On the Trivariate Polynomial Interpolation

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Abstract: This paper is concerned with the formulae for computing the coefficients of the trivariate polynomial interpolation (TPI) passing through $(m+1)(n+1)(r+1)$ distinct points in the solid rectangular region. The TPI is formulated as a matrix equation using Kronecker product and Khatri-Rao product of the matrices and the coefficients of the TPI are computed using the generalized inverse of a matrix. In addition, the closed formulae of the coefficients of the bivariate and univariate polynomial interpolations are obtained by the use of the inverse of the Vandermonde matrix. It is seen that the trivariate polynomial interpolation can be investigated as the matrix equation and the coefficients of the TPI can be computed directly from the solution of the matrix equation. Also, it is shown that the bivariate polynomial interpolation (BPI) is the special case of the TPI when $r = 0$. Numerical examples are represented.

Key-words: Polynomial interpolation, trivariate polynomial, bivariate polynomial, matrix equation.

1 Introduction

The polynomial interpolation plays an important role both in mathematics and applied sciences. The problem of interpolating is fairly common in many engineering and scientific applications [12-14, 22]. The polynomial interpolation is investigated using different forms and algorithms which produce the same polynomial. Also, it is easily solvable either numerically or by using a computer algebra package. The most commonly used polynomial interpolations are the Lagrange and Newton's forms [4, 8, 10, 12-14].

The polynomial interpolation in two or several variables is the most commonly investigated problem of interpolating by researchers [1, 2, 5, 9, 21, 22, 24, 26]. The results on the Hermite interpolation of two variables are given by using the points on different circles [1, 2]. In [2], the authors used the techniques introduced in [1] for considering the more general situation of Birkhoff interpolation. Polynomial interpolation of two variables based on points that are located on multiple circles was studied [2]. The available techniques for the polynomial interpolation and some criteria for uniqueness of interpolating polynomial were extended by generating them in certain directions and by giving variations on the fundamental formula [21]. Computational aspects of the interpolation in several variables are given by De Boor and Ron [5]. The cubature formula and Birkhoff interpolation of the interpolation in several variables were investigated [24, 26]. Finally, Gasca and Sauer gave a survey of main results on multivariate interpolation in the last twenty-five years [9].

In this paper, the problems of the trivariate, bivariate and univariate polynomial interpolations are considered. The main contribution of this paper is to present some formulae for computing the coefficients of the interpolating polynomials. This paper addresses the problem of finding the coefficients of an interpolating polynomial in one, two and three variables. The coefficients of the polynomial interpolations passing through given distinct points in two and three variables are formulated as the matrix equations. The matrix equations are solved by using the inverse of the Vandermonde matrix.

There is a large amount of literature on the computation of the inverse of the Vandermonde matrix and its applications [6, 7, 11, 19, 23]. The inverse of the Vandermonde matrix is investigated by using various methods and algorithms. LU factorization of the Vandermonde matrix and the inverse of the matrix using symmetric functions were investigated, and the applications were given in [16, 17, 19]. Explicit closed form expression for the inverse matrix and algorithms of generalized Vandermonde matrix were given by using the elementary symmetric functions [7]. In [18], the inverse matrix of lower and upper triangular factors of Vandermonde matrix using symmetric functions was investigated.

In this study, the coefficients of the trivariate polynomial interpolation passing through $(m+1)(n+1)(r+1)$ distinct points in the solid rectangular region are investigated as the matrix equation using Kronecker product and Khatri-Rao product of the matrices. It is shown that the trivariate polynomial interpolation can be formulated as a matrix equation and the coefficients of the trivariate polynomial interpolation can be computed directly from the matrix equation. In addition, it is shown that the bivariate polynomial interpolation is the special case of the TPI when $r=0$. Finally, the closed formulae of the coefficients of the bivariate and univariate polynomial interpolations which are the special cases of TPI are obtained using the inverse of the Vandermonde matrix.

2 Polynomial Interpolations

In this section, we give some basic definitions associated with univariate, bivariate and trivariate polynomial interpolations and the Vandermonde matrix. Further details can be found elsewhere [1, 4-10, 12-14].

The simplest and best known way to construct an mth-degree polynomial approximation $p(x)$ to a continuous function $y=f(x)$ in the interval $[a,b]\subset\Re$ is by interpolation. If $x_0,x_1,\ldots,x_m$ are $m+1$ distinct points in $[a,b]$ for arbitrary $m+1$ real values $y_0,y_1,\ldots,y_m$, then there exists a unique polynomial of degree at most m

$$p(x)=a_0+a_1x+a_2x^2+\cdots+a_mx^m \tag{1}$$

such that $p(x_i)=f(x_i)$ for (x_i,y_i), $0\le i\le m$ [4, 8, 10, 12, 14].

The coefficients a_i , $i=0,1,2,\ldots,m$ must satisfy the equations

$$\begin{aligned} a_0+a_1x_0+a_2{x_0}^2+\cdots+a_m{x_0}^m&=y_0\\ a_0+a_1x_1+a_2{x_1}^2+\cdots+a_m{x_1}^m&=y_1\\ &\vdots\\ a_0+a_1x_m+a_2{x_m}^2+\cdots+a_m{x_m}^m&=y_m. \end{aligned} \tag{2}$$

We can write this $(m+1)\times(m+1)$ system as follows:

$$V\mathbf{a}=\mathbf{b}, \tag{3}$$

where

$$V=\begin{bmatrix}1 & x_0 & {x_0}^2 & \cdots & {x_0}^m\\ 1 & x_1 & {x_1}^2 & \cdots & {x_1}^m\\ 1 & x_2 & {x_2}^2 & \cdots & {x_2}^m\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ 1 & x_m & {x_m}^2 & \cdots & {x_m}^m\end{bmatrix},\ \mathbf{a}=\begin{bmatrix}a_0\\a_1\\a_2\\ \vdots\\a_m\end{bmatrix},\ \mathbf{b}=\begin{bmatrix}y_0\\y_1\\y_2\\ \vdots\\y_m\end{bmatrix}$$

and the $(m+1)\times(m+1)$ matrix V is a Vandermonde matrix. When $m+1$ points are distinct, the Vandermonde matrix is nonsingular [4, 10, 12] and consequently the system (3) has a unique solution $\mathbf{a}=V^{-1}\mathbf{b}$, where V^{-1} is inverse of $(m+1)\times(m+1)$ matrix V. Note that this solution depends on the inverse of the Vandermonde matrix.

Let $X=\{x_0,x_1,\ldots,x_m\}$ and $Y=\{y_0,y_1,\ldots,y_n\}$.

Assumed that $z_{ij}=g(x_i,y_j)$ data are given for the function $z=g(x,y)$ of two variables at the points (x_i,y_j) in the rectangular array, there is a unique surface of the form

$$p(x,y)=\sum_{i=0}^{m}\sum_{j=0}^{n}a_{ij}x^iy^j \tag{4}$$

of degree at most $m+n$, namely x^my^n that passes through each point in the $X\times Y$, where $X\times Y$ is Cartesian product of the sets X and Y . The polynomial (4), which is called a bivariate polynomial interpolation, satisfies for $0\le i\le m$, $0\le j\le n$ $p(x_i,y_j)=g(x_i,y_j)$. Thus $g(x,y)$ at any point $(\hat{x},\hat{y})$ which is not in the $X\times Y$ can be estimated by $g(\hat{x},\hat{y})\approx p(\hat{x},\hat{y})$ [1, 5-7, 13].

Also, we now consider the trivariate polynomial interpolation. Given $u = h(x, y, z)$ to approximate over a solid rectangular region that is gridded by (x_i, y_j, z_k), $0 \le i \le m$, $0 \le j \le n$, $0 \le k \le r$ on $\Re^3$. Assumed that $u_{ijk} = h(x_i, y_j, z_k)$ data are given for the function of three variables at the $(m+1)(n+1)(r+1)$ distinct points in the solid rectangular region, there is a unique hyper surface on $\Re^4$ of the form

$$p(x, y, z) = \sum_{i=0}^{m} \sum_{j=0}^{n} \sum_{k=0}^{r} a_{ijk} x^i y^j z^k \tag{5}$$

of degree at most $m+n+r$, namely $x^m y^n z^r$ that passes through each point in the solid rectangular region. We say that (5) is a trivariate polynomial interpolation which satisfies

$$p(x_i, y_j, z_k) = h(x_i, y_j, z_k)$$

for all $0 \le i \le m$, $0 \le j \le n$ and $0 \le k \le r$.

It is clear that $h(x, y, z)$ at any point $(\hat{x}, \hat{y}, \hat{z})$, which is not in solid rectangular region on $\Re^3$, can be estimated by $h(\hat{x}, \hat{y}, \hat{z}) \approx p(\hat{x}, \hat{y}, \hat{z})$ [9, 12, 13].

In the next section, we first formulate a matrix equation to calculate the coefficients of the trivariate polynomial interpolation defined in (5) using Kronecker product and Khatri-Rao product, and then investigate the solution of this matrix equation.

3 The Trivariate Polynomial Interpolation

Let

$$V_x = \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \mathbf{x}_2 \\ \vdots \\ \mathbf{x}_m \end{bmatrix}, \; V_y = \begin{bmatrix} \mathbf{y}_0 \\ \mathbf{y}_1 \\ \mathbf{y}_2 \\ \vdots \\ \mathbf{y}_n \end{bmatrix}, \; V_z = \begin{bmatrix} \mathbf{z}_0 \\ \mathbf{z}_1 \\ \mathbf{z}_2 \\ \vdots \\ \mathbf{z}_r \end{bmatrix} \tag{6}$$

and

$$A = [A_0 \quad A_1 \quad \dots \quad A_r],$$
$$F = [F_0 \quad F_1 \quad \dots \quad F_m], \tag{7}$$

where

$$A_0 = \begin{bmatrix} a_{000} & a_{100} & \cdots & a_{m00} \\ a_{010} & a_{110} & \cdots & a_{m10} \\ \vdots & \vdots & \vdots & \vdots \\ a_{0n0} & a_{1n0} & \cdots & a_{mn0} \end{bmatrix}, \; A_1 = \begin{bmatrix} a_{001} & a_{101} & \cdots & a_{m01} \\ a_{011} & a_{111} & \cdots & a_{m11} \\ \vdots & \vdots & \vdots & \vdots \\ a_{0n1} & a_{1n1} & \cdots & a_{mn1} \end{bmatrix},$$

$\dots,$

$$A_r = \begin{bmatrix} a_{00r} & a_{10r} & \cdots & a_{m0r} \\ a_{01r} & a_{11r} & \cdots & a_{m1r} \\ \vdots & \vdots & \vdots & \vdots \\ a_{0nr} & a_{1nr} & \cdots & a_{mnr} \end{bmatrix},$$

$$F_0 = \begin{bmatrix} u_{000} & u_{001} & \cdots & u_{00r} \\ u_{010} & u_{011} & \cdots & u_{01r} \\ \vdots & \vdots & \vdots & \vdots \\ u_{0n0} & u_{0n1} & \cdots & u_{0nr} \end{bmatrix}, F_1 = \begin{bmatrix} u_{100} & u_{101} & \cdots & u_{10r} \\ u_{110} & u_{111} & \cdots & u_{11r} \\ \vdots & \vdots & \vdots & \vdots \\ u_{1n0} & u_{1n1} & \cdots & u_{1nr} \end{bmatrix},$$

$\dots,$

$$F_m = \begin{bmatrix} u_{m00} & u_{m01} & \cdots & u_{m0r} \\ u_{m10} & u_{m11} & \cdots & u_{m1r} \\ \vdots & \vdots & \vdots & \vdots \\ u_{mn0} & u_{mn1} & \cdots & u_{mnr} \end{bmatrix},$$

$$\mathbf{x}_i = [1 \quad x_i \quad x_i^2 \quad \dots \quad x_i^m],$$

$$\mathbf{y}_j = [1 \quad y_j \quad y_j^2 \quad \dots \quad y_j^n],$$

$$\mathbf{z}_k = [1 \quad z_k \quad z_k^2 \quad \dots \quad z_k^r],$$

and F_i is an $(n+1) \times (r+1)$ matrix and A_k is an $(n+1) \times (m+1)$ matrix.

Note that a_{ijk} and $h(x_i, y_j, z_k) = u_{ijk}$ are the elements of the $(n+1) \times (m+1)(r+1)$ matrices A and F defined in (7) respectively, where $0 \le i \le m$, $0 \le j \le n$ and $0 \le k \le r$.

Using (6) and (7), we can formulate the coefficients of the trivariate polynomial interpolation (5) as follows:

$$V_y A[\mathbf{x}_0^T \otimes I_{r+1} \; \mathbf{x}_1^T \otimes I_{r+1} \dots \mathbf{x}_m^T \otimes I_{r+1}](V_z^T \otimes I_{m+1}) = F \tag{8}$$

where I_{r+1} is the $(r+1) \times (r+1)$ identity matrix and $\otimes$ is the Kronecker product.

Also, we can express (8), which is the matrix equation of (5), as follows:

$$V_y \, A\left(V_x^T * (I_{r+1} \otimes \mathbf{1}_{m+1}^T)\right)\left(V_z^T \otimes I_{m+1}\right) = F, \tag{9}$$

where $*$ is the Khatri-Rao product and $\mathbf{1}_{m+1}$ is the $(m+1)\times 1$ vector whose entries are 1.

Corollary 1 Let $\left(V_x^{\,T} * (I_{r+1} \otimes \mathbf{1}_{m+1}^T)\right)$ be the matrix defined in (9). Then its inverse is

$$\left(V_x^{\,T} * (I_{r+1} \otimes \mathbf{1}_{m+1}^T)\right)^{-1} = \left(V_x * (I_{r+1} \otimes \mathbf{1}_{m+1}\right)\left((V_x^T V_x)^{-1} \otimes I_{r+1}\right). \quad (10)$$

Proof. Observing the rank of the matrix defined in (10) is $(m+1)(r+1)$, and using the generalized inverse of the matrix of full rank and Khatri-Rao product of the matrices [15, 25], (10) is easily obtained as

$$\left(V_x^{\,T} * (I_{r+1} \otimes \mathbf{1}_{m+1}^T)\right)^{-1} = \left[\boldsymbol{x}_0^T \otimes I_{r+1} \quad \boldsymbol{x}_1^T \otimes I_{r+1} \quad \ldots \quad \boldsymbol{x}_m^T \otimes I_{r+1}\right]^{-1}$$

$$= \begin{bmatrix} \boldsymbol{x}_0 \otimes I_{r+1} \\ \boldsymbol{x}_1 \otimes I_{r+1} \\ \vdots \\ \boldsymbol{x}_m \otimes I_{r+1} \end{bmatrix} \left(\left(V_x^T V_x\right)^{-1} \otimes I_{r+1}\right).$$

Thus the proof is completed.

We use the following Lemma to solve the matrix equation (9).

Lemma 1. A necessary and sufficient condition for the equation $BXC = D$ to have a solution is that

$$BB^{+}DC^{+}C = D$$

in this case, the general solution is

$$X = B^{+}DC^{+} + Y - B^{+}BYCC^{+},$$

where Y is an arbitrary matrix and B^{+} is the generalized inverse of the matrix B [3, 20].

Theorem 1 The equation (9) has a unique solution as follows:

$$A = V_y^{-1} F \left(V_z^{\,T} \otimes I_{m+1}\right)^{-1}\left(V_x^{\,T} * (I_{r+1} \otimes \mathbf{1}_{m+1}^T)\right)^{-1} \quad (11)$$

or equivalently

$$A = V_y^{-1} F \left(\left(V_z^{\,T}\right)^{-1} \otimes I_{m+1}\right)\left(V_x * (I_{r+1} \otimes \mathbf{1}_{m+1})\right)\left(\left(V_x^T V_x\right)^{-1} \otimes I_{r+1}\right) \quad (12)$$

Proof. Since

$$V_y V_y^{-1} F\left(V_z^{\,T} \otimes I_{m+1}\right)^{-1}\left(V_x^{\,T} * (I_{r+1} \otimes \mathbf{1}_{m+1}^T)\right)^{-1}\left(V_x^{\,T} * (I_{r+1} \otimes \mathbf{1}_{m+1}^T)\right)\left(V_z^{\,T} \otimes I_{m+1}\right)$$
$$= I_{n+1} F \left(V_z^{\,T} \otimes I_{m+1}\right)^{-1} I_{(m+1)(r+1)} \left(V_z^{\,T} \otimes I_{m+1}\right)$$

$$= I_{n+1} F I_{(m+1)(r+1)}$$
$$= F,$$

the equation (9) has a solution. Also using Lemma 1, we obtain the general solution as

$$A = A_p + A_h,$$

where

$$A_p = V_y^{-1} F \left(V_z^{\,T} \otimes I_{m+1}\right)^{-1}\left(V_x^{\,T} * (I_{r+1} \otimes \mathbf{1}_{m+1}^T)\right)^{-1},$$

$$A_h = W - V_y^{-1} V_y W \left(V_x^{\,T} * (I_{r+1} \otimes \mathbf{1}_{m+1}^T)\right) \left(V_z^{\,T} \otimes I_{m+1}\right)\left(V_z^{\,T} \otimes I_{m+1}\right)^{-1}\left(V_x^{\,T} * (I_{r+1} \otimes \mathbf{1}_{m+1}^T)\right)^{-1}$$

$$= W - I_{n+1} W \left(V_x^{\,T} * (I_{r+1} \otimes \mathbf{1}_{m+1}^T)\right) I_{(r+1)(m+1)} \left(V_x^{\,T} * (I_{r+1} \otimes \mathbf{1}_{m+1}^T)\right)^{-1}$$

$$= W - W I_{(m+1)(r+1)} = 0$$

W is an arbitrary matrix, and A_p and A_h are the particular and homogenous solutions, respectively. If $A_p = A$, then (9) has a unique solution. Thus the proof is completed.

Theorem 2 Let A_k and F_i be submatrices defined in (7). Then, the solution of the equation (9) is

$$A = V_y^{-1} \sum_{i=0}^{m} F_i \left(V_z^T\right)^{-1} \left[\mathbf{x}_i (V_x^T V_x)^{-1} \otimes I_{r+1}\right], \quad (13)$$

or

$$A_k = V_y^{-1} \sum_{i=0}^{m} F_i \left(V_z^T\right)^{-1} \left[\mathbf{x}_i (V_x^T V_x)^{-1} \otimes \mathbf{e}_{k+1}\right], \quad (14)$$

where

$$\mathbf{e}_{k+1}^{T} = [0 \cdots 0 \underbrace{1}_{(k+1)th\ entry} 0 \cdots 0],$$

for $i = 0,1,2,\ldots,m$ and $k = 0,1,2,\ldots,r$.

Proof. To prove the theorem, we can use Theorem 1 and the matrices defined in (7). Putting (7) in (12) and using Kronecker product and Khatri-Rao product, the equation (13) is obtained. Taking the identity matrix as in (13), we get the equation (14).

We show that whether the bivariate polynomial interpolation (4) is the special case of the trivariate polynomial interpolation (5) or not.

Let us assume that $r=0$ in (9) and $a_{ij0}=a_{ij}$, $u_{ij0}=z_{ij}$ for $0\le i\le m$, $0\le j\le n$ in (7). Taking $r=0$ and using the equations (7) and (9) we obtain

$$V_y A V_x^{\,T} = Z\,, \tag{15}$$

where

$$A=\begin{bmatrix} a_{00} & a_{10} & \cdots & a_{m0} \\ a_{01} & a_{11} & \cdots & a_{m1} \\ \vdots & \vdots & \vdots & \vdots \\ a_{0n} & a_{1n} & \cdots & a_{mn} \end{bmatrix},\; Z=\begin{bmatrix} z_{00} & z_{10} & \cdots & z_{m0} \\ z_{01} & z_{11} & \cdots & z_{m1} \\ \vdots & \vdots & \vdots & \vdots \\ z_{0n} & z_{1n} & \cdots & z_{mn} \end{bmatrix},$$

A and Z are the $(n+1)\times(m+1)$ matrices. In this case, it is clear that $I_{r+1}=1$, $V_z=1$ and

$$\left(V_x^{\,T} * (I_{r+1}\otimes \mathbf{1}_{m+1}^T)\right) = V_x^{\,T}\,.$$

The equation (15) can be defined as the matrix equation which is used to find the coefficients a_{ij} of the BPI (4) satisfying $p(x_i,y_j)=g(x_i,y_j)$ for all (x_i,y_j) in the rectangular array. In addition, we can state the BPI (4) as a matrix equation in the following form:

$$p(x,y)=\mathbf{y}A\,\mathbf{x}^T\,, \tag{16}$$

where A is defined above. This equation can be easily obtained from

$$p(x,y,z)=\mathbf{y}A\left(\mathbf{x}^T\otimes I_{r+1}\right)\mathbf{z}^T\,, \tag{17}$$

which is the matrix equation of (5), when $r=0$, where $\mathbf{z}=1$.

Note that the solution of the matrix equation (15),

$$A=V_y^{-1}\mathbf{Z}\left(V_x^{\,T}\right)^{-1} \tag{18}$$

can be found from Theorem.1 or Theorem.2, and the BPI can be investigated as a special case of the TPI.

Since Vandermonde matrices V_x and V_y are nonsingular [7, 12, 23], the system of matrix equation (15) has a unique solution [3, 20]. So there is one and only one solution set of coefficients for polynomial (4). It is well known that the equation (15) is solvable either numerically or using a computer algebra packages.

In this study, the coefficients of the polynomial interpolation are to be computed directly by generating special formulae, which can be applied easily to the polynomial interpolations satisfying the given distinct points.

We now consider the Vandermonde matrix V_x. We can rewrite the closed form of the inverse of V_x as

$$V_x^{-1}=\left[\mathbf{v}_0 \quad \mathbf{v}_1 \quad \cdots \quad \mathbf{v}_m\right], \tag{19}$$

where the columns vectors $\mathbf{v}_0,\mathbf{v}_1,\ldots,\mathbf{v}_m$ are

$$\mathbf{v}_0=\frac{1}{\prod_{i=1}^{m}\left(x_i-x_0\right)}\begin{bmatrix} \prod_{i=1}^{m} x_i \\ -\sum_{i_1=1}^{2}\sum_{i_2=2}^{3}\cdots\sum_{i_{m-1}=m-1}^{m} x_{i_1}x_{i_2}\cdots x_{i_{m-1}} \\ \vdots \\ (-1)^{m-2}\sum_{i_1=1}^{m-1}\sum_{i_2=2}^{m} x_{i_1}x_{i_2} \\ (-1)^{m-1}\sum_{i_1=1}^{m} x_{i_1} \\ (-1)^{m} \end{bmatrix}, \tag{20}$$

$$\mathbf{v}_1=\frac{1}{\prod_{\substack{i=0\\ i\neq 1}}^{m}\left(x_i-x_1\right)}\begin{bmatrix} \prod_{i=0,\,i\neq 1}^{m} x_i \\ -\sum_{i_0\neq 1,\,i_0=0}^{2}\sum_{i_2=2}^{3}\cdots\sum_{i_{m-1}=m-1}^{n} x_{i_0}x_{i_2}\cdots x_{i_{m-1}} \\ \vdots \\ (-1)^{m-2}\sum_{i_0\neq 1,\,i_0=0}^{m-1}\sum_{i_2=2}^{m} x_{i_0}x_{i_2} \\ (-1)^{m-1}\sum_{i_0=0,\,i_0\neq 1}^{m} x_{i_0} \\ (-1)^{m} \end{bmatrix},$$

$\ldots,$

$$\mathbf{v}_m = \frac{1}{\prod_{i=0}^{m-1}(x_i - x_n)} \begin{bmatrix} \prod_{i=0}^{m-1} x_i \\ -\sum_{i_0=0}^{1}\sum_{i_1=1}^{2}\cdots\sum_{i_{m-2}=m-2}^{m-1} x_{i_0}x_{i_1}\cdots x_{i_{m-2}} \\ \vdots \\ (-1)^{m-2}\sum_{i_0=0}^{m-2}\sum_{i_1=1}^{m-1} x_{i_0}x_{i_1} \\ (-1)^{m-1}\sum_{i_0=0}^{m-1} x_{i_0} \\ (-1)^m \end{bmatrix}.$$

All the formulae for the entries of the matrices L^{-1} and U^{-1}, being L and U the triangular matrices in the LU factorization of V_x can be found by replacing values of the elementary symmetric functions in [19]. The inverses of matrices L and U are formulated in an closed form as

$$U^{-1} = \begin{bmatrix} 1 & -x_0 & x_0x_1 & -\prod_{i_0=0}^{2} x_{i_0} & \dots & (-1)^m \prod_{i_0=0}^{m-1} x_{i_0} \\ 0 & 1 & -(x_0+x_1) & \sum_{i_0=0}^{1}\sum_{i_1=1}^{2} x_{i_0}x_{i_1} & \dots & (-1)^{m-1}\left(\sum_{i_0=0}^{1}\sum_{i_1=1}^{2}\cdots\sum_{i_{m-2}=m-2}^{m-1} x_{i_0}x_{i_1}\cdots x_{i_{m-2}}\right) \\ 0 & 0 & 1 & -\sum_{i_0=1}^{2} x_{i_0} & \dots & (-1)^{m-2}\left(\sum_{i_0=0}^{2}\sum_{i_1=1}^{3}\cdots\sum_{i_{m-3}=m-3}^{m-1} x_{i_0}x_{i_1}\cdots x_{i_{m-3}}\right) \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \dots & -\sum_{i_0=0}^{n-1} x_{i_0} \\ 0 & 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

and

$$L^{-1} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ -\frac{1}{x_1-x_0} & -\frac{1}{x_0-x_1} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \frac{(-1)^{m-1}}{\prod_{i=1}^{m-1}(x_i-x_0)} & \frac{(-1)^{m-1}}{\prod_{i=0,i\neq1}^{m-1}(x_i-x_1)} & \frac{(-1)^{m-1}}{\prod_{i=0,i\neq2}^{m-1}(x_i-x_2)} & \dots & 0 \\ \frac{(-1)^{m}}{\prod_{i=1}^{m}(x_i-x_0)} & \frac{(-1)^{m}}{\prod_{i=0,i\neq1}^{m}(x_i-x_1)} & \frac{(-1)^{m}}{\prod_{i=0,i\neq2}^{m}(x_i-x_2)} & \dots & \frac{(-1)^{m-1}}{\prod_{i=0}^{m-1}(x_i-x_m)} \end{bmatrix}$$

where $0 \le i_0 < i_1 < ... < i_m \le m$ and $i_l = i_{l-1} + 1$, $l = 1, 2, ..., m$. The inverse of the Vandermonde matrix defined as (19) can be used for computing the coefficients of the bivariate polynomial interpolation.

If (19) is applied to the inverses of the Vandermonde matrices V_x and V_y defined as (20), we obtain the closed formulae of the coefficients of the BPI.

This leads to the following results.

Corollary 2 Let B_{ji} be the product of $(j+1)^{th}$ row of V_y^{-1} with $(i+1)^{th}$ column of the matrix Z. Then B_{ji}'s are

$$B_{0i} = \frac{\prod_{j=1}^{n} y_j}{\prod_{j=1}^{n}(y_j - y_0)} g(x_i, y_0) + \frac{\prod_{j=0, j\neq1}^{n} y_j}{\prod_{j=0, j\neq1}^{n}(y_i - y_1)} g(x_i, y_1) + \cdots + \frac{\prod_{j=0}^{n-1} y_j}{\prod_{j=0}^{n-1}(y_j - y_n)} g(x_i, y_n),$$

$$B_{1i} = -\left(\frac{\sum_{j_1=1}^{2}\sum_{j_2=2}^{3}\cdots\sum_{j_{n-1}=n-1}^{n} y_{j_1}y_{j_2}\cdots y_{j_{n-1}}}{\prod_{j=1}^{n}(y_j - y_0)} g(x_i, y_0) + \frac{\sum_{j_0\neq1, j_0=0}^{2}\sum_{j_2=2}^{3}\cdots\sum_{j_{n-1}=n-1}^{n} y_{j_0}y_{j_2}\cdots y_{j_{n-1}}}{\prod_{j=0, j\neq1}^{n}(y_i - y_1)} g(x_i, y_1) + \cdots + \frac{\sum_{j_0=0}^{1}\sum_{j_1=1}^{2}\cdots\sum_{j_{n-2}=n-2}^{n-1} y_{j_0}y_{j_1}\cdots y_{j_{n-2}}}{\prod_{j=0}^{n-1}(y_j - y_n)} g(x_i, y_n) \right), ...,$$

$$B_{ni} = (-1)^n \left(\frac{1}{\prod_{j=1}^{n}(y_j - y_0)} g(x_i, y_0) + \frac{1}{\prod_{j=0, j\neq1}^{n}(y_i - y_1)} g(x_i, y_1) + \cdots + \frac{1}{\prod_{j=0}^{n-1}(y_j - y_n)} g(x_i, y_n) \right)$$

where $i = 0, 1, 2, .., m$ and $j = 0, 1, 2, .., n$.

Proof. To prove the corollary, we can apply (20) to the matrix V_y^{-1} defined in (6). Multiplying $(j+1)^{th}$ row of V_y^{-1} with $(i+1)^{th}$ column of the matrix Z, we obtain B_{ji} as $V_y^{-1}Z = [B_{ji}]$. Thus the proof is completed.

Corollary 3 Let a_{ij} be the coefficients of the polynomial interpolation (4) satisfying $(m+1)(n+1)$ distinct points in the rectangular array. Then the coefficients a_{ij} of the BPI are

$$a_{0j} = \frac{\prod_{i=1}^{m} x_i}{\prod_{i=1}^{m}(x_i - x_0)} B_{j0} + \frac{\prod_{i=0,i\neq 1}^{m} x_i}{\prod_{i=0,i\neq 1}^{m}(x_i - x_1)} B_{j1} + \cdots + \frac{\prod_{i=0}^{m-1} x_i}{\prod_{i=0}^{m-1}(x_i - x_m)} B_{jm} ,$$

$$a_{1j} = -\left(\frac{\sum_{i_1=1}^{2}\sum_{i_2=2}^{3}\cdots\sum_{i_{m-1}=m-1}^{m} x_{i_1}x_{i_2}\cdots x_{i_{m-1}}}{\prod_{i=1}^{m}(x_i - x_0)} B_{j0} + \frac{\sum_{i_0\neq 1, i_0=0}^{2}\sum_{i_2=2}^{3}\cdots\sum_{i_{m-1}=m-1}^{m} x_{i_0}x_{i_2}\cdots x_{i_{m-1}}}{\prod_{i=0,i\neq 1}^{m}(x_i - x_1)} B_{j1} + \cdots + \frac{\sum_{i_0=0}^{1}\sum_{i_1=1}^{2}\cdots\sum_{i_{m-2}=m-2}^{m-1} x_{i_0}x_{i_1}\cdots x_{i_{m-2}}}{\prod_{i=0}^{m-1}(x_i - x_m)} B_{jm} \right), \ldots,$$

$$a_{mj} = (-1)^m \left(\frac{1}{\prod_{i=1}^{m}(x_i - x_0)} B_{j0} + \frac{1}{\prod_{i=0,i\neq 1}^{m}(x_i - x_1)} B_{j1} + \cdots + \frac{1}{\prod_{i=0}^{m-1}(x_i - x_m)} B_{jm} \right)$$

for $i = 0, 1, 2, .., m$ and $j = 0, 1, 2, ..., n$.

Proof Using (20), the coefficients a_{ij} are easily computed from the product of $(j+1)^{th}$ row of $V_y^{-1}Z$ with $(i+1)^{th}$ column of $(V_x^{-1})^T$.

For example for $m = 2$ and $n = 2$, the coefficients of the polynomial interpolation in two variables are computed easily from the above formulae as follows:

$$B_{0i} = \frac{y_1 y_2}{(y_1 - y_0)(y_2 - y_0)} g(x_i, y_0) + \frac{y_0 y_2}{(y_0 - y_1)(y_2 - y_1)} g(x_i, y_1) + \frac{y_0 y_1}{(y_0 - y_2)(y_1 - y_2)} g(x_i, y_2)$$

$$B_{1i} = -\left(\frac{y_1 + y_2}{(y_1 - y_0)(y_2 - y_0)} g(x_i, y_0) + \frac{y_0 + y_2}{(y_0 - y_1)(y_2 - y_1)} g(x_i, y_1) + \frac{y_0 + y_1}{(y_0 - y_2)(y_1 - y_2)} g(x_i, y_2 \right)$$

$$B_{2i} = \frac{1}{(y_1 - y_0)(y_2 - y_0)} g(x_i, y_0) + \frac{1}{(y_0 - y_1)(y_2 - y_1)} g(x_i, y_1) + \frac{1}{(y_0 - y_2)(y_1 - y_2)} g(x_i, y_2)$$

$$a_{0j} = \frac{x_1 x_2}{(x_1 - x_0)(x_2 - x_0)} B_{j0} + \frac{x_0 x_2}{(x_0 - x_1)(x_2 - x_1)} B_{j1} + \frac{x_0 x_1}{(x_0 - x_2)(x_1 - x_2)} B_{j2}$$

$$a_{1j} = -\left(\frac{x_1 + x_2}{(x_1 - x_0)(x_2 - x_0)} B_{j0} + \frac{x_0 + x_2}{(x_0 - x_1)(x_2 - x_1)} B_{j1} + \frac{x_0 + x_1}{(x_0 - x_2)(x_1 - x_2)} B_{j2} \right)$$

$$a_{2j} = \frac{1}{(x_1 - x_0)(x_2 - x_0)} B_{j0} + \frac{1}{(x_0 - x_1)(x_2 - x_1)} B_{j1} + \frac{1}{(x_0 - x_2)(x_1 - x_2)} B_{j2}$$

where $i = 0, 1, 2$ and $j = 0, 1, 2$.

Example 1: Assume the following values for a function $g(x, y)$ in two variables of twelve distinct points: $g(-1,0) = 3$, $g(-1,1) = 2$, $g(-1,2) = -1$, $g(0,0) = 1$, $g(0,1) = 2$, $g(0,2) = 7$, $g(1,0) = -1$, $g(1,1) = 0$, $g(1,2) = 7$, $g(2,0) = -3$, $g(2,1) = 2$, $g(2,2) = 23$.

The coefficients of the polynomial interpolation in two variables are easily obtained by writing $m = 3$ and $n = 2$ in above formulae. Using these coefficients, the polynomial interpolation in two variables satisfying twelve points is written as $P(x, y) = 1 - y + 2y^2 - 2x + xy^2 - x^2y^2 + x^3y^2$.

Finally, taking $n = 0$, $a_{i0} = a_i$ and $z_{i0} = y_i$ from (15), we obtain (3) as follows:

$$\mathbf{a}^T V_x^{\ T} = \mathbf{b}^T , \tag{21}$$

where $V_y = 1$,

$\mathbf{a}^T = [a_0 \quad a_1 \quad \ldots \quad a_m]$

and

$\mathbf{b}^T = [y_0 \quad y_1 \quad \cdots \quad y_m]$.

It is clear that (21) has the same solution with (3) and consequently it is seen that the univariate polynomial interpolation is the special case of the BPI. Now, we can obtain the coefficients of the univariate polynomial defined in (1) or (21) from the closed form of the inverse of V_x. The following result is concerned with the coefficients of (1).

Corollary 4 Let a_i for $i = 0, 1, 2, .., m$ be the coefficients of the polynomial interpolation (1). Then

$$a_0 = \frac{\prod_{i=1}^{m} x_i}{\prod_{i=1}^{m}(x_i - x_0)} y_0 + \frac{\prod_{i=0,i\neq 1}^{m} x_i}{\prod_{i=0,i\neq 1}^{m}(x_i - x_1)} y_1 + \cdots + \frac{\prod_{i=0}^{m-1} x_i}{\prod_{i=0}^{m-1}(x_i - x_n)} y_m ,$$

$$a_1 = -\left(\frac{\sum_{i_1=1}^{2}\sum_{i_2=2}^{3}\cdots\sum_{i_{m-1}=m-1}^{m} x_{i_1}x_{i_2}\cdots x_{i_{m-1}}}{\prod_{i=1}^{m}(x_i - x_0)} y_0 + \frac{\sum_{i_0\neq 1, i_0=0}^{2}\sum_{i_2=2}^{3}\cdots\sum_{i_{m-1}=m-1}^{m} x_{i_0}x_{i_2}\cdots x_{i_{m-1}}}{\prod_{i=0,\neq 1}^{n}(x_i - x_1)} y_1 + \cdots + \frac{\sum_{i_0=0}^{1}\sum_{i_1=1}^{2}\cdots\sum_{i_{m-2}=m-2}^{m-1} x_{i_0}x_{i_1}\cdots x_{i_{m-2}}}{\prod_{i=0}^{m-1}(x_i - x_m)} y_m \right), ...,$$

$$a_{m-1} = (-1)^{m-1}\left(\frac{\sum_{i_1=1}^{m} x_{i_1}}{\prod_{i=1}^{m}(x_i - x_0)} y_0 + \frac{\sum_{i_0=0,i_0\neq 1}^{m} x_{i_0}}{\prod_{i=0,i\neq 1}^{m}(x_i - x_1)} y_1 + \cdots + \frac{\sum_{i_0=0}^{m-1} x_{i_0}}{\prod_{i=0}^{m-1}(x_i - x_m)} y_m \right)$$

$$a_m = (-1)^m \left(\frac{1}{\prod_{i=1}^{m}(x_i - x_0)} y_0 + \frac{1}{\prod_{i=0,i\neq 1}^{m}(x_i - x_1)} y_1 + \cdots + \frac{1}{\prod_{i=0}^{1}(x_i - x_m)} y_m \right)$$

Proof. Using (20) and applying the product of the matrix V^{-1} and the vector $\mathbf{b}$, it is easily proved.

Using Corollary 4, the following examples are presented.

Example 2 The linear equation satisfying (x_0, y_0) and (x_1, y_1) two distinct points is computed as

$$y = \left(\frac{x_1 y_0 - x_0 y_1}{x_1 - x_0} \right) + \left(\frac{y_1 - y_0}{x_1 - x_0} \right) x$$

and the quadratic equation passing through (x_0, y_0), (x_1, y_1) and (x_2, y_2) three distinct points is written as

$$P_2(x) = \left(\frac{x_1 x_2}{(x_1 - x_0)(x_2 - x_0)} y_0 + \frac{x_0 x_2}{(x_0 - x_1)(x_2 - x_1)} y_1 + \frac{x_0 x_1}{(x_0 - x_2)(x_1 - x_2)} y_2 \right)$$
$$- \left(\frac{x_1 + x_2}{(x_1 - x_0)(x_2 - x_0)} y_0 + \frac{x_0 + x_2}{(x_0 - x_1)(x_2 - x_1)} y_1 + \frac{x_0 + x_1}{(x_0 - x_2)(x_1 - x_2)} y_2 \right) x$$
$$+ \left(\frac{1}{(x_1 - x_0)(x_2 - x_0)} y_0 + \frac{1}{(x_0 - x_1)(x_2 - x_1)} y_1 + \frac{1}{(x_0 - x_2)(x_1 - x_2)} y_2 \right) x^2.$$

Example 3 Consider a set $S = \{(-1, 2), (0,1), (1,4), (3,-2)\}$

of four distinct points. The coefficients of third degree polynomial interpolation passing through four distinct points are obtained by writing $n = 3$ as

$$a_0 = \frac{x_1 x_2 x_3}{(x_1 - x_0)(x_2 - x_0)(x_3 - x_0)} y_0 + \frac{x_0 x_2 x_3}{(x_0 - x_1)(x_2 - x_1)(x_3 - x_1)} y_1 + \frac{x_0 x_1 x_3}{(x_0 - x_2)(x_1 - x_2)(x_3 - x_2)} y_2 + \frac{x_0 x_1 x_2}{(x_0 - x_3)(x_1 - x_3)(x_2 - x_3)} y_3$$

$$a_1 = -\left(\frac{x_1 x_2 + x_1 x_3 + x_2 x_3}{(x_1 - x_0)(x_2 - x_0)(x_3 - x_0)} y_0 + \frac{x_0 x_2 + x_0 x_3 + x_2 x_3}{(x_0 - x_1)(x_2 - x_1)(x_3 - x_1)} y_1 + \frac{x_0 x_1 + x_0 x_3 + x_1 x_3}{(x_0 - x_2)(x_1 - x_2)(x_3 - x_2)} y_2 + \frac{x_0 x_1 + x_0 x_2 + x_1 x_2}{(x_0 - x_3)(x_1 - x_3)(x_2 - x_3)} y_3 \right)$$

$$a_2 = \frac{x_1 + x_2 + x_3}{(x_1 - x_0)(x_2 - x_0)(x_3 - x_0)} y_1 + \frac{x_0 + x_2 + x_3}{(x_0 - x_1)(x_2 - x_1)(x_3 - x_1)} y_2 + \frac{x_0 + x_1 + x_3}{(x_0 - x_2)(x_1 - x_2)(x_3 - x_2)} y_3 + \frac{x_0 + x_1 + x_2}{(x_0 - x_3)(x_1 - x_3)(x_2 - x_3)} y_3$$

$$a_3 = -\left(\frac{1}{(x_1 - x_0)(x_2 - x_0)(x_3 - x_0)} y_0 + \frac{1}{(x_0 - x_1)(x_2 - x_1)(x_3 - x_1)} y_1 + \frac{1}{(x_0 - x_2)(x_1 - x_2)(x_3 - x_2)} y_2 + \frac{1}{(x_0 - x_3)(x_1 - x_3)(x_2 - x_3)} y_3 \right)$$

and $a_0 = 1$, $a_1 = 2$, $a_2 = 2$ and $a_3 = -1$ are calculated by using these formulae and so the polynomial interpolation satisfying four distinct points in the set S is $p_3(x) = 1 + 2x + 2x^2 - x^3$.

4 Conclusions

We consider the trivariate polynomial interpolation (TPI) passing through $(m+1)(n+1)(r+1)$ distinct points in the solid rectangular region. We conclude that the coefficients of TPI can be computed directly from the matrix equation by the use of generalized inverses, Kronecker product and Khatri-Rao product.

It is seen that the trivariate polynomial interpolation can be investigated as the matrix equation and its coefficients can be computed directly from this matrix equation. In addition, it is shown that the bivariate polynomial interpolation (BPI) is expressed as the special case of the TPI when $r = 0$ and the special formulae of the coefficients of the BPI are obtained using the inverse of the Vandermonde matrix. Also, the coefficients of the polynomial $p(x)$ are computed using the closed formulae.

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