

# Some New Results Related To A Generalized Ostrowski-Grüss Type Inequality On Time Scales

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*Abstract:* In this paper, we present some new bounds for a generalized Ostrowski-Grüss type integral inequality on time scales, which on one hand unify continuous and discrete analysis, on the other hand extend some known results in the literature. Some of the bounds for the presented Ostrowski-Grüss type inequality are sharp.

*Key-Words:* Ostrowski type inequality; Grüss type inequality; Time scales; Bounds; Numerical integration; Error estimate

## 1 Introduction

In recent years, the research for the Ostrowski type and Grüss type inequalities has been a hot topic in the literature. The Ostrowski type inequality can be used to estimate the absolute deviation of a function from its integral mean, and it was originally presented by Ostrowski in [1] as follows:

$$\begin{aligned} & \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \\ & \leq \left[ \frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right] (b-a)M, \quad x \in [a, b], \end{aligned}$$

where  $I \subset \mathbb{R}$  is an interval and  $\text{Int}I$  is the interior of  $I$ ,  $a, b \in \text{Int}I$  with  $a < b$ , and  $f : I \rightarrow \mathbb{R}$  is a differentiable mapping in  $\text{Int}I$  with  $|f'(t)| \leq M$ ,  $\forall t \in [a, b]$ .

The Grüss inequality, which can be used to estimate the absolute deviation of the integral of the product of two functions from the product of their respective integral, was originally presented by Grüss in [2] as follows

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(x)g(x)dx - \left( \frac{1}{b-a} \int_a^b f(x)dx \right) \times \right. \\ & \left. \left( \frac{1}{b-a} \int_a^b g(x)dx \right) \right| \leq \frac{1}{4}(M-m)(N-n), \end{aligned}$$

where  $f, g$  are integrable functions on  $[a, b]$  and satisfy  $m \leq f(x) \leq M$ ,  $n \leq g(x) \leq N$ .

In the last few decades, various generalizations of the Ostrowski inequality and the Grüss inequality including continuous and discrete versions have been established (for example, see [3-13] and the references therein), while some new inequalities are established, one of which is the inequalities of Ostrowski-Grüss type (for example, see [14-23]). The first Ostrowski-Grüss type inequality was presented by Dragomir and Wang in [14] as follows:

$$\begin{aligned} & \left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt - \frac{f(b) - f(a)}{b-a} \left( x - \frac{a+b}{2} \right) \right| \\ & \leq \frac{1}{4}(b-a)(\Gamma - \gamma), \end{aligned}$$

where  $f' \in L_1[a, b]$ , and  $\gamma \leq f'(x) \leq \Gamma$ .

Then in [15], Matić et al improved the above result to the following form:

$$\begin{aligned} & \left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt - \frac{f(b) - f(a)}{b-a} \left( x - \frac{a+b}{2} \right) \right| \\ & \leq \frac{1}{4\sqrt{3}}(b-a)(\Gamma - \gamma). \end{aligned}$$

In [16], Cheng presented a sharp version of the above inequality as follows

$$\begin{aligned} & \left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt - \frac{f(b) - f(a)}{b-a} \left( x - \frac{a+b}{2} \right) \right| \\ & \leq \frac{1}{8}(b-a)(\Gamma - \gamma). \end{aligned}$$

Recently, in [17], Ujević established a more generalized sharp inequality under more general conditions as follows:

$$|(b-a)f(x) - [f(b) - f(a)](x - \frac{a+b}{2}) - \int_a^b f(t)dt| \leq \frac{(b-a)^{\frac{3}{2}}}{2\sqrt{3}} \sqrt{\sigma(f')},$$

where  $f' \in L_2(a, b)$ , and  $\sigma(f') = \int_a^b (f'(s))^2 ds - \frac{1}{b-a} (\int_a^b f'(s) ds)^2$ .

More results related to the left-hand side of the above inequality can be found in [18,21-23].

On the other hand, Hilger [24] initiated the theory of time scales as a theory capable of treating continuous and discrete analysis in a consistent way, based on which some authors have studied the Ostrowski type and Grüss type inequalities on time scales. In [25], W. Liu et al established an Ostrowski-Grüss type inequality on time scales as follows, which is the time scale version of Dragomir and Wang's result in [14].

$$|f(t) - \frac{1}{b-a} \int_a^b f^\sigma(s) \Delta s - \frac{f(b) - f(a)}{(b-a)^2} (h_2(t, a) - h_2(t, b))| \leq \frac{1}{4} (b-a) (\Gamma - \gamma), \quad t \in [a, b] \cap \mathbb{T},$$

where  $\mathbb{T}$  is an arbitrary time scale,  $a, b, s, t \in \mathbb{T}$ ,  $f : [a, b] \rightarrow \mathbb{R}$  is differentiable,  $f^\Delta$  is rd-continuous and  $\gamma \leq f^\Delta(t) \leq \Gamma, \forall t \in [a, b] \cap \mathbb{T}$ .

For more Ostrowski type and Grüss type inequalities on time scales, we refer the reader to [26-36].

Motivated by the above work, in this paper, we will deal with a more generalized Ostrowski-Grüss type inequality on time scales whose left-hand side is of the following form

$$(1 - \lambda)(b-a)f(t) + (b-a)\lambda \frac{f(a) + f(b)}{2} - \int_a^b f(\sigma(s)) \Delta s - \frac{f(b) - f(a)}{b-a} \times [h_2(t, a + \lambda \frac{b-a}{2}) - h_2(a, a + \lambda \frac{b-a}{2}) + h_2(b, b - \lambda \frac{b-a}{2}) - h_2(t, b - \lambda \frac{b-a}{2})]$$

where  $\lambda \in [0, 1]$ . We will present some new estimates for it, some of which are sharp. The established results unify continuous and discrete analysis, and extend some known results in the literature.

Throughout this paper,  $\mathbb{R}$  denotes the set of real numbers and  $\mathbb{R}_+ = [0, \infty)$ , while  $\mathbb{Z}$  denotes

the set of integers, and  $\mathbb{N}_0$  denotes the set of non-negative integers. For a function  $f$  and two integers  $m_0, m_1$ , we have  $\sum_{s=m_0}^{m_1} f = 0$  provided  $m_0 > m_1$ .  $\mathbb{T}$  denotes an arbitrary time scale. On  $\mathbb{T}$  we define the forward jump operator  $\sigma \in (\mathbb{T}, \mathbb{T})$  such that  $\sigma(t) = \inf\{s \in \mathbb{T}, s > t\}$ . A point  $t \in \mathbb{T}$  is said to be right-dense if  $\sigma(t) = t$  and  $t \neq \sup \mathbb{T}$ . For an interval such as  $[a, b]$ , we denote  $[a, b] \cap \mathbb{T}$  by  $[a, b]_{\mathbb{T}}$ .

**Definition 1**  $h_k : \mathbb{T}^2 \rightarrow \mathbb{R}, k = 0, 1, 2, \dots$  are defined by

$$h_{k+1}(t, s) = \int_s^t h_k(\tau, s) \Delta \tau, \quad \forall s, t \in \mathbb{T},$$

where  $h_0(t, s) = 1$ .

**Remark 2** If  $\mathbb{T} = \mathbb{R}$ , then  $h_2(t, s) = \frac{(t-s)^2}{2}$ . If  $\mathbb{T} = \mathbb{Z}$ , then  $h_2(t, s) = \frac{(t-s)(t-s-1)}{2}$ . If  $\mathbb{T} = q^{\mathbb{N}_0}$ , then  $h_2(t, s) = \frac{(t-s)(t-qs)}{1+q}$ .

**Definition 3** For a function  $f \in (\mathbb{T}, \mathbb{R})$ , the delta derivative of  $f$  at  $t$  is denoted by  $f^\Delta(t)$  (provided it exists) with the property such that for every  $\varepsilon > 0$  there exists a neighborhood  $\mathfrak{U}$  of  $t$  satisfying

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \varepsilon |\sigma(t) - s|$$

for all  $s \in \mathfrak{U}$ .

**Remark 4** If  $\mathbb{T} = \mathbb{R}$ , then  $f^\Delta(t) = f'(t)$ . If  $\mathbb{T} = \mathbb{Z}$ , then  $f^\Delta(t) = f(t+1) - f(t)$ . If  $\mathbb{T} = q^{\mathbb{N}_0}$ , then  $f^\Delta(t) = \frac{f(qt) - f(t)}{t(q-1)}$ .

For more details about the calculus of time scales, we refer the reader to [37-38].

## 2 Main Results

**Lemma 5** [32, Lemma 1] (Generalized Montgomery Identity). Let  $a, b, s, t \in \mathbb{T}, a < b$  and  $f : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$  be  $\Delta$ -differentiable.  $\lambda \in [0, 1]$  such that  $a + \lambda \frac{b-a}{2}, b - \lambda \frac{b-a}{2} \in \mathbb{T}$  and  $t \in [a + \lambda \frac{b-a}{2}, b - \lambda \frac{b-a}{2}]_{\mathbb{T}}$ . Then

$$(1 - \lambda)f(t) + \lambda \frac{f(a) + f(b)}{2} = \frac{1}{b-a} \int_a^b f(\sigma(s)) \Delta s + \frac{1}{b-a} \int_a^b K(t, s) f^\Delta(s) \Delta s,$$

where

$$K(t, s) = \begin{cases} s - (a + \lambda \frac{b-a}{2}), & s \in [a, t]_{\mathbb{T}} \\ s - (b - \lambda \frac{b-a}{2}), & s \in [t, b]_{\mathbb{T}} \end{cases} \quad (1)$$

**Lemma 6** [35, Theorem 3.1]. Suppose  $a, b, s \in \mathbb{T}$ ,  $f, g \in C_{rd}$  and  $f, g : [a, b] \rightarrow \mathbb{R}$ . Then for  $m_1 \leq f(s) \leq M_1, m_2 \leq g(s) \leq M_2$ , we have

$$\left| \frac{1}{b-a} \int_a^b f^\sigma(s)g^\sigma(s)\Delta s - \frac{1}{b-a} \int_a^b f^\sigma(s)\Delta s \frac{1}{b-a} \int_a^b g^\sigma(s)\Delta s \right| \leq \frac{1}{4}(M_1 - m_1)(M_2 - m_2).$$

**Theorem 7** Let  $a, b, s, t \in \mathbb{T}, a < b. f : [a, b]_{\mathbb{T}} \rightarrow \mathbb{R}$  is  $\Delta$ -differentiable and furthermore, assume  $f^\Delta \in L_2(a, b)_{\mathbb{T}}$ .  $\lambda \in [0, 1]$  such that  $a + \lambda \frac{b-a}{2}, b - \lambda \frac{b-a}{2} \in \mathbb{T}$  and  $t \in [a + \lambda \frac{b-a}{2}, b - \lambda \frac{b-a}{2}]_{\mathbb{T}}$ . Then we have

$$\begin{aligned} & \left| (1-\lambda)(b-a)f(t) + (b-a)\lambda \frac{f(a)+f(b)}{2} - \int_a^b f(\sigma(s))\Delta s \right. \\ & \left. - \frac{f(b)-f(a)}{b-a} \times [h_2(t, a + \lambda \frac{b-a}{2}) - h_2(a, a + \lambda \frac{b-a}{2}) \right. \\ & \left. + h_2(b, b - \lambda \frac{b-a}{2}) - h_2(t, b - \lambda \frac{b-a}{2}) \right] \\ & \leq \left\{ \frac{b^3 - a^3}{3} - 2(a + \lambda \frac{b-a}{2}) [h_2(t, a + \lambda \frac{b-a}{2}) \right. \\ & \left. - h_2(a, a + \lambda \frac{b-a}{2})] - (a + \lambda \frac{b-a}{2})^2 (t-a) - 2(b - \lambda \frac{b-a}{2}) \right. \\ & \left. [h_2(b, b - \lambda \frac{b-a}{2}) - h_2(t, b - \lambda \frac{b-a}{2})] - (b - \lambda \frac{b-a}{2})^2 (b-t) \right. \\ & \left. - \frac{1}{b-a} [h_2(t, a + \lambda \frac{b-a}{2}) - h_2(a, a + \lambda \frac{b-a}{2}) \right. \\ & \left. + h_2(b, b - \lambda \frac{b-a}{2}) - h_2(t, b - \lambda \frac{b-a}{2}) \right]^2 \right\}^{\frac{1}{2}} \sqrt{T(f^\Delta)}, \quad (2) \end{aligned}$$

where  $T(f) = \int_a^b f^2(s)\Delta s - \frac{1}{b-a} (\int_a^b f(s)\Delta s)^2$ .

The inequality (2) is sharp in the sense that the coefficient constant 1 of the right-hand side of it can not be replaced by a smaller one.

**Proof.** From the definition of  $K(t, s)$  we obtain

$$\begin{aligned} & \int_a^b K(t, s)\Delta s \\ & = \int_a^t [s - (a + \lambda \frac{b-a}{2})]\Delta s + \int_t^b [s - (b - \lambda \frac{b-a}{2})]\Delta s \\ & = h_2(t, a + \lambda \frac{b-a}{2}) - h_2(a, a + \lambda \frac{b-a}{2}) \\ & \quad + h_2(b, b - \lambda \frac{b-a}{2}) - h_2(t, b - \lambda \frac{b-a}{2}), \quad (3) \end{aligned}$$

and

$$\int_a^b K^2(t, s)\Delta s$$

$$\begin{aligned} & = \int_a^t [s - (a + \lambda \frac{b-a}{2})]^2 \Delta s + \int_t^b [s - (b - \lambda \frac{b-a}{2})]^2 \Delta s \\ & = \int_a^t [s^2 - 2(a + \lambda \frac{b-a}{2})(s - (a + \lambda \frac{b-a}{2})) - (a + \lambda \frac{b-a}{2})^2] \Delta s \\ & \quad + \int_t^b [s^2 - 2(b - \lambda \frac{b-a}{2})(s - (b - \lambda \frac{b-a}{2})) - (b - \lambda \frac{b-a}{2})^2] \Delta s \\ & \leq \int_a^t [\frac{s^2 + s\sigma(s) + (\sigma(s))^2}{3} \\ & \quad - 2(a + \lambda \frac{b-a}{2})(s - (a + \lambda \frac{b-a}{2})) - (a + \lambda \frac{b-a}{2})^2] \Delta s \\ & \quad + \int_t^b [\frac{s^2 + s\sigma(s) + (\sigma(s))^2}{3} \\ & \quad - 2(b - \lambda \frac{b-a}{2})(s - (b - \lambda \frac{b-a}{2})) - (b - \lambda \frac{b-a}{2})^2] \Delta s \\ & = \frac{t^3 - a^3}{3} - 2(a + \lambda \frac{b-a}{2}) [h_2(t, a + \lambda \frac{b-a}{2}) \\ & \quad - h_2(a, a + \lambda \frac{b-a}{2})] - (a + \lambda \frac{b-a}{2})^2 (t-a) \\ & \quad + \frac{b^3 - t^3}{3} - 2(b - \lambda \frac{b-a}{2}) [h_2(b, b - \lambda \frac{b-a}{2}) \\ & \quad - h_2(t, b - \lambda \frac{b-a}{2})] - (b - \lambda \frac{b-a}{2})^2 (b-t). \quad (4) \end{aligned}$$

Furthermore, by Lemma 5 we have

$$\begin{aligned} & \int_a^b [K(t, s) - \frac{1}{b-a} \int_a^b K(t, s)\Delta s] \times \\ & \quad [f^\Delta(s) - \frac{1}{b-a} \int_a^b f^\Delta(s)\Delta s] \Delta s \\ & = \int_a^b K(t, s)f^\Delta(s)\Delta s - \frac{1}{b-a} \int_a^b K(t, s)\Delta s \int_a^b f^\Delta(s)\Delta s \\ & = (1-\lambda)(b-a)f(t) + (b-a)\lambda \frac{f(a)+f(b)}{2} \\ & \quad - \int_a^b f(\sigma(s))\Delta s - \frac{(f(b)-f(a))}{b-a} \times [h_2(t, a + \lambda \frac{b-a}{2}) \\ & \quad - h_2(a, a + \lambda \frac{b-a}{2}) + h_2(b, b - \lambda \frac{b-a}{2}) - h_2(t, b - \lambda \frac{b-a}{2})]. \quad (5) \end{aligned}$$

On the other hand,

$$\begin{aligned} & \left| \int_a^b [K(t, s) - \frac{1}{b-a} \int_a^b K(t, s)\Delta s] \times \right. \\ & \quad \left. [f^\Delta(s) - \frac{1}{b-a} \int_a^b f^\Delta(s)\Delta s] \Delta s \right| \\ & \leq \|K(t, \cdot) - \frac{1}{b-a} \int_a^b K(t, s)\Delta s\|_2 \times \end{aligned}$$

$$\begin{aligned} & \|f^\Delta(\cdot) - \frac{1}{b-a} \int_a^b f^\Delta(s) \Delta s\|_2 \\ &= \left[ \int_a^b K^2(t,s) \Delta s - \frac{1}{b-a} \left( \int_a^b K(t,s) \Delta s \right)^2 \right]^{\frac{1}{2}} \times \\ & \quad \left[ \int_a^b (f^\Delta(s))^2 \Delta s - \frac{1}{b-a} \left( \int_a^b f^\Delta(s) \Delta s \right)^2 \right]^{\frac{1}{2}} \\ &= \left[ \int_a^b K^2(t,s) \Delta s - \frac{1}{b-a} \left( \int_a^b K(t,s) \Delta s \right)^2 \right]^{\frac{1}{2}} \sqrt{T(f^\Delta)}. \end{aligned} \tag{6}$$

Combining (3)-(6) we can get the desired inequality (2).

To prove of the sharpness of (2), let  $\mathbb{T}$  be right dense and take

$$f(s) = \begin{cases} h_2(s, a + \lambda \frac{b-a}{2}) - h_2(t, a + \lambda \frac{b-a}{2}), & s \in [a, t]_{\mathbb{T}} \\ h_2(s, b - \lambda \frac{b-a}{2}) - h_2(t, b - \lambda \frac{b-a}{2}), & s \in [t, b]_{\mathbb{T}} \end{cases} \tag{7}$$

Then

$$f^\Delta(s) = \begin{cases} s - (a + \lambda \frac{b-a}{2}), & s \in [a, t]_{\mathbb{T}} \\ s - (b - \lambda \frac{b-a}{2}), & s \in (t, b]_{\mathbb{T}} \end{cases}. \tag{8}$$

So (4) and (6) hold equality, which implies (2) holds equality, and the proof is complete.  $\square$

**Remark 8** In Theorem 7, if we take  $\lambda$  for some certain values, then we can obtain some special results. For example, if we take  $\lambda = 0$ , then we have

$$\begin{aligned} & |(b-a)f(t) - \int_a^b f(\sigma(s)) \Delta s \\ & \quad - \frac{f(b) - f(a)}{b-a} \times [h_2(t, a) - h_2(t, b)]| \\ & \leq \left\{ -\frac{2}{3}(b^3 - a^3) + (b^2 - a^2)t - 2ah_2(t, a) \right. \\ & \quad \left. + 2bh_2(t, b) - \frac{1}{b-a} [h_2(t, a) - h_2(t, b)]^2 \right\}^{\frac{1}{2}} \sqrt{T(f^\Delta)}. \end{aligned}$$

If we take  $\lambda = 1$ , then we have

$$\begin{aligned} & |(b-a) \frac{f(a) + f(b)}{2} - \int_a^b f(\sigma(s)) \Delta s \\ & \quad - \frac{f(b) - f(a)}{b-a} [h_2(b, \frac{b+a}{2}) - h_2(a, \frac{b+a}{2})]| \\ & \leq \left\{ \frac{b^3 - a^3}{3} - (b+a) [h_2(b, \frac{b+a}{2}) - h_2(a, \frac{b+a}{2})] \right. \\ & \quad \left. - (\frac{b+a}{2})^2 (b-a) - \frac{1}{b-a} [h_2(b, \frac{b+a}{2}) \right. \\ & \quad \left. - h_2(a, \frac{b+a}{2})]^2 \right\}^{\frac{1}{2}} \sqrt{T(f^\Delta)}. \end{aligned}$$

If we take  $\lambda = \frac{1}{3}$ ,  $t = \frac{a+b}{2}$ , then we have

$$\begin{aligned} & \left| \frac{b-a}{6} [f(a) + 4f(\frac{a+b}{2}) + f(b)] - \int_a^b f(\sigma(s)) \Delta s \right. \\ & \quad \left. - \frac{f(b) - f(a)}{b-a} \times [h_2(\frac{a+b}{2}, \frac{b+5a}{6}) - h_2(a, \frac{b+5a}{6}) \right. \\ & \quad \left. + h_2(b, \frac{5b+a}{6}) - h_2(\frac{a+b}{2}, \frac{5b+a}{6})] \right| \\ & \leq \left\{ \frac{b^3 - a^3}{3} - (\frac{b+5a}{3}) [h_2(\frac{a+b}{2}, \frac{b+5a}{6}) - h_2(a, \frac{b+5a}{6})] \right. \\ & \quad \left. - [(\frac{b+5a}{6})^2 + (\frac{5b+a}{6})^2] (\frac{a+b}{2}) \right. \\ & \quad \left. - (\frac{5b+a}{3}) [h_2(b, \frac{5b+a}{6}) - h_2(\frac{a+b}{2}, \frac{5b+a}{6})] \right. \\ & \quad \left. - \frac{1}{b-a} [h_2(\frac{a+b}{2}, \frac{b+5a}{6}) - h_2(a, \frac{b+5a}{6}) \right. \\ & \quad \left. + h_2(b, \frac{5b+a}{6}) - h_2(\frac{a+b}{2}, \frac{5b+a}{6})]^2 \right\}^{\frac{1}{2}} \sqrt{T(f^\Delta)}. \end{aligned}$$

Since  $\mathbb{T}$  is an arbitrary time scale, if we take  $\mathbb{T}$  for some special cases in Theorem 7, we immediately obtain the following corollaries.

**Corollary 9** If we take  $\mathbb{T} = \mathbb{R}$  in Theorem 7, then we have

$$\begin{aligned} & |(1-\lambda)(b-a)f(t) + (b-a)\lambda \frac{f(a) + f(b)}{2} \\ & \quad - \int_a^b f(s) ds - (1-\lambda)(t - \frac{a+b}{2})(f(b) - f(a))| \\ & \leq \left\{ \frac{b^3 - a^3}{3} + (1-\lambda)(b-a)t^2 + (1-\lambda)(a^2 - b^2)t \right. \\ & \quad \left. - \frac{\lambda}{2}(b-a)(a^2 + b^2) + \frac{\lambda^2}{4}(b-a)^3 \right. \\ & \quad \left. - (1-\lambda)^2(b-a)(t - \frac{a+b}{2})^2 \right\}^{\frac{1}{2}} \sqrt{T(f')}, \end{aligned}$$

where  $T(f') = \int_a^b (f'(s))^2 ds - \frac{1}{b-a} (\int_a^b f'(s) ds)^2$ .

**Corollary 10** (Discrete case). Let  $\mathbb{T} = \mathbb{Z}$  in Theorem 7. Then we have

$$\begin{aligned} & |(1-\lambda)(b-a)f(t) + (b-a)\lambda \frac{f(a) + f(b)}{2} \\ & \quad - \sum_{s=a}^{nb-1} f(s+1) - [(1-\lambda)(t - \frac{a+b}{2}) - \frac{1}{2}](f(b) - f(a))| \\ & \leq \left\{ \frac{b^3 - a^3}{3} + (1-\lambda)(b-a)t^2 - (1-\lambda)(b-a)(a+b+1)t \right. \end{aligned}$$

$$-\frac{\lambda}{2}(b-a)(a^2+b^2) + \frac{\lambda^2}{4}(b-a)^3$$

$$+(1-\frac{\lambda}{2})(b^2-a^2)-(b-a)[(1-\lambda)(t-\frac{a+b}{2})-\frac{1}{2}]^2\}^{\frac{1}{2}}\sqrt{T(\Delta f)}$$

where

$$T(\Delta f) = \sum_{s=a}^{b-1} [f(s+1)-f(s)]^2 - \frac{1}{b-a} (\sum_{s=a}^{b-1} [f(s+1)-f(s)])^2.$$

**Corollary 11** (Quantum calculus case). Let  $\mathbb{T} = q^{\mathbb{N}_0}$ ,  $a = q^m$ ,  $b = q^n$  in Theorem 7, where  $m, n \in \mathbb{N}_0$  and  $q > 1$ . Suppose that  $t \in [q^m, q^n]_{q^{\mathbb{N}_0}}$ . Then we have

$$|(1-\lambda)(b-a)f(t) + (b-a)\lambda\frac{f(a)+f(b)}{2}$$

$$-a(q-1)\sum_{s=m}^{n-1} q^{s-m}f(q^{s+1}) - [\frac{2}{q+1}(1-\lambda)(t-\frac{a+b}{2})$$

$$-\frac{\lambda(q-1)}{q+1}t - (1-\frac{\lambda}{2})(a+b)\frac{q-1}{q+1}](f(b)-f(a))|$$

$$\leq \{\frac{b^3-a^3}{3} + \frac{2}{q+1}[(1-\lambda)(b-a)t^2 + 2(1-\lambda)(a^2-b^2)t$$

$$+ b^3 - a^3 - \frac{3\lambda}{2}(b-a)(a^2+b^2) + \frac{\lambda^2}{2}(b-a)^3]$$

$$+ \frac{2(q-1)}{q+1}[(1-\lambda)(a^2-b^2)t + b^3 - a^3 - \lambda(b-a)(a^2+b^2)$$

$$+ \frac{\lambda^2}{4}(b-a)^3] - (1-\lambda)(a^2-b^2)t + a^3 - b^3$$

$$+ \lambda(b-a)(a^2+b^2) - \frac{\lambda^2}{4}(b-a)^3$$

$$- (b-a)[\frac{2}{q+1}(1-\lambda)(t-\frac{a+b}{2})$$

$$-\frac{\lambda(q-1)}{q+1}t - (1-\frac{\lambda}{2})(a+b)\frac{q-1}{q+1}]^2\}^{\frac{1}{2}}\sqrt{T(D_q f)},$$

where

$$T(D_q f) = a(q-1)\sum_{s=m}^{n-1} q^{s-m}[\frac{f(q^{s+1})-f(q^s)}{q^s(q-1)}]^2$$

$$- \frac{1}{b-a}(a(q-1)\sum_{s=m}^{n-1} q^{s-m}[\frac{f(q^{s+1})-f(q^s)}{q^s(q-1)}])^2.$$

**Remark 12** After some basic computation, one can see Corollary 9 is in fact equivalent to [22, Theorem 5]. If we take  $\lambda = \frac{1}{3}$ ,  $t = \frac{a+b}{2}$  or  $\lambda = 0$  in Corollary 9, then Corollary 9 reduces to Theorem 1 and Theorem 4 in [17] respectively. So in this way, Theorem 7 presents better estimates, and is the further extension of the results in [17, 22] to arbitrary time scales.

**Theorem 13** Under the conditions of Theorem 7, if we assume  $f^\Delta \in L_\infty(a, b)_{\mathbb{T}}$  instead, then we have

$$|(1-\lambda)(b-a)f(t) + (b-a)\lambda\frac{f(a)+f(b)}{2}$$

$$- \int_a^b f(\sigma(s))\Delta s - \frac{f(b)-f(a)}{b-a} \times$$

$$[h_2(t, a + \lambda\frac{b-a}{2}) - h_2(a, a + \lambda\frac{b-a}{2})$$

$$+ h_2(b, b - \lambda\frac{b-a}{2}) - h_2(t, b - \lambda\frac{b-a}{2})]|$$

$$\leq \sqrt{b-a}\{\frac{b^3-a^3}{3} - 2(a+\lambda\frac{b-a}{2})[h_2(t, a+\lambda\frac{b-a}{2})$$

$$- h_2(a, a+\lambda\frac{b-a}{2})] - (a+\lambda\frac{b-a}{2})^2(t-a) - 2(b-\lambda\frac{b-a}{2})$$

$$[h_2(b, b-\lambda\frac{b-a}{2}) - h_2(t, b-\lambda\frac{b-a}{2})] - (b-\lambda\frac{b-a}{2})^2(b-t)$$

$$- \frac{1}{b-a}[h_2(t, a + \lambda\frac{b-a}{2}) - h_2(a, a + \lambda\frac{b-a}{2})$$

$$+ h_2(b, b - \lambda\frac{b-a}{2}) - h_2(t, b - \lambda\frac{b-a}{2})]^2\}^{\frac{1}{2}}\|f^\Delta\|_\infty. \quad (9)$$

**Proof.** First we have the following observation:

$$\int_a^b [K(t, s) - \frac{1}{b-a} \int_a^b K(t, s)\Delta s]f^\Delta(s)\Delta s$$

$$= \int_a^b K(t, s)f^\Delta(s)\Delta s - \frac{1}{b-a} \int_a^b K(t, s)\Delta s \int_a^b f^\Delta(s)\Delta s$$

$$= (1-\lambda)(b-a)f(t) + (b-a)\lambda\frac{f(a)+f(b)}{2}$$

$$- \int_a^b f(\sigma(s))\Delta s - \frac{(f(b)-f(a))}{b-a} \times$$

$$[h_2(t, a + \lambda\frac{b-a}{2}) - h_2(a, a + \lambda\frac{b-a}{2})$$

$$+ h_2(b, b - \lambda\frac{b-a}{2}) - h_2(t, b - \lambda\frac{b-a}{2})]. \quad (10)$$

Then

$$|(1-\lambda)(b-a)f(t) + (b-a)\lambda\frac{f(a)+f(b)}{2} - \int_a^b f(\sigma(s))\Delta s$$

$$- \frac{f(b)-f(a)}{b-a} \times [h_2(t, a + \lambda\frac{b-a}{2}) - h_2(a, a + \lambda\frac{b-a}{2})$$

$$+ h_2(b, b - \lambda\frac{b-a}{2}) - h_2(t, b - \lambda\frac{b-a}{2})]|$$

$$\leq \int_a^b |K(t, s) - \frac{1}{b-a} \int_a^b K(t, s)\Delta s|f^\Delta\|_\infty$$

$$\begin{aligned} &\leq \sqrt{b-a} \left[ \int_a^b \left| K(t,s) - \frac{1}{b-a} \int_a^b K(t,s) \right|^2 \Delta s \right]^{\frac{1}{2}} \|f^\Delta\|_\infty \\ &= \sqrt{b-a} \left[ \int_a^b K^2(t,s) \Delta s - \frac{1}{b-a} \left( \int_a^b K(t,s) \Delta s \right)^2 \right]^{\frac{1}{2}} \|f^\Delta\|_\infty. \end{aligned} \tag{11}$$

By a combination of (3), (4) and (11) we get the desired result.  $\square$

Now we present a sharp estimate for the left-hand side of (9).

**Theorem 14** *Under the conditions of Theorem 13, furthermore, assume  $\lambda \in [0, 1]$  such that  $a + \lambda \frac{b-a}{2}, b - \lambda \frac{b-a}{2} \in \mathbb{T}$ , and  $t \in [a + \lambda \frac{b-a}{2}, b - \lambda \frac{b-a}{2}]_{\mathbb{T}} \cap (a + \lambda \frac{b-a}{2} + \gamma(t), b - \lambda \frac{b-a}{2} + \gamma(t))_{\mathbb{T}}$ , where*

$$\begin{aligned} \gamma(t) &= \frac{h_2(t, a + \lambda \frac{b-a}{2}) - h_2(a, a + \lambda \frac{b-a}{2})}{b-a} + \\ &\quad \frac{h_2(b, b - \lambda \frac{b-a}{2}) - h_2(t, b - \lambda \frac{b-a}{2})}{b-a}. \end{aligned}$$

Then the following estimates hold:

$$\begin{aligned} &|(1-\lambda)(b-a)f(t) + (b-a)\lambda \frac{f(a) + f(b)}{2} \\ &- \int_a^b f(\sigma(s)) \Delta s - \frac{f(b) - f(a)}{b-a} \times [h_2(t, a + \lambda \frac{b-a}{2}) \\ &- h_2(a, a + \lambda \frac{b-a}{2}) + h_2(b, b - \lambda \frac{b-a}{2}) - h_2(t, b - \lambda \frac{b-a}{2})] \\ &\leq \begin{cases} [h_2(a, s_1^*) + h_2(t, s_1^*) + h_2(t, s_2^*) - h_2(b, s_2^*)] \|f^\Delta\|_\infty, \\ \quad \text{when } \gamma(t) \geq \lambda \frac{b-a}{2} \\ [h_2(a, s_1^*) + h_2(t, s_1^*) + h_2(t, s_2^*) + h_2(b, s_2^*)] \|f^\Delta\|_\infty, \\ \quad \text{when } -\lambda \frac{b-a}{2} < \gamma(t) < \lambda \frac{b-a}{2} \\ [h_2(t, s_1^*) - h_2(a, s_1^*) + h_2(t, s_2^*) + h_2(b, s_2^*)], \\ \quad \text{when } \gamma(t) \leq -\lambda \frac{b-a}{2} \end{cases} \end{aligned} \tag{12}$$

where  $s_1^* = a + \lambda \frac{b-a}{2} + \gamma(t), s_2^* = b - \lambda \frac{b-a}{2} + \gamma(t)$ .

The inequality (12) is sharp in the sense that the coefficient constant 1 of the right-hand side of it can not be replaced by a smaller one.

**Proof.** Define

$$P(t, s) = \begin{cases} s - s_1^*, & s \in [a, t]_{\mathbb{T}} \\ s - s_2^*, & s \in [t, b]_{\mathbb{T}} \end{cases}. \tag{13}$$

Then  $\int_a^b P(t, s) \Delta s = 0$ , and

$$\begin{aligned} &\int_a^b P(t, s) f^\Delta(s) \Delta s = (1-\lambda)(b-a)f(t) \\ &+ (b-a)\lambda \frac{f(a) + f(b)}{2} - \int_a^b f(\sigma(s)) \Delta s - \frac{f(b) - f(a)}{b-a} \times \end{aligned}$$

$$\begin{aligned} &[h_2(t, a + \lambda \frac{b-a}{2}) - h_2(a, a + \lambda \frac{b-a}{2}) \\ &+ h_2(b, b - \lambda \frac{b-a}{2}) - h_2(t, b - \lambda \frac{b-a}{2})]. \end{aligned} \tag{14}$$

Case 1: if  $\gamma(t) > \lambda \frac{b-a}{2}$ , then one can see  $P(t, s) \geq 0$  for  $t \in [s_1^*, t]_{\mathbb{T}}$ . So from (14) we obtain

$$\begin{aligned} &|(1-\lambda)(b-a)f(t) + (b-a)\lambda \frac{f(a) + f(b)}{2} - \int_a^b f(\sigma(s)) \Delta s \\ &- \frac{f(b) - f(a)}{b-a} \times [h_2(t, a + \lambda \frac{b-a}{2}) \\ &- h_2(a, a + \lambda \frac{b-a}{2}) + h_2(b, b - \lambda \frac{b-a}{2}) - h_2(t, b - \lambda \frac{b-a}{2})] \\ &= \left| \int_a^b P(t, s) f^\Delta(s) \Delta s \right| \leq \int_a^b |P(t, s)| \|f^\Delta(s)\| \Delta s \\ &\leq \int_a^b |P(t, s)| \Delta s \|f^\Delta\|_\infty \\ &= \left[ \int_a^{s_1^*} (s_1^* - s) \Delta s + \int_{s_1^*}^t (s - s_1^*) \Delta s + \int_t^{s_2^*} (s_2^* - s) \Delta s \right] \|f^\Delta\|_\infty \\ &= [h_2(a, s_1^*) + h_2(t, s_1^*) + h_2(t, s_2^*) - h_2(b, s_2^*)] \|f^\Delta\|_\infty. \end{aligned} \tag{15}$$

Case 2: if  $-\lambda \frac{b-a}{2} \leq \gamma(t) \leq \lambda \frac{b-a}{2}$ , then  $P(t, s) \geq 0$  for  $t \in [s_1^*, t]_{\mathbb{T}} \cup [s_2^*, b]_{\mathbb{T}}$ . So from (14) we obtain

$$\begin{aligned} &|(1-\lambda)(b-a)f(t) + (b-a)\lambda \frac{f(a) + f(b)}{2} - \int_a^b f(\sigma(s)) \Delta s \\ &- \frac{f(b) - f(a)}{b-a} \times [h_2(t, a + \lambda \frac{b-a}{2}) \\ &- h_2(a, a + \lambda \frac{b-a}{2}) + h_2(b, b - \lambda \frac{b-a}{2}) - h_2(t, b - \lambda \frac{b-a}{2})] \\ &\leq \int_a^b |P(t, s)| \Delta s \|f^\Delta\|_\infty = \left[ \int_a^{s_1^*} (s_1^* - s) \Delta s \right. \\ &+ \int_{s_1^*}^t (s - s_1^*) \Delta s + \int_t^{s_2^*} (s_2^* - s) \Delta s + \left. \int_{s_2^*}^b (s - s_2^*) \Delta s \right] \|f^\Delta\|_\infty \\ &= [h_2(a, s_1^*) + h_2(t, s_1^*) + h_2(t, s_2^*) + h_2(b, s_2^*)] \|f^\Delta\|_\infty. \end{aligned} \tag{16}$$

Case 3: if  $\gamma(t) < -\lambda \frac{b-a}{2}$ , then  $P(t, s) \geq 0$  for  $t \in [a, t]_{\mathbb{T}} \cup [s_2^*, b]_{\mathbb{T}}$ . So from (14) we obtain

$$\begin{aligned} &|(1-\lambda)(b-a)f(t) + (b-a)\lambda \frac{f(a) + f(b)}{2} - \int_a^b f(\sigma(s)) \Delta s \\ &- \frac{f(b) - f(a)}{b-a} \times [h_2(t, a + \lambda \frac{b-a}{2}) \end{aligned}$$

$$\begin{aligned}
 & -h_2(a, a + \lambda \frac{b-a}{2}) + h_2(b, b - \lambda \frac{b-a}{2}) - h_2(t, b - \lambda \frac{b-a}{2}) \\
 & \leq \int_a^b |P(t, s)| \Delta s \|f^\Delta\|_\infty \\
 & = [\int_a^t (s - s_1^*) \Delta s + \int_t^{s_2^*} (s_2^* - s) \Delta s + \int_{s_2^*}^b (s - s_2^*) \Delta s] \|f^\Delta\|_\infty \\
 & = [h_2(t, s_1^*) - h_2(a, s_1^*) + h_2(t, s_2^*) + h_2(b, s_2^*)] \|f^\Delta\|_\infty. \tag{17}
 \end{aligned}$$

Now we will prove the sharpness of (12) in three cases:

Case 1: if  $\gamma(t) > \lambda \frac{b-a}{2}$ , we take

$$f(s) = \begin{cases} -s, & s \in [a, s_1^*]_{\mathbb{T}} \\ s - 2s_1^*, & s \in [s_1^*, t]_{\mathbb{T}} \\ -s - 2s_1^* + 2t, & s \in [t, b]_{\mathbb{T}} \end{cases} \tag{18}$$

Then

$$f^\Delta(s) = \begin{cases} -1, & s \in [a, s_1^*]_{\mathbb{T}} \cup [t, b]_{\mathbb{T}} \\ 1, & s \in [s_1^*, t]_{\mathbb{T}} \end{cases} \tag{19}$$

So  $\|f^\Delta\|_\infty = 1$ , and  $P(t, s)f^\Delta(s) \geq 0$ , which implies

$$\begin{aligned}
 |\int_a^b P(t, s) f^\Delta(s) \Delta s| &= \int_a^b |P(t, s) f^\Delta(s)| \Delta s \\
 &= \int_a^b |P(t, s)| \Delta s \|f^\Delta\|_\infty.
 \end{aligned}$$

Then (15) holds equality, and the sharpness of (12) under the condition  $\gamma(t) > \lambda \frac{b-a}{2}$  is proved.

In Case 2 and Case 3, we take

$$f(s) = \begin{cases} -s, & s \in [a, s_1^*]_{\mathbb{T}} \\ s - 2s_1^*, & s \in [s_1^*, t]_{\mathbb{T}} \\ -s - 2s_1^* + 2t, & s \in [t, s_2^*]_{\mathbb{T}} \\ s - 2s_2^* - 2s_1^* + 2t, & s \in (s_2^*, b]_{\mathbb{T}} \end{cases} \tag{20}$$

and

$$f(s) = \begin{cases} s, & s \in [a, t]_{\mathbb{T}} \\ -s + 2t, & s \in [t, s_2^*]_{\mathbb{T}} \\ s - 2s_2^* + 2t, & s \in (s_2^*, b]_{\mathbb{T}} \end{cases} \tag{21}$$

respectively. Then in both cases, we always have  $\|f^\Delta\|_\infty = 1$ ,  $P(t, s)f^\Delta(s) \geq 0$ , and

$$\begin{aligned}
 |\int_a^b P(t, s) f^\Delta(s) \Delta s| &= \int_a^b |P(t, s) f^\Delta(s)| \Delta s \\
 &= \int_a^b |P(t, s)| \Delta s \|f^\Delta\|_\infty.
 \end{aligned}$$

So (16) and (17) hold equality, and furthermore, (12) holds equality. Then the sharpness of (12) is proved, and the proof is complete.  $\square$

**Theorem 15** Under the conditions of Theorem 7, if there exist constants  $K_1, K_2$  such that  $K_1 \leq f^\Delta(s) \leq K_2$ , then we have

$$\begin{aligned}
 & |(1-\lambda)(b-a)f(t) + (b-a)\lambda \frac{f(a) + f(b)}{2} - \int_a^b f(\sigma(s)) \Delta s \\
 & \quad - \frac{f(b) - f(a)}{b-a} \times [h_2(t, a + \lambda \frac{b-a}{2}) \\
 & \quad - h_2(a, a + \lambda \frac{b-a}{2}) + h_2(b, b - \lambda \frac{b-a}{2}) - h_2(t, b - \lambda \frac{b-a}{2})]| \\
 & \leq \frac{1}{4}(b-a)(K_2 - K_1). \tag{22}
 \end{aligned}$$

**Proof.** From the definition of  $K(t, s)$  we have  $\sup(K(t, s)) - \inf(K(t, s)) = (t-a) - (t-b) = b-a$ . Then the desired result can be obtained by the combination of (5) and Lemma 6.  $\square$

**Remark 16** If we take  $\lambda = 0$  in Theorem 15, then Theorem 15 reduces to [25, Theorem 4].

The sharp estimate for the left-hand side of (22) is presented in the following theorem.

**Theorem 17** Under the conditions of Theorem 2.11, we have

$$\begin{aligned}
 & |(1-\lambda)(b-a)f(t) + (b-a)\lambda \frac{f(a) + f(b)}{2} - \int_a^b f(\sigma(s)) \Delta s \\
 & \quad - \frac{f(b) - f(a)}{b-a} \times [h_2(t, a + \lambda \frac{b-a}{2}) \\
 & \quad - h_2(a, a + \lambda \frac{b-a}{2}) + h_2(b, b - \lambda \frac{b-a}{2}) - h_2(t, b - \lambda \frac{b-a}{2})]| \\
 & \leq \begin{cases} h_2(t, s_1^*)(K_2 - K_1), & \text{when } \gamma(t) > \lambda \frac{b-a}{2} \\ \frac{[h_2(t, s_1^*) + h_2(t, s_2^*) + h_2(a, s_2^*) + h_2(b, s_2^*)](K_2 - K_1)}{2}, & \text{when } -\lambda \frac{b-a}{2} \leq \gamma(t) \leq \lambda \frac{b-a}{2} \\ h_2(t, s_2^*)(K_2 - K_1), & \text{when } \gamma(t) < -\lambda \frac{b-a}{2} \end{cases}, \tag{23}
 \end{aligned}$$

where  $s_1^*, s_2^*$  are defined as in Theorem 14.

The inequality (23) is sharp in the sense that the coefficient constant 1 of the right-hand side of it can not be replaced by a smaller one.

**Proof.** Let  $P(t, s)$  be defined as in Theorem 14. Then we have  $\int_a^b P(t, s) \Delta s = 0$ .

Case 1: if  $\gamma(t) > \lambda \frac{b-a}{2}$ , then one can see  $P(t, s) \geq 0$  for  $t \in [s_1^*, t]_{\mathbb{T}}$ , and

$$\int_{s_1^*}^t P(t, s) \Delta s = -[\int_a^{s_1^*} P(t, s) \Delta s + \int_t^b P(t, s) \Delta s].$$

So we can obtain

$$\int_a^b P(t, s) f^\Delta(s) \Delta s$$

$$\begin{aligned} &\leq K_2 \int_{s_1^*}^t P(t, s) \Delta s + K_1 \left[ \int_a^{s_1^*} P(t, s) \Delta s + \int_t^b P(t, s) \Delta s \right] \\ &= \left[ \int_{s_1^*}^t P(t, s) \Delta s \right] (K_2 - K_1) = h_2(t, s_1^*) (K_2 - K_1) \end{aligned} \tag{24}$$

and

$$\begin{aligned} & - \int_a^b P(t, s) f^\Delta(s) \Delta s \\ &\leq -K_1 \int_{s_1^*}^t P(t, s) \Delta s - K_2 \left[ \int_a^{s_1^*} P(t, s) \Delta s + \int_t^b P(t, s) \Delta s \right] \\ &= h_2(t, s_1^*) (K_2 - K_1). \end{aligned} \tag{25}$$

Combining (24) and (25) we have

$$\left| \int_a^b P(t, s) f^\Delta(s) \Delta s \right| \leq h_2(t, s_1^*) (K_2 - K_1). \tag{26}$$

Case 2: if  $-\lambda \frac{b-a}{2} \leq \gamma(t) \leq \lambda \frac{b-a}{2}$ , then  $P(t, s) \geq 0$  for  $t \in [s_1^*, t]_{\mathbb{T}} \cup [s_2^*, b]_{\mathbb{T}}$ , and

$$\begin{aligned} &\int_{s_1^*}^t P(t, s) \Delta s + \int_{s_2^*}^b P(t, s) \Delta s \\ &= - \left[ \int_a^{s_1^*} P(t, s) \Delta s + \int_t^{s_2^*} P(t, s) \Delta s \right]. \end{aligned}$$

So we can obtain

$$\begin{aligned} &\int_a^b P(t, s) f^\Delta(s) \Delta s \\ &\leq K_2 \left[ \int_{s_1^*}^t P(t, s) \Delta s + \int_{s_2^*}^b P(t, s) \Delta s \right] \\ &+ K_1 \left[ \int_a^{s_1^*} P(t, s) \Delta s + \int_t^{s_2^*} P(t, s) \Delta s \right] \\ &= \left[ \int_{s_1^*}^t P(t, s) \Delta s + \int_{s_2^*}^b P(t, s) \Delta s \right] (K_2 - K_1) \\ &= [h_2(t, s_1^*) + h_2(b, s_2^*)] (K_2 - K_1) \end{aligned} \tag{27}$$

and

$$\begin{aligned} & - \int_a^b P(t, s) f^\Delta(s) \Delta s \\ &\leq -K_1 \left[ \int_{s_1^*}^t P(t, s) \Delta s + \int_{s_2^*}^b P(t, s) \Delta s \right] \\ &- K_2 \left[ \int_a^{s_1^*} P(t, s) \Delta s + \int_t^{s_2^*} P(t, s) \Delta s \right] \\ &= \left[ \int_{s_1^*}^t P(t, s) \Delta s + \int_{s_2^*}^b P(t, s) \Delta s \right] (K_2 - K_1) \end{aligned}$$

$$= [h_2(t, s_1^*) + h_2(b, s_2^*)] (K_2 - K_1). \tag{28}$$

Combining (27) and (28) we have

$$\begin{aligned} &\left| \int_a^b P(t, s) f^\Delta(s) \Delta s \right| \leq [h_2(t, s_1^*) + h_2(b, s_2^*)] (K_2 - K_1) \\ &= \frac{[h_2(t, s_1^*) + h_2(t, s_2^*) + h_2(a, s_2^*) + h_2(b, s_2^*)]}{2} (K_2 - K_1). \end{aligned} \tag{29}$$

Case 3: if  $\gamma(t) < -\lambda \frac{b-a}{2}$ , then  $P(t, s) \geq 0$  for  $t \in [a, t]_{\mathbb{T}} \cup [s_2^*, b]_{\mathbb{T}}$ , and  $\int_a^t P(t, s) \Delta s + \int_{s_2^*}^b P(t, s) \Delta s = -\int_t^{s_2^*} P(t, s) \Delta s$ . Similar to Case 1 and Case 2, we can obtain

$$\left| \int_a^b P(t, s) f^\Delta(s) \Delta s \right| \leq h_2(t, s_2^*) (K_2 - K_1). \tag{30}$$

To prove the sharpness of (19), in Case 1 we can take

$$f(s) = \begin{cases} K_1 s, & s \in [a, s_1^*]_{\mathbb{T}} \\ K_2 s - K_2 s_1^* + K_1 s_1^*, & s \in [s_1^*, t]_{\mathbb{T}} \\ K_1 s - K_1 t + K_2 t - K_2 s_1^* + K_1 s_1^*, & s \in [t, b]_{\mathbb{T}} \end{cases} \tag{31}$$

Then

$$f^\Delta(s) = \begin{cases} K_1, & s \in [a, s_1^*]_{\mathbb{T}} \cup [t, b]_{\mathbb{T}} \\ K_2, & s \in [s_1^*, t]_{\mathbb{T}} \end{cases} \tag{32}$$

So (26) holds equality, and the sharpness of (19) under the condition  $\gamma(t) > \lambda \frac{b-a}{2}$  can be obtained.

In Case 2 and Case 3, we take

$$f(s) = \begin{cases} K_1 s, & s \in [a, s_1^*]_{\mathbb{T}} \\ K_2 s - K_2 s_1^* + K_1 s_1^*, & s \in [s_1^*, t]_{\mathbb{T}} \\ K_1 s + (K_2 - K_1)t - K_2 s_1^* + K_1 s_1^*, & s \in [t, s_2^*]_{\mathbb{T}} \\ K_2 s - (K_2 - K_1)s_2^* + (K_2 - K_1)t \\ - K_2 s_1^* + K_1 s_1^*, & s \in (s_2^*, b]_{\mathbb{T}} \end{cases} \tag{33}$$

and

$$f(s) = \begin{cases} K_2 s, & s \in [a, t]_{\mathbb{T}} \\ K_1 s - K_1 t + K_2 t, & s \in [t, s_2^*]_{\mathbb{T}} \\ K_2 s - K_2 s_2^* + K_1 s_1^* - K_1 t + K_2 t, & s \in (s_2^*, b]_{\mathbb{T}} \end{cases} \tag{34}$$

respectively. Then similar to (31)-(32) we can also obtain that (29) and (30) hold equality. So (23) holds equality, and the proof is complete.  $\square$

**Remark 18** If we take  $\mathbb{T} = \mathbb{R}$ ,  $\lambda = 0$  in Theorem 2.13, then Theorem 2.13 reduces to [16, Theorem 1.5].

**Remark 19** In Theorem 17, if we take  $\mathbb{T} = \mathbb{R}$ ,  $t = \frac{a+b}{2}$ ,  $\lambda = 0$ , then we obtain the following mid-point inequality:

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(s) ds \right| \leq \frac{(b-a)}{8} (K_2 - K_1). \tag{35}$$



If we take  $\mathbb{T} = \mathbb{R}$ ,  $t = \frac{a+b}{2}$ ,  $\lambda = \frac{1}{3}$ , then we obtain the following Simpson type inequality:

$$\begin{aligned} & \left| \frac{b-a}{6} [f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)] - \int_a^b f(s) ds \right| \\ & \leq \frac{5(b-a)^2}{72} (K_2 - K_1). \end{aligned} \quad (36)$$

We note (36) improves the results related to Simpson type inequalities in [39, (2.7)].

**Remark 20** For Theorems 13,14,15 and 17, we can obtain similar corollaries as Corollaries 9–11, which are omitted here.

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