\Delta t| + \int_y^\infty \hat{B}_2(x, t) v(X, t) |\Delta t. \]
\[
\hat{B}_1(x, y) + \int_y^\infty e^{-\hat{B}_2(y, \sigma(t))} \hat{B}_2(x, t) \hat{B}_1(x, t) \Delta t = \hat{\mu} \hat{B}_3(x, y) + \hat{B}_4(x, y), \\
(x, y) \in ([M, \infty) \cap T) \times ([N, \infty) \cap T),
\]

(48)

On the other hand, from Lemma 8, (44) and (45) we have

\[
\hat{\mu} \leq \left( \int_N \int_M [L(s, t, v(t))] \Delta t \right) + \left( \int_t \int_s h_2(x, \eta)(q \frac{K^{\frac{q-p}{p}}}{p} v(x, \eta)) + \frac{p-q}{p} K^{\frac{q}{p}} \Delta \xi \Delta \eta \right) \Delta s \Delta t
\]

\[
= \left( \int_N \int_M [L(s, t, v(s, t)) + \frac{p-1}{p} K^{\frac{1}{p}}] \right) \Delta s \Delta t
\]

\[
+ \left( \int_t \int_s h_2(x, \eta)(q \frac{K^{\frac{q-p}{p}}}{p} v(x, \eta)) + \frac{p-q}{p} K^{\frac{q}{p}} \Delta \xi \Delta \eta \right) \Delta s \Delta t
\]

\[
\leq \left( \int_N \int_M [A(s, t, v(s, t)) + \frac{p-1}{p} K^{\frac{1}{p}}] - L(s, t, v(s, t)) \right) \Delta s \Delta t
\]

\[
+ \left( \int_t \int_s h_2(x, \eta)(q \frac{K^{\frac{q-p}{p}}}{p} v(x, \eta)) + \frac{p-q}{p} K^{\frac{q}{p}} \Delta \xi \Delta \eta \right) \Delta s \Delta t
\]

\[
= \hat{\lambda} + \left( \int_N \int_M [A(s, t, v(s, t)) + \frac{p-1}{p} K^{\frac{1}{p}}] - L(s, t, v(s, t)) \right) \Delta s \Delta t
\]

\[
+ \left( \int_t \int_s h_2(x, \eta)(q \frac{K^{\frac{q-p}{p}}}{p} v(x, \eta)) + \frac{p-q}{p} K^{\frac{q}{p}} \Delta \xi \Delta \eta \right) \Delta s \Delta t.
\]

(49)

where \(\hat{\lambda}\) is defined in (37). Then using (48) in (49) yields

\[
\hat{\mu} \leq \hat{\lambda} + \left( \int_N \int_M [A(s, t, v(s, t)) + \frac{p-1}{p} K^{\frac{1}{p}}] - L(s, t, v(s, t)) \right) \Delta s \Delta t
\]

\[
+ \left( \int_t \int_s h_2(x, \eta)(q \frac{K^{\frac{q-p}{p}}}{p} v(x, \eta)) + \frac{p-q}{p} K^{\frac{q}{p}} \Delta \xi \Delta \eta \right) \Delta s \Delta t.
\]

Combining (45), (48) and (51) we can obtain the desired result.

Remark 15 In [31-32], the authors researched some Gronwall-Bellman type inequalities in two independent variables on time scales. We note that the presented inequalities in (8), (18) and (35) established here are of different forms from the main results in [31-32].

3 Some Applications

In this section, we present some applications for the results established above. New explicit bounds for solutions for certain dynamic equations are derived in the first two examples, while the quantitative property of solutions is concerned in the final example.

Example 1: Consider the following dynamic differential equation

\[
(u^p(x, y))_y^\Delta = F(x, y, u(x, y), \int_y^\infty \int_x^\infty W(\xi, \eta, u(\xi, \eta)) \Delta \xi \Delta \eta), \quad (x, y) \in T_0 \times T_0,
\]

(52)

with the initial condition \(u^p(\infty, y))_y^\Delta = b^\Delta(y), \quad u^p(x, \infty) = a(x),\) where \(u \in C_{rd}(T_0 \times \hat{T}_0, \mathbb{R}), \quad a \in C_{rd}(\hat{T}_0, \mathbb{R}), \quad b \in C_{rd}(\hat{T}_0, \mathbb{R}), \) \(b\) is delta differential, and \(b(\infty) = 0,\) \(k \in C_{rd}(\hat{T}_0 \times \hat{T}_0, \mathbb{R}), \) \(p \geq 0\) is a constant, \(F \in (\mathbb{T}_0 \times \hat{T}_0, \mathbb{R}^2, \mathbb{R}), \) \(W \in (\mathbb{T}_0 \times \hat{T}_0, \mathbb{R}).\)
Theorem 16 Suppose \( u(x,y) \) is a solution of (52), \( |a(x) + b(y)| \leq k(x,y) \), and \( |F(x, y, u, v)| \leq f(x, y)|u|^q + |v|, \) \( |W(\xi, \eta, u)| \leq h(\xi, \eta)|u|^m \), where \( f, \, k, \, q, \, m \) are defined as in Theorem 9. Then

\[
|u(x,y)| \leq [B_1(x,y) + \int_y^\infty e^{-B_2(y,\sigma(t))}]
\]

\[
B_2(x,t)B_1(x,t)\Delta t^\frac{1}{2}, \; (x,y) \in T_0 \times \tilde{T}_0,
\]  

(53)

where

\[
B_1(x,y) = k(x,y) + \int_y^\infty \int_x^\infty [f(s,t)\frac{p-q}{p}K^\frac{q}{2}]
\]

\[
+ \int_t^\infty \int_s^\infty h(\xi, \eta)\frac{p-m}{p}K^\frac{m}{2}\Delta \xi \Delta \eta] \Delta s \Delta t, \forall K > 0,
\]

and \( B_2(x,y) \) is defined as in Theorem 9 (with \( g(x,y) \equiv 0 \)).

Proof: The desired inequality can be obtained by an application of Theorem 10 to (55).

Example 2: Consider the following dynamic integral equation

\[
u^p(x,y) = C + \int_y^\infty \int_x^\infty F_1(s,t,u(s,t)),
\]

\[
\int_t^\infty \int_s^\infty W_1(\xi, \eta, u(\xi, \eta))\Delta \xi \Delta \eta] \Delta s \Delta t
\]

\[
+ \int_t^\infty \int_N^\infty F_2(s,t,u(s,t)),
\]

\[
\int_t^\infty \int_M^\infty W_2(\xi, \eta, u(\xi, \eta))\Delta \xi \Delta \eta] \Delta s \Delta t,
\]

\[
(x,y) \in (M, \infty) \cap (T) \times (N, \infty) \cap (T),
\]  

(57)

where \( u \in C_r\left(T_0 \times \tilde{T}_0, \mathbb{R}\right), \) \( p > 0 \) is a constant, \( C = \nu^p(\infty, \infty), \) \( M \in T_0, \) \( N \in \tilde{T}_0 \) are two fixed numbers, \( F_i \in (T_0 \times \tilde{T}_0 \times \mathbb{R}^2, \mathbb{R}), \) \( W_i \in (T_0 \times \tilde{T}_0 \times \mathbb{R}, \mathbb{R}), \) \( i = 1, 2. \)

Theorem 17 Under the conditions of Theorem 16, furthermore, we have

\[
|u(x,y)| \leq [B_1(x,y)e^{-B_2(y,\infty)}]^{\frac{1}{2}}, \; (x,y) \in T_0 \times \tilde{T}_0,
\]  

(56)

where \( B_1, \, B_2 \) are defined as in Theorem 16.

Proof: The desired inequality can be obtained by an application of Theorem 10 to (55).

Example 2: Consider the following dynamic integral equation

\[
u^p(x,y) = C + \int_y^\infty \int_x^\infty F_1(s,t,u(s,t)),
\]

\[
\int_t^\infty \int_s^\infty W_1(\xi, \eta, u(\xi, \eta))\Delta \xi \Delta \eta] \Delta s \Delta t
\]

\[
+ \int_t^\infty \int_N^\infty F_2(s,t,u(s,t)),
\]

\[
\int_t^\infty \int_M^\infty W_2(\xi, \eta, u(\xi, \eta))\Delta \xi \Delta \eta] \Delta s \Delta t,
\]

\[
(x,y) \in (M, \infty) \cap (T) \times (N, \infty) \cap (T),
\]  

(57)

where \( u \in C_r\left(T_0 \times \tilde{T}_0, \mathbb{R}\right), \) \( p > 0 \) is a constant, \( C = \nu^p(\infty, \infty), \) \( M \in T_0, \) \( N \in \tilde{T}_0 \) are two fixed numbers, \( F_i \in (T_0 \times \tilde{T}_0 \times \mathbb{R}^2, \mathbb{R}), \) \( W_i \in (T_0 \times \tilde{T}_0 \times \mathbb{R}, \mathbb{R}), \) \( i = 1, 2. \)
\[
\int_t^\infty \int_s^\infty W_2(\xi, \eta, u(\xi, \eta))|\Delta \xi \Delta \eta| \Delta s \Delta t
\]
\[
\leq |C| + \int_y^\infty \int_x^\infty |L(s, t, |u(s, t)|) + \int_t^\infty \int_s^\infty W_1(\xi, \eta, u(\xi, \eta))|\Delta \xi \Delta \eta| \Delta s \Delta t
\]
\[
+ \int_t^\infty \int_s^\infty W_2(\xi, \eta, u(\xi, \eta))|\Delta \xi \Delta \eta| \Delta s \Delta t
\]
\[
\leq |C| + \int_y^\infty \int_x^\infty |L(s, t, |u(s, t)|) + \int_t^\infty \int_s^\infty h_1(\xi, \eta)|u(\xi, \eta)|^\alpha |\Delta \xi \Delta \eta| \Delta s \Delta t + \int_t^\infty \int_s^\infty L(s, t, |u(s, t)|)
\]
\[
+ \int_t^\infty \int_s^\infty h_2(\xi, \eta)|u(\xi, \eta)|^\beta |\Delta \xi \Delta \eta| \Delta s \Delta t.
\]
So by use of Theorem 14 we can obtain the desired inequality (58).

**Example 3:** Consider the following dynamic integral equation
\[
u(x, y) = C + \int_y^\infty \int_x^\infty F(s, t, u(s, t)),
\]
(59)
where \(u \in C_{rd}(\bar{T}_0 \times \bar{T}_0, \mathbb{R}), C = u^p(\mathbb{R}, \mathbb{R}, F \in (T_0 \times T_0 \times \mathbb{R}^2, \mathbb{R}), \mathbb{W} \in (T_0 \times T_0 \times \mathbb{R}^2, \mathbb{R}).

**Theorem 19** Assume \(|F(s, t, u_1, v_1) - F(s, t, u_2, v_2)| \leq |f(s, t)||u_1 - u_2| + |v_1 - v_2|, |W(s, t, u_1) - W(s, t, u_2)| \leq |h(s, t)||u_1 - u_2|, where \(f, h\) are defined as in Theorem 9, and furthermore, assume \(\tau_1(x) \geq x_0, \tau_2(y) \geq y_0\), then Eq. (62) has at most one solution.

**Proof:** Suppose \(u_1(x, y), u_2(x, y)\) are two solutions of (59). Then we have
\[
|u_1(x, y) - u_2(x, y)| \leq |\int_y^\infty \int_x^\infty |F(s, t, u_1(s, t)) - F(s, t, u_2(s, t))|\Delta s \Delta t|
\]
\[
\int_t^\infty \int_s^\infty W(\xi, \eta, u_1(\xi, \eta))|\Delta \xi \Delta \eta| - F(s, t, u_2(s, t))|\Delta s \Delta t|
\]
\[
\int_t^\infty \int_s^\infty W(\xi, \eta, u_2(\xi, \eta))|\Delta \xi \Delta \eta| |\Delta s \Delta t|
\]
\[
\leq \int_y^\infty \int_x^\infty |F(s, t, u_1(s, t)) - F(s, t, u_2(s, t))| \Delta s \Delta t
\]
\[
+ \int_y^\infty \int_x^\infty \int_t^\infty \int_s^\infty |W(\xi, \eta, u_1(\xi, \eta)) - W(\xi, \eta, u_2(\xi, \eta))| |\Delta s \Delta t|
\]
\[
\int_t^\infty \int_s^\infty |W(\xi, \eta, u_1(\xi, \eta))| |\Delta s \Delta t|
\]
\[
\leq \int_y^\infty \int_x^\infty |f(s, t)| |u_1(s, t) - u_2(s, t)| |\Delta s \Delta t|
\]
\[
+ \int_y^\infty \int_x^\infty \int_t^\infty \int_s^\infty |h(s, t)| |u_1(s, t) - u_2(s, t)| |\Delta s \Delta t| + \int_y^\infty \int_x^\infty \int_t^\infty \int_s^\infty |h(s, t)| |u_1(s, t) - u_2(s, t)| |\Delta s \Delta t|.
\]
(60)
A suitable application of Theorem 11 yields \(|u_1(x, y) - u_2(x, y)| \leq 0\), that is, \(u_1(x, y) \equiv u_2(x, y)\), and the proof is complete.

### 4 Conclusions

We have established some new Gronwall-Bellman-type dynamic inequalities in two independent variables containing integration on infinite intervals on time scales. As applications, we apply the results established to research boundedness and quantitative property for the solutions to some certain dynamic equations on time scales. In fact, the motive to establish Gronwall-Bellman type inequalities with new forms mostly comes from the research for the properties of solutions to various differential equations, difference equations, and dynamic equations on time scales. It is worth to note that in order to fulfill analysis for the properties of solutions to some fractional differential equations, it is necessary to investigate how to establish new Gronwall-Bellman type fractional inequalities, which are supposed to further research.

**References:**


