Existence of positive solutions to a four-point boundary value problems

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Abstract: By using the fixed point theorem of cone expansion and compression of norm type and monotone iterative technique, we study the following equation

$$(\phi_p(u'(t)))' + \lambda q(t) f(t, u(t)) = 0, \quad t \in (0,1),$$

and

$$(\phi_p(u'(t)))' + q(t) f(t, u(t), u'(t)) = 0, \quad t \in (0,1),$$

subject to boundary conditions:

$$u'(0) - \alpha u(\xi) = 0, \qquad u'(1) + \beta u(\eta) = 0,$$

where $\phi_p(s) = |s|^{p-2} \cdot s,\ p > 1$, the existence and iteration of positive solutions are proved. The interesting point is the nonlinear term f is involved with the first-order derivative explicitly in section 3.

Key–Words: Positive solutions; $p-$Laplacian; Boundary value problem; Monotone iterative technique; Completely continuous; Cone.

1 Introduction

In this paper, we study the existence of positive solutions for the four-point boundary value problem (BVP for short) with $p-$Laplacian

$$(\phi_p(u'(t)))' + \lambda q(t) f(t, u(t)) = 0, \quad t \in (0,1), \quad (1)$$

and

$$(\phi_p(u'(t)))' + q(t) f(t, u(t), u'(t)) = 0, \quad t \in (0,1), \quad (2)$$

subject to boundary conditions:

$$u'(0) - \alpha u(\xi) = 0, \qquad u'(1) + \beta u(\eta) = 0, \quad (3)$$

where $\phi_p(s) = |s|^{p-2} \cdot s,\ p > 1,\ (\phi_p)^{-1} = \phi_q,\ \frac{1}{p} + \frac{1}{q} = 1,\ \xi, \eta \in (0,1),\ \lambda > 0$ and $\xi < \eta,\ \alpha \in (0, \frac{1}{\xi}),\ \beta \in (0, \frac{1}{1-\eta})$. By determining the range of λ and using the fixed point theorem of cone expansion and compression of norm type, we get the positive solutions for (1), (3). Furthermore, by using monotone iterative technique, we not only obtain the existence of positive solutions for (2), (3), but also construct some iterative schemes to find the solutions.

Equations of the above form occur in the study of the n-dimensional $p-$Laplace equation, non-Newtonian fluid theory and the turbulent flow of a gas in a porous medium [11]. When the nonlinear term f does not depend on the first-order derivative, Eq.(2) has been studied extensively, and the existence and multiplicity results are available in the literature [1-4,13-18]. However, there are few papers dealing with the iteration of positive solutions when the nonlinear term f is involved in first-order derivative explicitly. In [5], the authors considered the triple positive solution for two-point boundary value problems with one-dimensional $p-$Laplacian

$$(\phi_p(x'(t)))' + q(t) f(t, x(t), x'(t)) = 0, \quad t \in (0,1),$$
$$x(0) = 0 = x(1)$$

or

$$x(0) = 0 = x'(1).$$

Bai [6] considered the following boundary value problem

$$(\phi_p(x'(t)))' + q(t) f(t, x(t), x'(t)) = 0, \quad t \in (0,1),$$

together with one of the following boundary conditions:

$$\alpha\phi_p(x(0)) - \beta\phi_p(x'(0)) = 0,$$
$$\gamma\phi_p(x(1)) + \delta\phi_p(x'(1)) = 0,$$

or

$$x(0) - g_1(x'(0)) = 0,$$
$$x(1) + g_2(x'(1)) = 0,$$

by means of the Avery-Peterson fixed point theorem, sufficient conditions are obtained that guarantee the existence of at least three positive solutions. The solution is nonnegative, if it has, under the condition f is nonnegative, but this result can't be generalized to four-point BVP (1),(3) and BVP (2),(3). That is the main difficulty to us. In our paper, we give a condition $\alpha \in (0, \frac{1}{\xi}),\ \beta \in (0, \frac{1}{1-\eta})$ to overcome this difficulty. Recently, in [7], using a fixed point theorem due to Avery and Peterson, which we can refer the reader to [10], Wang studied the following boundary value problems

$$(\phi_p(x'(t)))' + q(t)f(t, x(t), x(t-1), x'(t)) = 0,$$

subject to one of the following two pairs of boundary conditions:

$$\begin{cases} x(t) = \xi(t), & -1 \le t \le 0, \\ x(1) = 0, \end{cases}$$

$$\begin{cases} x(t) = \xi(t), & -1 \le t \le 0, \\ x'(1) = 0. \end{cases}$$

More recently, in [8], using the fixed point theorem of cone expansion and compression of norm type, we discussed the positive solution for the following boundary value problems with sign changing nonlinearity

$$(\phi_p(u'))' + f(t, u, u') = 0, \quad t \in [0, 1],$$

subject to the boundary value conditions:

$$u'(0) = \sum_{i=1}^{n} \alpha_i u'(\xi_i), \qquad u(1) = \sum_{i=1}^{n} \beta_i u(\xi_i).$$

We can see that all the results obtained in the above papers are only the existence of positive solutions under some conditions, but it is more useful to us is give a way to find the solution. So in the section 3 of this paper, we not only obtain the existence of positive solutions for (2), (3), but also construct some iterative schemes to find the solutions. To our knowledge, this is the first paper to obtain iteration of positive solutions to a four-point p-Laplacian boundary value problems (2),(3). The emphasis here is that the nonlinear term is involved explicitly with the first-order derivative explicitly.

Throughout, it is assumed that:

(H_1) $f \in C([0, 1] \times [0, +\infty), (0, +\infty)),\ \ \lambda > 0$;

(H_2) $f \in C([0, 1] \times [0, +\infty) \times (-\infty, +\infty),\ (0, +\infty))$;

(H_3) $q(t)$ is a nonnegative measurable function defined on $(0, 1)$, $q(t) \not\equiv 0$ on any subinterval of $(0, 1)$. In addition, $\int_0^1 q(t)dt < +\infty$;

(H_4) $\xi, \eta \in (0, 1)$ and $\xi < \eta,\ \alpha \in (0, \frac{1}{\xi}),\ \beta \in (0, \frac{1}{1-\eta})$.

2 Existence of Positive Solutions to BVP (1),(3)

In this section, by determining the range of λ, and using the fixed point theorem of cone expansion and compression of norm type, we study the existence of positive solutions for the four-point boundary value problem with $p-$Laplacian

$$(\phi_p(u'(t)))' + \lambda q(t)f(t, u(t)) = 0, \quad t \in (0, 1),$$

subject to boundary conditions:

$$u'(0) - \alpha u(\xi) = 0, \qquad u'(1) + \beta u(\eta) = 0,$$

where $\phi_p(s) = |s|^{p-2} \cdot s,\ p > 1,\ (\phi_p)^{-1} = \phi_q,\ \frac{1}{p} + \frac{1}{q} = 1,\ \lambda > 0,\ \xi, \eta \in (0, 1)$ and $\xi < \eta,\ \alpha \in (0, \frac{1}{\xi}),\ \beta \in (0, \frac{1}{1-\eta})$.

Let $X = C[0, 1]$ be endowed with the maximum norm,

$$\|u\| = \max_{0 \le t \le 1} |u(t)|.$$

From the fact $(\phi_p(u'(t)))' = -\lambda q(t)f(t, u(t)) \le 0$, we know that u is concave on $[0, 1]$. So, define the cone K by

$$K = \{u \in X |\, u(t) \ge 0,\ u \text{ is concave on } [0, 1]\} \subset X.$$

For any $x \in C[0, 1],\ x(t) \ge 0$, we consider the following boundary value problem:

$$(\phi_p(u'(t)))'(t) + \lambda q(t)f(t, x(t)) = 0, \quad t \in (0, 1), \tag{4}$$

$$u'(0) - \alpha u(\xi) = 0,\ \ u'(1) + \beta u(\eta) = 0. \tag{5}$$

Lemma 1 *For any* $x \in C[0, 1],\ x(t) \ge 0$, *BVP* (4),(5) *has a unique solution* $u(t)$ *which can be ex-*

pressed in the form

$$u(t)=\begin{cases}\frac{1}{\alpha}\phi_q(\int_0^\sigma \lambda q(\tau)f(\tau,x(\tau))d\tau)\\ -\int_0^\xi \phi_q(\int_s^\sigma \lambda q(\tau)f(\tau,x(\tau))d\tau)ds\\ +\int_0^t \phi_q(\int_s^\sigma \lambda q(\tau)f(\tau,x(\tau))d\tau)ds, \\ \qquad\qquad\qquad\qquad 0\le t\le\sigma,\\ \frac{1}{\beta}\phi_q(\int_\sigma^1 \lambda q(\tau)f(\tau,x(\tau))d\tau)\\ -\int_\eta^1 \phi_q(\int_\sigma^s \lambda q(\tau)f(\tau,x(\tau))d\tau)ds\\ +\int_t^1 \phi_q(\int_\sigma^s \lambda q(\tau)f(\tau,x(\tau))d\tau)ds,\\ \qquad\qquad\qquad\qquad \sigma\le t\le 1,\end{cases} \tag{6}$$

where σ is the unique solution of $Q(t)=\upsilon_1(t)-\upsilon_2(t)=0,\ 0<t<1$, in which

$$\upsilon_1(t)=\frac{1}{\alpha}\phi_q\left(\int_0^t \lambda q(\tau)f(\tau,x(\tau))d\tau\right) + \int_\xi^t \phi_q\left(\int_s^t \lambda q(\tau)f(\tau,x(\tau))d\tau\right)ds, \tag{7}$$

$$\upsilon_2(t)=\frac{1}{\beta}\phi_q\left(\int_t^1 \lambda q(\tau)f(\tau,x(\tau))d\tau\right) + \int_t^\eta \phi_q\left(\int_t^s \lambda q(\tau)f(\tau,x(\tau))d\tau\right)ds. \tag{8}$$

Proof: Firstly, we consider $Q(t)$ is a strictly increasing continuous function defined on $[0,1]$ three possibilities.

(i) If $\xi\le t\le\eta$, it is obvious that $Q(t)$ is a strictly increasing continuous function.

(ii) If $t<\xi$, we have

$$\begin{aligned}Q(t) &= \tfrac{1}{\alpha}\phi_q\left(\int_0^t \lambda q(\tau)f(\tau,x(\tau))d\tau\right)\\ &\quad+\int_\xi^t \phi_q\left(\int_s^t \lambda q(\tau)f(\tau,x(\tau))d\tau\right)ds\\ &\quad-\tfrac{1}{\beta}\phi_q\left(\int_t^1 \lambda q(\tau)f(\tau,x(\tau))d\tau\right)\\ &\quad-\int_t^\eta \phi_q\left(\int_t^s \lambda q(\tau)f(\tau,x(\tau))d\tau\right)ds\\ &= \tfrac{1}{\alpha}\phi_q\left(\int_0^t \lambda q(\tau)f(\tau,x(\tau))d\tau\right)\\ &\quad-\tfrac{1}{\beta}\phi_q\left(\int_t^1 \lambda q(\tau)f(\tau,x(\tau))d\tau\right)\\ &\quad-\int_\xi^\eta \phi_q\left(\int_t^s \lambda q(\tau)f(\tau,x(\tau))d\tau\right)ds.\end{aligned}$$

We can see that $Q(t)$ is a strictly increasing continuous function.

(iii) If $t>\eta$, we have

$$\begin{aligned}Q(t) &= \tfrac{1}{\alpha}\phi_q\left(\int_0^t \lambda q(\tau)f(\tau,x(\tau))d\tau\right)\\ &\quad+\int_\xi^t \phi_q\left(\int_s^t \lambda q(\tau)f(\tau,x(\tau))d\tau\right)ds\\ &\quad-\tfrac{1}{\beta}\phi_q\left(\int_t^1 \lambda q(\tau)f(\tau,x(\tau))d\tau\right)\\ &\quad-\int_t^\eta \phi_q\left(\int_t^s \lambda q(\tau)f(\tau,x(\tau))d\tau\right)ds\\ &= \tfrac{1}{\alpha}\phi_q\left(\int_0^t \lambda q(\tau)f(\tau,x(\tau))d\tau\right)\\ &\quad-\tfrac{1}{\beta}\phi_q\left(\int_t^1 \lambda q(\tau)f(\tau,x(\tau))d\tau\right)\\ &\quad+\int_\xi^\eta \phi_q\left(\int_s^t \lambda q(\tau)f(\tau,x(\tau))d\tau\right)ds.\end{aligned}$$

We can see that $Q(t)$ is a strictly increasing continuous function. $Q(0)=\upsilon_1(0)-\upsilon_2(0)<0$, $Q(1)=\upsilon_1(1)-\upsilon_2(1)>0$ implies a unique $\sigma\in(0,1)$ such that $Q(\sigma)=0$. So, $u(t)$ is continuous at $t=\sigma$ and $u(t)$ satisfies the equation (4),(5), therefore, $u(t)$ is a solution of BVP (4),(5).

Then we will prove that the solution $u(t)$ of BVP (4),(5) can be expressed in the form (6). We claim that for the unique $\sigma\in(0,1)$ we have $u'(\sigma)=0$. If not, without loss of generality, we assume that $u'(t)>0$. This implies $u(\xi)=\frac{1}{\alpha}u'(0)>0$, $u(\eta)=-\frac{1}{\beta}u'(1)<0$, which is a contradiction.

First, by integrating the equation of the problem (4) on $(\sigma,1)$ we have

$$u'(t)=-\phi_q(\int_\sigma^t \lambda q(s)f(s,x(s))ds), \tag{9}$$

thus,

$$u(1)-u(t)=-\int_t^1 \phi_q(\int_\sigma^s \lambda q(\tau)f(\tau,x(\tau))d\tau)ds),$$

i.e.,

$$u(t)=u(1)+\int_t^1 \phi_q(\int_\sigma^s \lambda q(\tau)f(\tau,x(\tau))d\tau)ds), \tag{10}$$

let $t=1$ on (9), we have $u'(1)=-\phi_q(\int_\sigma^1 \lambda q(s)f(s,x(s))ds)$. By the equation of the boundary condition (5), we have

$$\begin{aligned}u(\eta) &= \tfrac{1}{\beta}\phi_q(\int_\sigma^1 \lambda q(s)f(s,x(s))ds)\\ &= u(1)+\int_\eta^1 \phi_q(\int_\sigma^s \lambda q(\tau)f(\tau,x(\tau))d\tau)ds),\end{aligned}$$

so we have

$$\begin{aligned}u(1) &= \tfrac{1}{\beta}\phi_q(\int_\sigma^1 \lambda q(s)f(s,x(s))ds)\\ &\quad-\int_\eta^1 \phi_q(\int_\sigma^s \lambda q(\tau)f(\tau,x(\tau))d\tau)ds),\end{aligned} \tag{11}$$

by (10),(11), for $t\in(\sigma,1)$ we have

$$\begin{aligned}u(t) &= \tfrac{1}{\beta}\phi_q(\int_\sigma^1 \lambda q(\tau)f(\tau,x(\tau))d\tau)\\ &\quad-\int_\eta^1 \phi_q(\int_\sigma^s \lambda q(\tau)f(\tau,x(\tau))d\tau)ds\\ &\quad+\int_t^1 \phi_q(\int_\sigma^s \lambda q(\tau)f(\tau,x(\tau))d\tau)ds.\end{aligned}$$

Similarly, for $t\in(0,\sigma)$, by integrating the equation of problem (4) on $(0,\sigma)$, we have

$$\begin{aligned}u(t) &= \tfrac{1}{\alpha}\phi_q(\int_0^\sigma \lambda q(\tau)f(\tau,x(\tau))d\tau)\\ &\quad-\int_0^\xi \phi_q(\int_s^\sigma \lambda q(\tau)f(\tau,x(\tau))d\tau)ds\\ &\quad+\int_0^t \phi_q(\int_s^\sigma \lambda q(\tau)f(\tau,x(\tau))d\tau)ds.\end{aligned}$$

Define an operator $T : K \to X$ by

$$(Tu)(t) = \begin{cases} \frac{1}{\alpha}\phi_q(\int_0^\sigma \lambda q(\tau)f(\tau,u(\tau))d\tau) \\ -\int_0^\xi \phi_q(\int_s^\sigma \lambda q(\tau)f(\tau,u(\tau))d\tau)ds \\ +\int_0^t \phi_q(\int_s^\sigma \lambda q(\tau)f(\tau,u(\tau))d\tau)ds, \\ \qquad\qquad 0 \le t \le \sigma, \\ \frac{1}{\beta}\phi_q(\int_\sigma^1 \lambda q(\tau)f(\tau,u(\tau))d\tau) \\ -\int_\eta^1 \phi_q(\int_\sigma^s \lambda q(\tau)f(\tau,u(\tau))d\tau)ds \\ +\int_t^1 \phi_q(\int_\sigma^s \lambda q(\tau)f(\tau,u(\tau))d\tau)ds, \\ \qquad\qquad \sigma \le t \le 1. \end{cases} \tag{12}$$

Lemma 2 *Assume that* $(\mathrm{H}_1), (\mathrm{H}_3), (\mathrm{H}_4)$ *hold. Then* $T : K \to K$ *is completely continuous.*

Proof: From the definition of T, we deduce that

$$(\phi_p(Tu)'(t))' = -\lambda q(t)f(t,u(t)) \le 0,\ t \in (0,1),$$

$$(Tu)'(0) - \alpha(Tu)(\xi) = 0,$$

$$(Tu)'(1) + \beta(Tu)(\eta) = 0,$$

and

$$\begin{aligned}(Tu)(\xi) - (Tu)(0) &= \textstyle\int_0^\xi (Tu)'(s)ds \\ &< \xi(Tu)'(0) = \alpha\xi(Tu)(\xi),\end{aligned}$$

using (H_3), we have $(Tu)(\xi) - (Tu)(0) < (Tu)(\xi)$, which implies that $(Tu)(0) > 0$. Similarly, we can prove that $(Tu)(1) > 0$. This shows that $T(K) \subset K$, and each fixed point of T is a solution of problem (1),(3). We claim that $T : K \to K$ is completely continuous. The continuity of T is obvious since $\phi_q,\ f$ is continuous. Now, we prove T is compact.

Let $\Omega \subset K$ be an bounded set. Then, there exists $R > 0$, such that $\Omega \subset \{u \in K | \|u\| \le R\}$. For any $u \in \Omega$, we have

$$\begin{aligned}&0 \le \lambda \textstyle\int_0^1 q(r)f(r,x(r))dr \\ &< \lambda \max_{s\in[0,1],u\in[0,R]} f(s,u) \textstyle\int_0^1 q(r)dr =: \overline{M}.\end{aligned}$$

From the definition of T, we get

$$|Tu| < \begin{cases} \frac{1}{\alpha}\phi_q(\overline{M}) + \phi_q(\overline{M}), & 0 \le t \le \sigma, \\ \frac{1}{\beta}\phi_q(\overline{M}) + \phi_q(\overline{M}), & \sigma \le t \le 1. \end{cases}$$

On the other hand, for all $u \in \Omega$, we find

$$|(Tu)'(t)| < \phi_q(\overline{M}) \qquad 0 \le t \le 1.$$

In view of the above two equations, we have $T\Omega$ is uniformly bounded and equicontinuous, so we have $T\Omega$ is compact on $C[0,1]$.
Therefore, we have $T : K \to K$ is completely continuous.

Lemma 3 ([9]) *Suppose E is a real Banach space, $K \subset E$ is a cone, let Ω_1, Ω_2 be two bounded open sets of E such that $\theta \in \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$. Let operator $T : K \cap (\overline{\Omega_2}\backslash\Omega_1) \to K$ be completely continuous. Suppose that one of two conditions hold*
(i) $\|Tx\| \le \|x\|,\ \forall x \in K \cap \partial\Omega_1,\quad \|Tx\| \ge \|x\|,\ \forall x \in K \cap \partial\Omega_2;$
(ii) $\|Tx\| \ge \|x\|,\ \forall x \in K \cap \partial\Omega_1,\quad \|Tx\| \le \|x\|,\ \forall x \in K \cap \partial\Omega_2,$
then T has at least one fixed point in $K \cap (\overline{\Omega_2}\backslash\Omega_1)$.

Lemma 4 ([12]) *Let $u \in K$, $\theta \in (0, \frac{1}{2})$ is a constant, then $u(t) \ge \theta\|u\|$, $t \in [\theta, 1-\theta]$.*

Lemma 5 *Suppose that conditions $(\mathrm{H}_1), (\mathrm{H}_3), (\mathrm{H}_4)$ hold, then there exists a constant $\theta \in (0, \frac{1}{2})$ which satisfies*

$$0 < \int_\theta^{1-\theta} q(t)dt < \infty.$$

Furthermore, the function

$$\begin{aligned}A(t) &= \textstyle\int_\theta^t \phi_q\left(\int_s^t q(\tau)d\tau\right)ds \\ &+ \textstyle\int_t^{1-\theta} \phi_q\left(\int_t^s q(\tau)d\tau\right)ds,\ \ t \in [\theta, 1-\theta],\end{aligned}$$

is positive continuous function on $[\theta, 1-\theta]$, therefore $A(t)$ has minimum on $[\theta, 1-\theta]$. Hence we suppose that there exists $L > 0$ such that $A(t) \ge L$.

Let

$$f_\beta = \lim_{|u|\to\beta} \inf \min_{t\in[0,1]} \frac{f(t,u)}{(|u|)^{p-1}},$$
$$f^\beta = \lim_{|u|\to\beta} \sup \max_{t\in[0,1]} \frac{f(t,u)}{(|u|)^{p-1}}, \qquad \beta = (0^+ \ or\ \infty).$$

Theorem 6 *Suppose that conditions $(\mathrm{H}_1), (\mathrm{H}_3), (\mathrm{H}_4)$ hold, and let $f_\infty > 0$, $f^0 < \infty$. Then problem (1),(3) has at least one positive solution if*

$$\frac{2^{p-1}}{\theta^{p-1}f_\infty L^{p-1}} < \lambda < \frac{1}{f^0 M^{p-1}}, \tag{13}$$

where $M = \frac{1}{2}[\frac{1}{\alpha} + \frac{1}{\beta} + 2]\phi_q\left(\int_0^1 q(s)ds\right)$.

Proof: From (13), there exists $\varepsilon > 0$, such that

$$\frac{2^{p-1}}{\theta^{p-1}(f_\infty - \varepsilon)L^{p-1}} \le \lambda \le \frac{1}{(f^0 + \varepsilon)M^{p-1}} \tag{14}$$

(I) For fixed ε. In view of $f^0 < \infty$, there exists $H_1 > 0$, such that for $u :\ 0 < |u| \le H_1$, we have

$$f(t,u) \le (f^0 + \varepsilon)(|u|)^{p-1}.$$

Let $\Omega_1 = \{u \in X : \|u\| < H_1\}$, when $u \in K \cap \partial\Omega_1$, we have

$$\begin{aligned}
&\|Tu\| = (Tu)(\sigma)\\
&= \tfrac{1}{2}\Big[\tfrac{1}{\alpha}\phi_q\left(\int_0^\sigma \lambda q(\tau)f(\tau,u(\tau))d\tau\right)\\
&+\int_\xi^\sigma \phi_q\left(\int_s^\sigma \lambda q(\tau)f(\tau,u(\tau))d\tau\right)ds\\
&+\tfrac{1}{\beta}\phi_q\left(\int_\sigma^1 \lambda q(\tau)f(\tau,u(\tau))d\tau\right)\\
&+\int_\sigma^\eta \phi_q\left(\int_\sigma^s \lambda q(\tau)f(\tau,u(\tau))d\tau\right)ds\Big]\\
&\le \tfrac{1}{2}\Big[\tfrac{1}{\alpha}\phi_q\left(\int_0^\sigma \lambda q(\tau)(f^0+\varepsilon)\|u\|^{p-1}d\tau\right)\\
&+\int_\xi^\sigma \phi_q\left(\int_s^\sigma \lambda q(\tau)(f^0+\varepsilon)\|u\|^{p-1}d\tau\right)ds\\
&+\tfrac{1}{\beta}\phi_q\left(\int_\sigma^1 \lambda q(\tau)(f^0+\varepsilon)\|u\|^{p-1}d\tau\right)\\
&+\int_\sigma^\eta \phi_q\left(\int_\sigma^s \lambda q(\tau)(f^0+\varepsilon)\|u\|^{p-1}d\tau\right)ds\Big]\\
&\le \tfrac{1}{2}\Big[\tfrac{1}{\alpha}\lambda^{q-1}(f^0+\varepsilon)^{q-1}\|u\|\phi_q\left(\int_0^\sigma q(\tau)d\tau\right)\\
&+\int_\xi^\sigma \lambda^{q-1}(f^0+\varepsilon)^{q-1}\|u\|\phi_q\left(\int_s^\sigma q(\tau)d\tau\right)ds\\
&+\tfrac{1}{\beta}\lambda^{q-1}(f^0+\varepsilon)^{q-1}\|u\|\phi_q\left(\int_\sigma^1 q(\tau)d\tau\right)\\
&+\int_\sigma^\eta \lambda^{q-1}(f^0+\varepsilon)^{q-1}\|u\|\phi_q\left(\int_\sigma^s q(\tau)d\tau\right)ds\Big]\\
&\le \tfrac{1}{2}[\tfrac{1}{\alpha}+\tfrac{1}{\beta}+2]\phi_q\left(\int_0^1 q(\tau)d\tau\right)\lambda^{q-1}(f^0+\varepsilon)^{q-1}\|u\|\\
&= M\lambda^{q-1}(f^0+\varepsilon)^{q-1}\|u\|\\
&\le \|u\|,
\end{aligned}$$

therefore, we have $\|Tu\| \le \|u\|$.

(II) In view of $f_\infty > 0$, there exists $\bar{H}_2 > 0$, such that for $u : |u| \ge \bar{H}_2$, we have

$$f(t,u) \ge (f_\infty - \varepsilon)(|u|)^{p-1}.$$

Let $H_2 = \max\left\{\frac{H_1}{\theta}, \frac{\bar{H}_2}{\theta}\right\}$, $\Omega_2 = \{u \in X : \|u\| < H_2\}$. By lemma 4, we have when $u \in K \cap \partial\Omega_2$, we have

$$\theta\|u\| \le |u| \le \|u\|, \qquad t \in [\theta, 1-\theta].$$

We consider three possibilities.

(i) If $\sigma \in [\theta, 1-\theta]$, then for $u \in K \cap \partial\Omega_2$, by Lemma 5, we have

$$\begin{aligned}
&2\|Tu\| = 2(Tu)(\sigma)\\
&\ge \int_0^\sigma \phi_q\left(\int_s^\sigma \lambda q(\tau)f(\tau,u(\tau))d\tau\right)ds\\
&+\int_\sigma^1 \phi_q\left(\int_\sigma^s \lambda q(\tau)f(\tau,u(\tau))d\tau\right)ds\\
&\ge \int_\theta^\sigma \phi_q\left(\int_s^\sigma \lambda q(\tau)f(\tau,u(\tau))d\tau\right)ds\\
&+\int_\sigma^{1-\theta} \phi_q\left(\int_\sigma^s \lambda q(\tau)f(\tau,u(\tau))d\tau\right)ds\\
&\ge \int_\theta^\sigma \phi_q\left(\int_s^\sigma \lambda q(\tau)(f_\infty-\varepsilon)\theta^{p-1}\|u\|^{p-1}d\tau\right)ds\\
&+\int_\sigma^{1-\theta} \phi_q\left(\int_\sigma^s \lambda q(\tau)(f_\infty-\varepsilon)\theta^{p-1}\|u\|^{p-1}d\tau\right)ds\\
&= \lambda^{q-1}(f_\infty-\varepsilon)^{q-1}\theta\|u\|[\int_\theta^\sigma \phi_q\left(\int_s^\sigma q(\tau)d\tau\right)ds\\
&+\int_\sigma^{1-\theta} \phi_q\left(\int_\sigma^s q(\tau)d\tau\right)ds]\\
&= \lambda^{q-1}(f_\infty-\varepsilon)^{q-1}\theta A(\theta)\|u\|\\
&\ge \lambda^{q-1}(f_\infty-\varepsilon)^{q-1}\theta L\|u\| \ge 2\|u\|.
\end{aligned}$$

(ii) If $\sigma \in (1-\theta, 1)$, then for $u \in K \cap \partial\Omega_2$, by Lemma 5, we have

$$\begin{aligned}
&\|Tu\| = (Tu)(\sigma)\\
&\ge \int_0^\sigma \phi_q\left(\int_s^\sigma \lambda q(\tau)f(\tau,u(\tau))d\tau\right)ds\\
&\ge \int_\theta^{1-\theta} \phi_q\left(\int_s^{1-\theta} \lambda q(\tau)f(\tau,u(\tau))d\tau\right)ds\\
&\ge \int_\theta^{1-\theta} \phi_q\left(\int_s^{1-\theta} \lambda q(\tau)(f_\infty-\varepsilon)\theta^{p-1}\|u\|^{p-1}d\tau\right)ds\\
&\ge \lambda^{q-1}(f_\infty-\varepsilon)^{q-1}\theta\|u\|[\int_\theta^{1-\theta} \phi_q\left(\int_s^{1-\theta} q(\tau)d\tau\right)ds\\
&= \lambda^{q-1}(f_\infty-\varepsilon)^{q-1}\theta\|u\|A(1-\theta)\\
&\ge \lambda^{q-1}(f_\infty-\varepsilon)^{q-1}\theta\|u\|L \ge \|u\|.
\end{aligned}$$

(iii) If $\sigma \in (0, \theta)$, then for $u \in K \cap \partial\Omega_2$, by Lemma 5, we have

$$\begin{aligned}
&\|Tu\| = (Tu)(\sigma)\\
&\ge \int_\sigma^1 \phi_q\left(\int_\sigma^s \lambda q(\tau)f(\tau,u(\tau))d\tau\right)ds\\
&\ge \int_\theta^{1-\theta} \phi_q\left(\int_\theta^s \lambda q(\tau)f(\tau,u(\tau))d\tau\right)ds\\
&\ge \int_\theta^{1-\theta} \phi_q\left(\int_\theta^s \lambda q(\tau)(f_\infty-\varepsilon)\theta^{p-1}\|u\|^{p-1}d\tau\right)ds\\
&\ge \lambda^{q-1}(f_\infty-\varepsilon)^{q-1}\theta\|u\|[\int_\theta^{1-\theta} \phi_q\left(\int_\theta^s q(\tau)d\tau\right)ds\\
&= \lambda^{q-1}(f_\infty-\varepsilon)^{q-1}\theta\|u\|A(\theta)\\
&\ge \lambda^{q-1}(f_\infty-\varepsilon)^{q-1}\theta\|u\|L \ge \|u\|.
\end{aligned}$$

Therefore, for any $u \in K \cap \partial\Omega_2$, we all have $\|Tu\| \ge \|u\|$.

By Lemma 3, we have operator T has a fixed point $u \in K \cap (\bar{\Omega}_2 \setminus \Omega_1)$, and $H_1 \le \|u\| \le H_2$, so u is a positive solution of BVP (1), (3).

Theorem 7 *Suppose that conditions* (H_1), (H_3), (H_4) *hold, and let* $f_0 > 0,\ f^\infty < \infty$. *Then problem* (1),(3) *has at least one positive solution if*

$$\frac{2^{p-1}}{\theta^{p-1}f_0L^{p-1}} < \lambda < \frac{1}{f^\infty M^{p-1}}. \tag{15}$$

Proof: From (15), there exists $\varepsilon > 0$, such that

$$\frac{2^{p-1}}{\theta^{p-1}(f_0-\varepsilon)L^{p-1}} \le \lambda \le \frac{1}{(f^\infty+\varepsilon)M^{p-1}}. \tag{16}$$

(I) For fixed ε. In view of $f_0 > 0$, there exists $H_1 > 0$, such that for $u : 0 < |u| \le H_1$, we have

$$f(t,u) \ge (f_0 - \varepsilon)(|u|)^{p-1}.$$

Let $\Omega_1 = \{u \in X : \|u\| < H_1\}$, when $u \in K \cap \partial\Omega_1$, we have

$$\theta\|u\| \le |u| \le \|u\|, \qquad t \in [\theta, 1-\theta].$$

We consider three possibilities.

(i) If $\sigma \in [\theta, 1-\theta]$, then for $u \in K \cap \partial\Omega_1$, by Lemma

5, we have

$$\begin{aligned}
&2\|Tu\| = 2(Tu)(\sigma)\\
&\geq \int_0^\sigma \phi_q\left(\int_s^\sigma \lambda q(\tau)f(\tau,u(\tau))d\tau\right)ds\\
&+\int_\sigma^1 \phi_q\left(\int_\sigma^s \lambda q(\tau)f(\tau,u(\tau))d\tau\right)ds\\
&\geq \int_\theta^\sigma \phi_q\left(\int_s^\sigma \lambda q(\tau)f(\tau,u(\tau))d\tau\right)ds\\
&+\int_\sigma^{1-\theta} \phi_q\left(\int_\sigma^s \lambda q(\tau)f(\tau,u(\tau))d\tau\right)ds\\
&\geq \int_\theta^\sigma \phi_q\left(\int_s^\sigma \lambda q(\tau)(f_0-\varepsilon)\theta^{p-1}\|u\|^{p-1}d\tau\right)ds\\
&+\int_\sigma^{1-\theta} \phi_q\left(\int_\sigma^s \lambda q(\tau)(f_0-\varepsilon)\theta^{p-1}\|u\|^{p-1}d\tau\right)ds\\
&=\lambda^{q-1}(f_0-\varepsilon)^{q-1}\theta\|u\|[\int_\theta^\sigma \phi_q\left(\int_s^\sigma q(\tau)d\tau\right)ds\\
&+\int_\sigma^{1-\theta}\phi_q\left(\int_\sigma^s q(\tau)d\tau\right)ds]\\
&\geq \lambda^{q-1}(f_0-\varepsilon)^{q-1}\theta A(\theta)\|u\|\\
&\geq \lambda^{q-1}(f_0-\varepsilon)^{q-1}\theta L\|u\| \geq 2\|u\|.
\end{aligned}$$

(ii) If $\sigma\in(1-\theta,1)$, then for $u\in K\cap\partial\Omega_1$, by Lemma 5, we have

$$\begin{aligned}
&\|Tu\| = (Tu)(\sigma)\\
&\geq \int_0^\sigma \phi_q\left(\int_s^\sigma \lambda q(\tau)f(\tau,u(\tau))d\tau\right)ds\\
&\geq \int_\theta^{1-\theta} \phi_q\left(\int_s^{1-\theta} \lambda q(\tau)f(\tau,u(\tau))d\tau\right)ds\\
&\geq \int_\theta^{1-\theta} \phi_q\left(\int_s^{1-\theta} \lambda q(\tau)(f_0-\varepsilon)\theta^{p-1}\|u\|^{p-1}d\tau\right)ds\\
&\geq \lambda^{q-1}(f_0-\varepsilon)^{q-1}\theta\|u\|[\int_\theta^{1-\theta}\phi_q\left(\int_s^{1-\theta} q(\tau)d\tau\right)ds\\
&=\lambda^{q-1}(f_0-\varepsilon)^{q-1}\theta\|u\|A(1-\theta)\\
&\geq \lambda^{q-1}(f_0-\varepsilon)^{q-1}\theta\|u\|L \geq \|u\|.
\end{aligned}$$

(iii) If $\sigma\in(0,\theta)$, then for $u\in K\cap\partial\Omega_1$, by Lemma 5, we have

$$\begin{aligned}
&\|Tu\| = (Tu)(\sigma)\\
&\geq \int_\sigma^1 \phi_q\left(\int_\sigma^s \lambda q(\tau)f(\tau,u(\tau))d\tau\right)ds\\
&\geq \int_\theta^{1-\theta} \phi_q\left(\int_\theta^s \lambda q(\tau)f(\tau,u(\tau))d\tau\right)ds\\
&\geq \int_\theta^{1-\theta} \phi_q\left(\int_\theta^s \lambda q(\tau)(f_0-\varepsilon)\theta^{p-1}\|u\|^{p-1}d\tau\right)ds\\
&\geq \lambda^{q-1}(f_0-\varepsilon)^{q-1}\theta\|u\|[\int_\theta^{1-\theta}\phi_q\left(\int_\theta^s q(\tau)d\tau\right)ds\\
&=\lambda^{q-1}(f_0-\varepsilon)^{q-1}\theta\|u\|A(\theta)\\
&\geq \lambda^{q-1}(f_0-\varepsilon)^{q-1}\theta\|u\|L \geq \|u\|.
\end{aligned}$$

Therefore, for any $u\in K\cap\partial\Omega_1$, we all have $\|Tu\|\geq\|u\|$.

(II) In view of $f^\infty<\infty$, there exists $\bar{H}_2>0$, such that for $u:\ |u|\geq\bar{H}_2$, we have

$$f(t,u)\leq(f^\infty+\varepsilon)(|u|)^{p-1}.$$

Here there are two cases to consider, namely, where f is bounded and where f is unbounded.

Case 1. Suppose f is bounded. There exists $\Lambda>0$, such that $f(t,u)\leq\Lambda^{p-1}$. Let $H_2=\max\left\{\frac{H_1}{\theta},\ \lambda^{q-1}\Lambda M\right\}$, $\Omega_2=\{u\in X:\ \|u\|<H_2\}$, when $u\in K\cap\partial\Omega_2$, we have

$$\begin{aligned}
&\|Tu\| = (Tu)(\sigma) =\\
&\frac{1}{2}\Big[\frac{1}{\alpha}\phi_q\left(\int_0^\sigma \lambda q(\tau)f(\tau,u(\tau))d\tau\right)\\
&+\int_\xi^\sigma \phi_q\left(\int_s^\sigma \lambda q(\tau)f(\tau,u(\tau))d\tau\right)ds\\
&+\frac{1}{\beta}\phi_q\left(\int_\sigma^1 \lambda q(\tau)f(\tau,u(\tau))d\tau\right)\\
&+\int_\sigma^\eta \phi_q\left(\int_\sigma^s \lambda q(\tau)f(\tau,u(\tau))d\tau\right)ds\Big]\\
&\leq\frac{1}{2}\Big[\frac{1}{\alpha}\phi_q\left(\int_0^\sigma \lambda q(\tau)\Lambda^{p-1}d\tau\right)\\
&+\int_\xi^\sigma \phi_q\left(\int_s^\sigma \lambda q(\tau)\Lambda^{p-1}d\tau\right\}ds\\
&+\frac{1}{\beta}\phi_q\left(\int_\sigma^1 \lambda q(\tau)\Lambda^{p-1}d\tau\right)\\
&+\int_\sigma^\eta \phi_q\left(\int_\sigma^s \lambda q(\tau)\Lambda^{p-1}d\tau\right)ds\Big]\\
&\leq\frac{1}{2}\Big[\frac{1}{\alpha}\lambda^{q-1}\Lambda\phi_q\left(\int_0^\sigma q(\tau)d\tau\right)\\
&+\int_\xi^\sigma \lambda^{q-1}\Lambda\phi_q\left(\int_s^\sigma q(\tau)d\tau\right)ds\\
&+\frac{1}{\beta}\lambda^{q-1}\Lambda\phi_q\left(\int_\sigma^1 q(\tau)d\tau\right)\\
&+\int_\sigma^\eta \lambda^{q-1}\Lambda\phi_q\left(\int_\sigma^s q(\tau)d\tau\right)ds\Big]\\
&\leq\frac{1}{2}[\frac{1}{\alpha}+\frac{1}{\beta}+2]\phi_q\left(\int_0^1 q(\tau)d\tau\right)\lambda^{q-1}\Lambda\\
&=M\lambda^{q-1}\Lambda\leq H_2=\|u\|.
\end{aligned}$$

Case 2. Suppose f is unbounded. Let $H_2>\max\{H_1,\ \bar{H}_2\}$, when $t\in[0,1]$, and $0<|u|\leq H_2$, we have $f(t,u(t))\leq f(t,H_2)$. Let $\Omega_2=\{u\in X:\ \|u\|<H_2\}$, when $u\in K\cap\partial\Omega_2$, we have

$$\begin{aligned}
&\|Tu\| = (Tu)(\sigma)\\
&=\frac{1}{2}\Big[\frac{1}{\alpha}\phi_q\left(\int_0^\sigma \lambda q(\tau)f(\tau,u(\tau))d\tau\right)\\
&+\int_\xi^\sigma \phi_q\left(\int_s^\sigma \lambda q(\tau)f(\tau,u(\tau))d\tau\right)ds\\
&+\frac{1}{\beta}\phi_q\left(\int_\sigma^1 \lambda q(\tau)f(\tau,u(\tau))d\tau\right)\\
&+\int_\sigma^\eta \phi_q\left(\int_\sigma^s \lambda q(\tau)f(\tau,u(\tau))d\tau\right)ds\Big]\\
&\leq\frac{1}{2}\Big[\frac{1}{\alpha}\phi_q\left(\int_0^\sigma \lambda q(\tau)f(\tau,H_2)d\tau\right)\\
&+\int_\xi^\sigma \phi_q\left(\int_s^\sigma \lambda q(\tau)f(\tau,H_2)d\tau\right)ds\\
&+\frac{1}{\beta}\phi_q\left(\int_\sigma^1 \lambda q(\tau)f(\tau,H_2)d\tau\right)\\
&+\int_\sigma^\eta \phi_q\left(\int_\sigma^s \lambda q(\tau)f(\tau,H_2)d\tau\right)ds\Big]\\
&\leq\frac{1}{2}\Big[\frac{1}{\alpha}\phi_q\left(\int_0^\sigma \lambda q(\tau)(f^\infty+\varepsilon)(H_2)^{p-1}d\tau\right)\\
&+\int_\xi^\sigma \phi_q\left(\int_s^\sigma \lambda q(\tau)(f^\infty+\varepsilon)(H_2)^{p-1}d\tau\right)ds\\
&+\frac{1}{\beta}\phi_q\left(\int_\sigma^1 \lambda q(\tau)(f^\infty+\varepsilon)(H_2)^{p-1}d\tau\right)\\
&+\int_\sigma^\eta \phi_q\left(\int_\sigma^s \lambda q(\tau)(f^\infty+\varepsilon)(H_2)^{p-1}d\tau\right)ds\Big]
\end{aligned}$$

$$\leq \tfrac{1}{2}\Big[\tfrac{1}{\alpha}\lambda^{q-1}(f^\infty+\varepsilon)^{q-1}(H_2)\phi_q\left(\int_0^1 q(\tau)d\tau\right)$$
$$+\lambda^{q-1}(f^\infty+\varepsilon)^{q-1}(H_2)\phi_q\left(\int_0^1 q(\tau)d\tau\right)$$
$$+\tfrac{1}{\beta}\lambda^{q-1}(f^\infty+\varepsilon)^{q-1}(H_2)\phi_q\left(\int_0^1 q(\tau)d\tau\right)$$
$$+\lambda^{q-1}(f^\infty+\varepsilon)^{q-1}(H_2)\phi_q\left(\int_0^1 q(\tau)d\tau\right)\Big]$$
$$=\lambda^{q-1}(f^\infty+\varepsilon)^{q-1}(H_2)$$
$$\tfrac{1}{2}[\tfrac{1}{\alpha}+\tfrac{1}{\beta}+2]\phi_q\left(\int_0^1 q(\tau)d\tau\right)$$
$$=\lambda^{q-1}(f^\infty+\varepsilon)^{q-1}(H_2)M$$
$$\leq H_2=\|u\|.$$

Therefore, we have $\|Tu\|\leq\|u\|$.

By Lemma 3, we have operator T has a fixed point $u\in K\cap(\bar{\Omega}_2\setminus\Omega_1)$, and $H_1\leq\|u\|\leq H_2$, so u is a positive solution of BVP (1), (3).

3 Existence and Iteration of Positive Solutions to BVP (2),(3)

Let $X=C^1[0,1]$ be endowed with the maximum norm,

$$\|u\|=\max\left\{\max_{0\leq t\leq 1}|u(t)|,\max_{0\leq t\leq 1}|u'(t)|\right\}.$$

From the fact $(\phi_p(u'(t)))'=-q(t)f(t,u(t),u'(t))\leq 0$, we know that u is concave on $[0,1]$. So, define the cone K by

$$K=\{u\in X|\,u(t)\geq 0,\ u\,\text{is concave on}\ [0,1]\}\subset X.$$

For any $x\in C^1[0,1],\ x(t)\geq 0$, we consider the following boundary value problem:

$$(\phi_p(u'(t)))'(t)+q(t)f(t,x(t),x'(t))=0,\quad t\in(0,1),\tag{17}$$
$$u'(0)-\alpha u(\xi)=0,\ \ u'(1)+\beta u(\eta)=0.\tag{18}$$

Lemma 8 *For any $x\in C^1[0,1],\ x(t)\geq 0$, BVP (17),(18) has a unique solution $u(t)$ which can be expressed in the form*

$$u(t)=\begin{cases}\frac{1}{\alpha}\phi_q(\int_0^\sigma q(\tau)f(\tau,x(\tau),x'(\tau))d\tau)\\ -\int_0^\xi\phi_q(\int_s^\sigma q(\tau)f(\tau,x(\tau),x'(\tau))d\tau)ds\\ +\int_0^t\phi_q(\int_s^\sigma q(\tau)f(\tau,x(\tau),x'(\tau))d\tau)ds,\\ \qquad\qquad 0\leq t\leq\sigma,\\ \frac{1}{\beta}\phi_q(\int_\sigma^1 q(\tau)f(\tau,x(\tau),x'(\tau))d\tau)\\ -\int_\eta^1\phi_q(\int_\sigma^s q(\tau)f(\tau,x(\tau),x'(\tau))d\tau)ds\\ +\int_t^1\phi_q(\int_\sigma^s q(\tau)f(\tau,x(\tau),x'(\tau))d\tau)ds,\\ \qquad\qquad\sigma\leq t\leq 1,\end{cases}\tag{19}$$

where σ is the unique solution of $Q(t)=\upsilon_1(t)-\upsilon_2(t)=0,\ 0<t<1$, in which

$$\upsilon_1(t)=\frac{1}{\alpha}\phi_q\left(\int_0^t q(\tau)f(\tau,x(\tau),x'(\tau))d\tau\right)$$
$$+\int_\xi^t\phi_q\left(\int_s^t q(\tau)f(\tau,x(\tau),x'(\tau))d\tau\right)ds,\tag{20}$$
$$\upsilon_2(t)=\frac{1}{\beta}\phi_q\left(\int_t^1 q(\tau)f(\tau,x(\tau),x'(\tau))d\tau\right)$$
$$+\int_t^\eta\phi_q\left(\int_t^s q(\tau)f(\tau,x(\tau),x'(\tau))d\tau\right)ds.\tag{21}$$

Proof: The proof of this Lemma is similar to Lemma 1, so we omit here.

Define an operator $\bar{T}:K\to X$ by

$$(\bar{T}u)(t)=\begin{cases}\frac{1}{\alpha}\phi_q(\int_0^\sigma q(\tau)f(\tau,u(\tau),u'(\tau))d\tau)\\ -\int_0^\xi\phi_q(\int_s^\sigma q(\tau)f(\tau,u(\tau),u'(\tau))d\tau)ds\\ +\int_0^t\phi_q(\int_s^\sigma q(\tau)f(\tau,u(\tau),u'(\tau))d\tau)ds,\\ \qquad\qquad 0\leq t\leq\sigma,\\ \frac{1}{\beta}\phi_q(\int_\sigma^1 q(\tau)f(\tau,u(\tau),u'(\tau))d\tau)\\ -\int_\eta^1\phi_q(\int_\sigma^s q(\tau)f(\tau,u(\tau),u'(\tau))d\tau)ds\\ +\int_t^1\phi_q(\int_\sigma^s q(\tau)f(\tau,u(\tau),u'(\tau))d\tau)ds,\\ \qquad\qquad\sigma\leq t\leq 1.\end{cases}\tag{22}$$

Lemma 9 *Assume that* $(\mathrm{H}_2)-(\mathrm{H}_4)$ *hold. Then $\bar{T}:K\to K$ is completely continuous.*

Proof: From the definition of $\bar{T}$, we deduce that

$$(\phi_p(\bar{T}u)'(t))'=-q(t)f(t,u(t),u'(t))\leq 0,$$
$$t\in(0,1),$$
$$(\bar{T}u)'(0)-\alpha(\bar{T}u)(\xi)=0,$$
$$(\bar{T}u)'(1)+\beta(\bar{T}u)(\eta)=0,$$

and

$$\begin{aligned}(\bar{T}u)(\xi)-(\bar{T}u)(0)&=\textstyle\int_0^\xi(\bar{T}u)'(s)ds<\xi(\bar{T}u)'(0)\\ &=\alpha\xi(\bar{T}u)(\xi),\end{aligned}$$

using (H_3), we have $(\bar{T}u)(\xi)-(\bar{T}u)(0)<(\bar{T}u)(\xi)$, which implies that $(\bar{T}u)(0)>0$. Similarly, we can prove that $(\bar{T}u)(1)>0$. This shows that $\bar{T}(K)\subset K$, and each fixed point of $\bar{T}$ is a solution of problem (2),(3). We claim that $\bar{T}:K\to K$ is completely continuous. The continuity of $\bar{T}$ is obvious since $\phi_q,\ f$ is continuous. Now, we prove $\bar{T}$ is compact.

Let $\Omega\subset K$ be an bounded set. Then, there exists $R>0$, such that $\Omega\subset\{u\in K|\|u\|\leq R\}$. For any $u\in\Omega$, we have

$$0\leq\int_0^1 q(r)f(r,u(r),u'(r))dr<$$
$$\max_{s\in[0,1],u\in[0,R],v\in[-R,R]}f(s,u,v)\int_0^1 q(r)dr$$
$$=:\overline{M}.$$

From the definition of $\bar{T}$, we get

$$|\bar{T}u|<\begin{cases}\frac{1}{\alpha}\phi_q(\overline{M})+\phi_q(\overline{M}), & 0\leq t\leq\sigma,\\ \frac{1}{\beta}\phi_q(\overline{M})+\phi_q(\overline{M}), & \sigma\leq t\leq 1.\end{cases}$$

On the other hand, for all $u \in \Omega$, we find

$$|(\bar{T}u)'(t)| < \phi_q(\overline{M}) \qquad 0 \le t \le 1.$$

In view of the above two equations, we have $\bar{T}\Omega$ is uniformly bounded and equicontinuous, so we have $\bar{T}\Omega$ is compact on $C^0[0,1]$.

From (22), we know for $\forall\varepsilon > 0,\ \exists\delta > 0$, such that when $|t_1 - t_2| < \delta$, we have

$$\begin{aligned}&|\phi_p(\bar{T}u)'(t_1) - \phi_p(\bar{T}u)'(t_2)|\\ &= \textstyle\int_{t_1}^{t_2} q(r)f(r, u(r), u'(r))\\ &< \varepsilon.\end{aligned}$$

So $\phi_p(\bar{T}\Omega)'$ is uniformly bounded and equicontinuous, so $\phi_p(\bar{T}\Omega)'$ is compact on $C^0[0,1]$, therefore we have $(\bar{T}\Omega)'$ is compact on $C^0[0,1]$.
The Arzela-Ascoli theorem guarantees that $\bar{T}\Omega$ is relatively compact on $C^1[0,1]$, which means $\bar{T}$ is compact. Therefore, we have $\bar{T} : K \to K$ is completely continuous.

Let

$$\begin{aligned}A &= \max\Big\{\phi_q(\textstyle\int_0^1 q(\tau)d\tau)(1 + \frac{1}{\alpha}),\\ &\quad \phi_q(\textstyle\int_0^1 q(\tau)d\tau)(1 + \frac{1}{\beta})\Big\}\end{aligned}$$

We will prove the following result.

Theorem 10 *Assume that* $(\mathrm{H}_2)-(\mathrm{H}_4)$ *hold, and there exists* $a > 0$, *such that*

$$\begin{aligned}&(C_1)\ f(t,\ x_1,\ y_1) \le f(t,\ x_2,\ y_2),\\ &\qquad \text{for any } 0 \le t \le 1, \quad 0 \le x_1 \le x_2 \le a,\\ &\qquad 0 \le |y_1| \le |y_2| \le a;\\ &(C_2)\ \max_{0\le t\le 1} f(t,a,a) \le \phi_p\left(\tfrac{a}{A}\right);\\ &(C_3)\ f(t,\ 0,\ 0) \not\equiv 0 \text{ for } 0 \le t \le 1.\end{aligned}$$

Then the boundary value problem (2),(3) *has one positive solution* $\omega^* \in K$ *such that* $0 < \omega^* \le a$, $0 < |(\omega^*)'| \le a$ *and* $\lim\limits_{n\to\infty} \bar{T}^n\omega_0 = \omega^*$, $\lim\limits_{n\to\infty} (\bar{T}^n\omega_0)' = (\omega^*)'$ *where*

$$\omega_0(t) = a\frac{\max\left\{(\frac{1}{\alpha} + t),\ (\frac{1}{\beta} + (1-t))\right\}}{\max\left\{(\frac{1}{\alpha} + 1),\ (\frac{1}{\beta} + 1)\right\}},\ \ 0 \le t \le 1.$$

Proof: We denote

$$K_a = \{u \in K|\ \|u\| < a\},$$

and

$$\overline{K_a} = \{u \in K|\ \|u\| \le a\}.$$

We first prove $\bar{T} :\ \overline{K_a} \to \overline{K_a}$. Let $u \in \overline{K_a}$, then

$$0 \le u(t) \le \max_{0\le t\le 1} |u(t)| \le \|u\| \le a, \tag{23}$$

$$|u'(t)| \le \max_{0\le t\le 1} |u'(t)| \le \|u\| \le a. \tag{24}$$

So, by assumptions (C_1) and (C_2), we have

$$\begin{aligned}&0 \le f(t, u(t), u'(t)) \le f(t, a, a) \le \max_{0\le t\le 1} f(t, a, a)\\ &\le \phi_p\left(\tfrac{a}{A}\right),\ 0 \le t \le 1.\end{aligned} \tag{25}$$

In fact,

$$\begin{aligned}\|\bar{T}u\| &= \max\left\{\max_{0\le t\le 1} |(\bar{T}u)(t)|, \max_{0\le t\le 1} |(\bar{T}u)'(t)|\right\}\\ &= \max\left\{(\bar{T}u)(\sigma),\ (\bar{T}u)'(0),\ -(\bar{T}u)'(1)\right\}.\end{aligned}$$

Therefore, for $u \in \overline{K_a}$, we have

$$\begin{aligned}(\bar{T}u)(\sigma) &= \tfrac{1}{\alpha}\phi_q(\textstyle\int_0^\sigma q(\tau)f(\tau, u(\tau), u'(\tau))d\tau)\\ &\quad + \textstyle\int_\xi^\sigma \phi_q(\int_s^\sigma q(\tau)f(\tau, u(\tau), u'(\tau))d\tau)ds,\\ &= \tfrac{1}{\beta}\phi_q(\textstyle\int_\sigma^1 q(\tau)f(\tau, u(\tau), u'(\tau))d\tau)\\ &\quad + \textstyle\int_\sigma^\eta \phi_q(\int_\sigma^s q(\tau)f(\tau, u(\tau), u'(\tau))d\tau)ds,\\ &\le \tfrac{a}{A}\max\Big\{\phi_q(\textstyle\int_0^1 q(\tau)d\tau)(1 + \frac{1}{\alpha}),\\ &\qquad\qquad \phi_q(\textstyle\int_0^1 q(\tau)d\tau)(1 + \frac{1}{\beta})\Big\}\\ &= a,\end{aligned}$$

$$\begin{aligned}(\bar{T}u)'(0) &= \phi_q\left(\textstyle\int_0^\sigma q(\tau)f(\tau, u(\tau), u'(\tau))d\tau\right)\\ &\le \tfrac{a}{A}\phi_q\left(\textstyle\int_0^1 q(\tau)d\tau\right) \le a,\end{aligned}$$

$$\begin{aligned}-(\bar{T}u)'(1) &= \phi_q\left(\textstyle\int_\sigma^1 q(\tau)f(\tau, u(\tau), u'(\tau))d\tau\right)\\ &\le \tfrac{a}{A}\phi_q\left(\textstyle\int_0^1 q(\tau)d\tau\right) \le a.\end{aligned}$$

Thus, we obtain that

$$\|\bar{T}u\| \le a.$$

This means $\bar{T} :\ \overline{K_a} \to \overline{K_a}$. Let

$$\omega_0(t) = a\frac{\max\left\{(\frac{1}{\alpha} + t),\ (\frac{1}{\beta} + (1-t))\right\}}{\max\left\{(\frac{1}{\alpha} + 1),\ (\frac{1}{\beta} + 1)\right\}},$$

$$0 \le t \le 1,$$

$$\Phi = \phi_q(\int_0^1 q(\tau)d\tau)(\frac{1}{\alpha} + t),$$

$$\Psi = \phi_q(\int_0^1 q(\tau)d\tau)(\frac{1}{\beta} + (1-t)),$$

$$\Gamma = \phi_q(\int_0^1 q(\tau)d\tau)(1 + \frac{1}{\alpha}),$$

$$\Upsilon = \phi_q(\int_0^1 q(\tau)d\tau)(1+\frac{1}{\beta}).$$

Let $\omega_1 = \bar{T}\omega_0$, then $\omega_1 \in \overline{K_a}$, we denote

$$\omega_{n+1} = \bar{T}\omega_n = \bar{T}^{n+1}\omega_0, \quad (n = 0, 1, 2, \cdots). \quad (26)$$

Since $\bar{T} : \overline{K_a} \to \overline{K_a}$, we have $\omega_n \in \bar{T}\overline{K_a} \subseteq \overline{K_a}$, $n = 0, 1, 2, \cdots$.
Since $\bar{T}$ is completely continuous, $\{\omega_n\}_{n=0}^{\infty}$ is a sequentially compact set,

$$\begin{aligned}
&\omega_1(t) = \bar{T}\omega_0(t)\\
&= \begin{cases} \frac{1}{\alpha}\phi_q(\int_0^\sigma q(\tau)f(\tau,\omega_0(\tau),\omega_0'(\tau))d\tau) \\ + \int_\xi^t \phi_q(\int_s^\sigma q(\tau)f(\tau,\omega_0(\tau),\omega_0'(\tau))d\tau)ds, \\ \qquad 0 \le t \le \sigma \\ \frac{1}{\beta}\phi_q(\int_\sigma^1 q(\tau)f(\tau,\omega_0(\tau),\omega_0'(\tau))d\tau) \\ + \int_t^\eta \phi_q(\int_\sigma^s q(\tau)f(\tau,\omega_0(\tau),\omega_0'(\tau))d\tau)ds, \\ \qquad \sigma \le t \le 1 \end{cases}\\
&\le \frac{a}{A}\max\Big\{\frac{1}{\alpha}\phi_q(\int_0^1 q(\tau)d\tau) + \int_\xi^t \phi_q(\int_s^1 q(\tau)d\tau)ds,\\
&\frac{1}{\beta}\phi_q(\int_0^1 q(\tau)d\tau) + \int_t^\eta \phi_q(\int_0^s q(\tau)d\tau)ds\Big\}\\
&\le a\frac{\max\{\Phi,\ \Psi\}}{\max\{\Gamma,\ \Upsilon\}}\\
&= \omega_0(t),
\end{aligned}$$

$$\begin{aligned}
&|\omega_1'(t)| = |(\bar{T}\omega_0)'(t)|\\
&= \begin{cases} |\phi_q(\int_t^\sigma q(\tau)f(\tau,\omega_0(\tau),\omega_0'(\tau))d\tau)|, \\ 0 \le t \le \sigma, \\ |-\phi_q(\int_\sigma^t q(\tau)f(\tau,\omega_0(\tau),\omega_0'(\tau))d\tau)|, \\ \sigma \le t \le 1 \end{cases}\\
&\le \frac{a}{A}\phi_q(\int_0^1 q(\tau)d\tau)\\
&= a\frac{\phi_q(\int_0^1 q(\tau)d\tau)}{\max\Big\{\phi_q(\int_0^1 q(\tau)d\tau)(1+\frac{1}{\alpha}), \phi_q(\int_0^1 q(\tau)d\tau)(1+\frac{1}{\beta})\Big\}}\\
&= |\omega_0'(t)|,
\end{aligned}$$

then we obtain

$$\omega_1(t) \le \omega_0(t), \quad |\omega_1'(t)| \le |\omega_0'(t)|, \quad 0 \le t \le 1.$$

So,

$$\omega_2(t) = \bar{T}\omega_1(t) \le \bar{T}\omega_0(t) = \omega_1(t), \quad 0 \le t \le 1,$$

$$|\omega_2'(t)| = |(\bar{T}\omega_1)'(t)| \le |(\bar{T}\omega_0)'(t)| = |\omega_1'(t),$$
$$0 \le t \le 1.$$

Hence by induction, we have

$$\omega_{n+1} \le \omega_n, \quad |\omega_{n+1}'(t)| \le |\omega_n'(t)|,$$
$$0 \le t \le 1, \quad n = 1, 2, \cdots.$$

Thus, there exists $\omega^* \in \overline{K_a}$ such that $\omega_n \to \omega^*$. Letting $n \to \infty$ in (26), we obtain $\bar{T}\omega^* = \omega^*$ since T is continuous.

If $f(t,\ 0,\ 0) \not\equiv 0,\ 0 \le t \le 1$, then the zero function is not the solution of (2),(3). Therefore, ω^* is a positive solution of (2),(3).

Corollary 11 *Assume* $(\mathrm{H}_2) - (\mathrm{H}_4)$, (C_1), (C_3) *hold, and there exist* $0 < a_1 < a_2 < \cdots < a_n$, *such that*

$$(C_2') \quad \max_{0\le t\le 1} f(t, a_k, a_k) \le \phi_p(\frac{a_k}{A}),\ k = 1, 2, \cdots, n,$$

$$(particularly,\ \underline{lim}_{r\to+\infty}\max_{0\le t\le 1} f(t, r, a_k) = 0,$$
$$k = 1, 2, \cdots, n).$$

Then the boundary value problem (2),(3) *has* n *positive solutions* $\omega_k^* \in K$ *such that* $0 < \omega_k^* \le a_k$, $0 < |(\omega_k^*)'| \le a_k$ *and* $\lim_{n\to\infty}\bar{T}^n\omega_{k_0} = \omega_k^*$, $\lim_{n\to\infty}(\bar{T}^n\omega_{k_0})' = (\omega_k^*)'$*where*

$$\omega_{k_0}(t) = a_k\frac{\max\Big\{(\frac{1}{\alpha}+t),\ (\frac{1}{\beta}+(1-t))\Big\}}{\max\Big\{(\frac{1}{\alpha}+1),\ (\frac{1}{\beta}+1)\Big\}},$$
$$0 \le t \le 1.$$

Example: Consider the following boundary value problem

$$(|u'|^{-\frac{1}{2}}u')' + f(t, u(t), u'(t)) = 0, \quad 0 < t < 1, \quad (27)$$

$$u'(0) - u(\frac{1}{3}) = 0, \quad u'(1) + u(\frac{1}{2}) = 0, \quad (28)$$

where

$$f(t,x,y) = -t^2 + t + \frac{1}{4}x + \frac{1}{8}y^2.$$

We notice that $p = \frac{3}{2},\ q(t) = 1,\ \alpha = \beta = 1,\ \xi = \frac{1}{3},\ \eta = \frac{1}{2}$.

If we take $a = 1$, then we have $A = 2$.

Furthermore, we get $f(t,x,y)$ satisfies:

(C_1) $f(t,\ x_1,\ y_1) \le f(t,\ x_2,\ y_2),\ for\ any$ $0 \le t \le 1,\ \ 0 \le x_1 \le x_2 \le 1,$ $0 \le |y_1| \le |y_2| \le 1;$

(C_2) $\max_{0\le t\le 1} f(t,a,a) = f(\frac{1}{2},1,1) < \phi_{\frac{3}{2}}(\frac{a}{A}) = \frac{\sqrt{2}}{2};$

(C_3) $f(t,\ 0,\ 0) \not\equiv 0\ for\ 0 \le t \le 1.$

Then by Theorem 10, the boundary value problem (27),(28) has one positive solution $\omega^* \in K$ such that $0 < \omega^* \le 1,\ 0 < |(\omega^*)'| \le 1$ and $\lim_{n\to\infty}\bar{T}^n\omega_0 = \omega^*,\ \lim_{n\to\infty}(\bar{T}^n\omega_0)' = (\omega^*)'$.

4 Conclusion

In this paper, firstly, by determining the range of λ, we obtain at least one positive solution for the BVP (1), (3) using the fixed point theorem of cone expansion and compression of norm type. Secondly, we use monotone iterative technique to study problems (2), (3), we not only obtain the existence of positive solutions for (2), (3), but also construct some iterative schemes to find the solutions. To our knowledge, this is the first paper to obtain iteration of positive solutions to a four-point p-Laplacian boundary value problems (1),(3) and (2), (3). The solution is nonnegative, if it has, under the condition f is nonnegative, but this result can't be generalized to four-point BVP (1),(3) and BVP (2),(3). That is the main difficulty to us. The novelty of this paper is that we give a condition $\alpha \in (0, \frac{1}{\xi})$, $\beta \in (0, \frac{1}{1-\eta})$ to overcome this difficulty. To the author's knowledge, this is the first paper can be found in the literature on the existence of positive solutions and iteration of positive solutions to a four-point p-Laplacian boundary value problems.

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